RIGIDITY OF CONVEX DIVISIBLE DOMAINS IN FLAG MANIFOLDS

WOUTER VAN LIMBEEK AND ANDREW ZIMMER

Abstract. In contrast to the many examples of convex divisible domains in real projective space, we prove that up to projective isomorphism there is only one convex divisible domain in the Grassmannian of $p$-planes in $\mathbb{R}^{2p}$ when $p > 1$. Moreover, this convex divisible domain is a model of the symmetric space associated to the simple Lie group $\text{SO}(p, p)$.

Contents

1. Introduction 2
2. Preliminaries 7
3. The Grassmannians 8

Part 1. An invariant metric 12
4. The metric 12
5. Limits in the local Hausdorff topology and rescaling 15
6. The geometry near the boundary 16

Part 2. The automorphism group is non-discrete 18
7. Extreme points and symmetry 18

Part 3. The automorphism group is simple 22
8. Initial reduction 22
9. The centralizer 27
10. Unipotent subgroups 38
11. When $p = 2$ 41
12. Finishing the proof of Theorem 1.3 43

Part 4. Appendices 46
Appendix A. Proof of Theorem 4.5 46
Appendix B. Proof of Theorem 5.1 48
References 50
1. Introduction

The Lie group $\text{PGL}_{d+1}(\mathbb{R})$ acts naturally on real projective space $\mathbb{P}(\mathbb{R}^{d+1})$ and for an open set $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ we define the automorphism group of $\Omega$ as

$$\text{Aut}(\Omega) = \{ \varphi \in \text{PGL}_{d+1}(\mathbb{R}) : \varphi \Omega = \Omega \}.$$

An open set $\Omega$ is then called a convex divisible domain if it is a bounded convex open set in some affine chart of $\mathbb{P}(\mathbb{R}^{d+1})$ and there exists a discrete group $\Gamma \leq \text{Aut}(\Omega)$ which acts properly, freely, and cocompactly on $\Omega$. The fundamental example of a convex divisible domain comes from the Klein-Beltrami model of real hyperbolic $d$-space $\mathbb{H}^d_R$:

**Example 1.1.** Let $B \subset \mathbb{P}(\mathbb{R}^{d+1})$ be the unit ball in some affine chart. The group $\text{PSO}(d,1) \leq \text{PGL}_d(\mathbb{R})$ acts transitively (and by projective transformations) on $B$, and the stabilizer of a point is $\text{PSO}(d)$. This gives a natural identification $B \cong \text{PSO}(d,1)/\text{PSO}(d)$.

Further, there is a natural metric on $B$, called the Hilbert metric, such that $\text{PSO}(d,1) = \text{Isom}(B)^0$. Equipped with this metric, $B$ is isometric to hyperbolic $d$-space. Any torsion-free cocompact lattice $\Gamma$ in $\text{PSO}(d,1)$ will act properly discontinuously, freely, and cocompactly on $B$.

There are many more examples of convex divisible domains, for instance:

1. The symmetric spaces associated to $\text{SL}_d(\mathbb{R})$, $\text{SL}_d(\mathbb{C})$, $\text{SL}_d(\mathbb{H})$, and $E_6(-26)$ can all be realized as convex divisible domains. For instance, consider the convex set

$$\mathcal{P} = \{ [X] \in \mathbb{P}(S_{d,d}) : X \text{ is positive definite} \}$$

where $S_{d,d}$ is the vector space of real symmetric $d$-by-$d$ matrices. Then the group $\text{SL}_d(\mathbb{R})$ acts transitively on $\mathcal{P}$ by $g \cdot [X] = [gXg^t]$ and the stabilizer of a point is $\text{SO}(d)$. Hence, if $\Gamma \leq \text{PSL}_d(\mathbb{R})$ is a cocompact torsion-free lattice then $\Gamma$ acts properly, freely, and cocompactly on $\mathcal{P}$.

2. Let $B \subset \mathbb{P}(\mathbb{R}^{d+1})$ be the Klein-Beltrami model of $\mathbb{H}^d_R$. Results of Johnson-Millson [JM87] and Koszul [Kos68] imply that the domain $B$ can be deformed to a divisible convex domain $\Omega$ where $\text{Aut}(\Omega)$ is discrete (see [Ben00, Section 1.3] for $d > 2$ and [Gol10] for $d = 2$).

3. There are many examples in low dimensions (see for instance [Vin71, VK67]).

4. For every $d \geq 4$, Kapovich [Kap07] has constructed divisible convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^{d+1})$ such that $\text{Aut}(\Omega)$ is discrete, Gromov hyperbolic, and not quasi-isometric to any symmetric space.

5. Benoist [Ben06] and Ballas, Danciger, and Lee [BDL15] have constructed divisible convex domains $\Omega \subset \mathbb{P}(\mathbb{R}^4)$ such that $\text{Aut}(\Omega)$ is discrete, not Gromov hyperbolic, and not quasi-isometric to any symmetric space.

More background can be found in the survey papers by Benoist [Ben08], Marquis [Mar14], and Quint [Qui10].

There is a more general setting in which convex divisible domains can be studied, namely in flag manifolds: Suppose $G$ is a connected semi-simple Lie group with trivial center and compact factors. If $P \leq G$ is a parabolic subgroup then $G$ acts by diffeomorphisms on the compact manifold $G/P$, which is called a flag manifold. Given an open set $\Omega \subset G/P$ we define the automorphism group of $\Omega$ to be

$$\text{Aut}(\Omega) = \{ g \in G : g\Omega = \Omega \}.$$
The manifold $G/P$ admits natural affine charts given by translates of a Bruhat big cell, and a domain $\Omega$ is convex if it is convex in some affine chart. There are many examples of convex divisible domains in flag manifolds coming from symmetric spaces: The Harish-Chandra embedding shows that every noncompact Hermitian symmetric space $X$ embeds as a domain $\Omega_X$ into a flag manifold $G/P$ (and this flag manifold can be identified with the compact dual of $X$) such that $\text{Aut}(\Omega_X) = \text{Isom}_0(X)$, see e.g. [Hel78, 8.7.14]. More generally, Nagano [Nag65, Theorem 6.1] has characterized all the noncompact symmetric spaces $X$ whose compact dual $X^*$ can be identified with a flag manifold $G/P$ and $X$ embeds as a domain $\Omega_X$ into $G/P$ such that $\text{Aut}(\Omega_X) = \text{Isom}_0(X)$. In all these examples the images are bounded convex domains in some affine chart of $G/P$ [Nag65, Theorem 6.2].

There also exist examples of symmetric spaces which embed into a flag manifold which cannot be identified with their compact dual. In particular, we have already seen above that the symmetric spaces associated to $\text{SL}_{d-2}(\mathbb{C})$, $\text{SL}_{d-2}(\mathbb{H})$, and $E_{6(-26)}$ can all be realized as convex divisible domains in real projective spaces.

Given these examples it is natural to ask:

**Question 1.2.** If $G/P$ is a flag manifold, are there non-symmetric convex divisible domains in $G/P$?

Outside of the case when $G/P$ can be identified with real projective space or the complex projective plane we suspect that the answer is no. In particular, outside of those two cases the action of $G$ on $G/P$ usually preserves some special structure. For instance: if $G = \text{PSL}_{p+q}(\mathbb{R})$ and $P$ is the stabilizer of a $p$-plane then $G/P$ can be identified with $\text{Gr}_p(\mathbb{R}^{p+q})$ the Grassmanians of $p$-planes in $\mathbb{R}^{p+q}$. In this case the action of $G$ on $G/P$ preserves an “algebraic distance” given by $d(V,W) = \dim(V \cap W)$. Despite this source of rigidity, the above question seems difficult to answer in full generality.

In this paper we specialize to the particular case of real Grassmannians. As above let $G = \text{PSL}_{p+q}(\mathbb{R})$ and $P$ is the stabilizer of a $p$-plane then $G/P$ can be identified with $\text{Gr}_p(\mathbb{R}^{p+q})$ the Grassmanians of $p$-planes in $\mathbb{R}^{p+q}$. The set of $q$-by-$p$ real matrices $M_{q,p}(\mathbb{R})$ can be naturally identified with an affine chart of $\text{Gr}_q(\mathbb{R}^{p+q})$ via $X \leftrightarrow \text{Im} \left( \begin{pmatrix} \text{Id}_p \\ X \end{pmatrix} \right)$. Now let $\mathcal{B}_{q,p}$ be the unit ball (with respect to the Euclidean operator norm) in $M_{q,p}(\mathbb{R})$. As in the real projective setting $\mathcal{B}_{q,p}$ is a symmetric domain: In fact $\mathcal{B}_{q,p}$ can be identified with the symmetric space $\text{PSO}(p,q)/\text{PS}(\text{O}(p) \times \text{O}(q))$. Further under the above identification we have $\text{Aut}(\mathcal{B}_{q,p}) \cong \text{PSO}(p,q)$.

In contrast to the many examples of convex divisible domains in real projective space, we prove that every convex divisible domain in $\text{Gr}_p(\mathbb{R}^{2p})$ is symmetric and even more precisely that up to projective isomorphism $\mathcal{B}_{p,p}$ is the only convex divisible domain in $\text{Gr}_p(\mathbb{R}^{2p})$. The following is our main result.

**Theorem 1.3.** Suppose $p > 1$ and $\Omega \subset \text{Gr}_p(\mathbb{R}^{2p})$ is a bounded convex open subset of some affine chart, and there exists a discrete group $\Gamma \leq \text{Aut}(\Omega)$ such that $\Gamma$ acts cocompactly on $\Omega$. Then $\Omega$ is projectively isomorphic to $\mathcal{B}_{p,p}$.

**Remark 1.4.** There is much more flexibility for domains which are not bounded in an affine chart:

1. If $\Omega$ is an entire affine chart, there exists a discrete group $\Gamma \leq \text{Aut}(\Omega)$ which acts freely, properly, and cocompactly on $\Omega$ (see Subsection 3.5 below).
(2) If $P \subseteq Q$ are parabolic subgroups in $G$, then there is a natural projection $\pi : G/P \to G/Q$. Then for any divisible domain $\Omega \subset G/Q$, the preimage $\pi^{-1}(\Omega)$ is a divisible domain in $G/P$. This shows that for many flag manifolds, classifying divisible domains is at least as difficult as classifying divisible domains in real projective spaces.

(3) There are recent constructions by Guichard-Wienhard [GW08, GW12], Guéritaud-Guichard-Kassel-Wienhard [GGKW17], and by Kapovich-Leeb-Porti [KLP13] of open domains $\Omega$ in certain flag manifolds where there exists a discrete group $\Gamma \leq \text{Aut}(\Omega)$ that acts properly, freely, and cocompactly on $\Omega$. These constructions come from the theory of Anosov representations, and give many examples of nonsymmetric divisible domains $\Omega$. However, these constructions almost never produce domains which are bounded in any affine chart.

Remark 1.5. It is well-known that convex domains in real projective space are very similar to nonpositively curved Riemannian manifolds (see for instance [Ben04, Ben06, Cra09, CLT15]). In particular the flexibility of domains in real projective space and the rigidity of domains in $\text{Gr}_p(\mathbb{R}^{2p})$ when $p > 1$ can be compared to the well-known dichotomy for the rigidity of a nonpositively curved metric based on its Euclidean rank. Nonpositively curved metrics of rank one are very flexible (e.g. negatively curved metrics), but in higher rank there is an amazing amount of rigidity. Namely, the Higher Rank Rigidity Theorem of Ballmann [Bal85] and Burns-Spatzier [BS87a, BS87b] states that any nonpositively curved, irreducible, closed Riemannian manifold whose Euclidean rank is at least two, is isometric to a locally symmetric space. In this sense convex divisible domains in $\text{Gr}_p(\mathbb{R}^{2p})$ behave like irreducible nonpositively curved manifolds of higher Euclidean rank.

Remark 1.6. In Theorem 1.3 we only assume that there is a discrete group $\Gamma \leq \text{Aut}(\Omega)$ acting cocompactly on $\Omega$. However this implies that there exists a discrete group $\Gamma_0 \leq \text{Aut}(\Omega)$ that acts freely, properly discontinuously, and cocompactly on $\Omega$. Namely, we construct an invariant metric for the action of $\text{Aut}(\Omega)$ (see Step 1 below and Proposition 4.7), and hence $\text{Aut}(\Omega)$ acts properly on $\Omega$. Thus if $\Gamma \leq \text{Aut}(\Omega)$ is a discrete group and $\Gamma$ acts cocompactly on $\Omega$ then $\Gamma$ is finitely generated (by the Švarc-Milnor lemma, see [BH99, Chapter I.8 Proposition 8.19]). Then Selberg’s lemma (see [Alp87]) implies that $\Gamma$ has a finite-index torsion-free subgroup $\Gamma_0 \leq \Gamma$. Then $\Gamma_0$ acts freely, properly discontinuously, and cocompactly on $\Omega$.

1.1. Outline of the proof of Theorem 1.3. The proof of Theorem 1.3 uses a variety of techniques from real projective geometry, several complex variables, Riemannian geometry, Lie theory, and algebraic topology. Here is an outline of the three main steps:

Step 1: Constructing an invariant metric. A convex domain $\Omega$ in an affine chart of $\mathbb{P}(\mathbb{R}^{d+1})$ that is proper (that is, does not contain any affine real lines) has a complete metric called the Hilbert metric. One of the main steps in the proof is the construction of a metric $K_0$ that generalizes this classical construction.

We say a convex domain $\Omega$ in an affine chart of $\text{Gr}_p(\mathbb{R}^{p+q})$ is $\mathcal{R}$-proper if it does not contain any “rank-one affine real lines” (see Definition 4.3 below).
Theorem 1.7. (Theorem 4.5 and Theorem 5.1 below) Suppose \( \mathbb{M} \subset \text{Gr}_p(\mathbb{R}^{p+q}) \) is an affine chart and \( \Omega \subset \mathbb{M} \) is an \( R \)-proper convex open subset of \( \mathbb{M} \). Then there exists a complete length metric \( K_\Omega \) with the following properties:

1. (Invariance) the group \( \text{Aut}(\Omega) \) acts by isometries on \( (\Omega, K_\Omega) \).
2. (Equivariance) if \( \Phi \in \text{PGL}_{p+q}(\mathbb{R}) \), then
   \[ K_\Omega(x, y) = K_{\Phi \Omega}(\Phi x, \Phi y). \]
3. (Continuity in the local Hausdorff topology) if \( \Omega_n \subset \mathbb{M} \) is a sequence of \( R \)-proper convex open sets converging in the local Hausdorff topology to an \( R \)-proper convex open set \( \Omega \subset \mathbb{M} \), then \( K_{\Omega_n} \) converges to \( K_\Omega \) uniformly on compact subsets of \( \Omega \), and
4. if \( p = 1 \), then \( K_\Omega \) coincides with the classical Hilbert metric.

The above theorem allows us to establish an analogue of the powerful “rescaling” method from several complex variables (see the survey articles [Fia91, KK08]). See Remark 1.13 below for further details on this analogy (or lack thereof). We prove:

Theorem 1.8. (Theorem 5.2 below) Suppose \( \mathbb{M} \subset \text{Gr}_p(\mathbb{R}^{p+q}) \) is an affine chart, \( \Omega \subset \mathbb{M} \) is an \( R \)-proper convex open subset of \( \mathbb{M} \), and \( \text{Aut}(\Omega) \) acts cocompactly on \( \Omega \). If \( A_n \in \text{Aff}(\mathbb{M}) \cap \text{PGL}_{p+q}(\mathbb{R}) \) and \( A_n \Omega \) is a sequence of \( R \)-proper convex sets converging in the local Hausdorff topology to an \( R \)-proper convex open set \( \hat{\Omega} \), then there exists some \( \Phi \in \text{PGL}_{p+q}(\mathbb{R}) \) such that \( \Phi(\Omega) = \hat{\Omega} \).

Remark 1.9. An affine chart \( \mathbb{M} \subset \text{Gr}_p(\mathbb{R}^{p+q}) \) can be identified with the vector space \( M_{q,p}(\mathbb{R}) \) of \( q \)-by-\( p \) real matrices in a way that is unique up to an affine automorphism of \( M_{q,p}(\mathbb{R}) \) (see Subsection 3.3 for details). In particular, the group \( \text{Aff}(\mathbb{M}) \) of affine transformations of \( \mathbb{M} \) is well-defined (see Definition 3.3).

To explain how the properties of the metric \( K_\Omega \) imply Theorem 1.8 let us sketch the proof:

Proof Sketch. Suppose that \( A_n \Omega \to \hat{\Omega} \). Fix a point \( x_0 \in \Omega \). Since \( \text{Aut}(\Omega) \) acts cocompactly on \( \Omega \), we can pass to a subsequence and find \( \varphi_n \in \text{Aut}(\Omega) \) such that \( A_n \varphi_n x_0 \to \hat{x}_0 \in \hat{\Omega} \). Now consider the maps \( f_n := A_n \varphi_n \). By part (1) and (2) of Theorem 1.7, each \( f_n \) induces an isometry \( (\Omega, K_\Omega) \to (\Omega_n, K_{\Omega_n}) \). Then by part (3) of Theorem 1.7, one can pass to a subsequence such that \( f_n \to f \) and \( f \) will be an isometry \( (\Omega, K_\Omega) \to (\hat{\Omega}, K_{\hat{\Omega}}) \). A simple argument then shows that \( f \) is actually the restriction of a element in \( \text{PGL}_{p+q}(\mathbb{R}) \).

Theorem 1.8 should also be compared to a theorem of Benzécri from real projective geometry. Let \( X_d \) be the space of proper convex open sets in \( \mathbb{P}(\mathbb{R}^d) \) with the Hausdorff topology. Then \( X_d \) is closed in the Hausdorff topology and \( \text{PGL}_d(\mathbb{R}) \) acts on \( X_d \). With this notation Benzécri proved:

Theorem 1.10. Benzécri Suppose \( \Omega \) is a proper convex open set in \( \mathbb{P}(\mathbb{R}^d) \). If \( \text{Aut}(\Omega) \) acts cocompactly on \( \Omega \), then \( \text{PGL}_d(\mathbb{R}) \cdot \Omega \) is a closed subset of \( X_d \).

It is important to note that unlike in the real projective setting, when \( p, q > 1 \), convexity is not invariant under the action of \( \text{PGL}_{p+q}(\mathbb{R}) \) on \( \text{Gr}_p(\mathbb{R}^{p+q}) \): If \( \Omega \) is a convex subset of some affine chart \( \mathbb{M} \subset \text{Gr}_p(\mathbb{R}^{p+q}) \) and \( \phi(\Omega) \subset \mathbb{M} \) for some \( \phi \in \text{PGL}_p(\mathbb{R}^{p+q}) \), then \( \phi(\Omega) \) may not be a convex subset of \( \mathbb{M} \). Thus to preserve convexity we are forced to consider the orbit of \( \Omega \) under the group \( \text{Aff}(\mathbb{M}) \cap \text{PGL}_{p+q}(\mathbb{R}) \).
Step 2: The automorphism group is non-discrete. In the second step of the proof we use the rescaling theorem (Theorem 1.8) from Step 1 to show that $\text{Aut}(\Omega)$ is non-discrete when $\Omega \subset \text{Gr}_p(\mathbb{R}^{2p})$ is a convex divisible domain.

We can identify $M_{p,p}(\mathbb{R})$ with the affine chart \[
\left\{ \begin{bmatrix} \text{Id}_p & X \\ 0 & \text{Id}_p \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}
\]
of $\text{Gr}_p(\mathbb{R}^{2p})$. Recall that the unit ball $B_{p,p} \subset M_{p,p}(\mathbb{R})$ (with respect to the Euclidean operator norm) can be identified with the symmetric domain $\text{PSO}(p,p)/\text{PSO}(p) \times \text{O}(p)$. Note that $B_{p,p}$ is a convex set and the extreme points of $B_{p,p}$ are exactly the orthogonal matrices. Given an orthogonal matrix $A \in \partial B_{p,p}$, define the projective transformation \[
F(X) := \begin{bmatrix} -\text{Id}_p & A^{-1} \\ \text{Id}_p & A^{-1} \end{bmatrix} \cdot X = (A^{-1}X + \text{Id}_p)(A^{-1}X - \text{Id}_p)^{-1}.
\]
Then we see that $F(B_{p,p}) = \{ X \in M_{p,p}(\mathbb{R}) : X^t + X > 0 \}$ and $F(A) = 0$. Now $F(B_{p,p})$ is a cone and in particular $\text{Aut}(F(B_{p,p}))$ contains a one-parameter group of homotheties. Translating this back to $B_{p,p}$ shows that $A \in \partial B_{p,p}$ is the attracting fixed point of a one-parameter group of automorphisms of $B_{p,p}$.

Using the rescaling theorem (Theorem 1.8) from Step 1 we will recover these one-parameter groups for a general divisible domain. The key result is the following:

**Theorem 1.11.** (Theorem 7.4 below) Suppose $M \subset \text{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, $\Omega \subset M$ is an $R$-proper convex subset of $M$, and $\text{Aut}(\Omega)$ acts cocompactly on $\Omega$. If $e \in \partial \Omega$ is an extreme point, then the tangent cone of $\Omega$ at $e$ is $R$-proper.

Now the tangent cone of $\Omega$ at $e$ is precisely the limit of the rescaled domains
\[
n(\Omega - e) + e
\]
in the local Hausdorff topology. In particular combining Theorem 1.8 and Theorem 1.11 implies the following:

**Corollary 1.12.** (Corollary 7.11 below) Suppose $M \subset \text{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, $\Omega \subset M$ is an $R$-proper convex subset of $M$, and $\text{Aut}(\Omega)$ acts cocompactly on $\Omega$. Then $\text{Aut}(\Omega)$ is non-discrete.

**Remark 1.13.** In the several complex variable setting, rescaling can also be used to find one-parameter groups of automorphisms (see [Fra89, Section 6] or [Kim04]). However, in this setting one obtains these automorphisms by rescaling at a point in the boundary with either $C^1$ or $C^2$ regularity. This procedure actually finds automorphisms because a complex line has two real dimensions (see the proof of [Fra91, Lemma 6.8]). In contrast we find a one-parameter group of automorphisms by rescaling at a point where the tangent cone is $R$-proper and hence very far from being $C^1$. Finally, we should observe that the rescaling method cannot be used to find one-parameter groups of automorphisms in the real projective setting.

**Remark 1.14.** If $p \neq q$, an explicit computation for $B_{p,q}$ shows that Theorem 1.11 fails in this setting. This is one of the main problems that prevent us from extending our methods to the general case.
Step 3: Showing the automorphism group is simple and acts transitively

In the final part of the proof we show that Aut_0(Ω), the connected component of the identity of Aut(Ω), is a simple Lie group which acts transitively on Ω.

Our approach for this step is based on work of Farb and Weinberger [FW08] who prove a number of remarkable rigidity results for compact aspherical Riemannian manifolds whose universal covers have non-discrete isometry groups. In particular, we combine their approach with the representation theory of Lie groups to establish: Whenever Ω is a bounded and convex domain in an affine chart and Γ ⊆ Aut(Ω) is discrete such that Γ \ Ω is compact, at least one of the following holds (see Theorem 8.1):

1. a finite-index subgroup of Γ has non-trivial centralizer in PGL_2p(ℝ),
2. there exists a nontrivial abelian normal unipotent group U ≤ Aut(Ω) such that Γ ∩ U is a cocompact lattice in U,
3. p = 2 and there exists a finite-index subgroup G’ of Aut(Ω) such that G’ = Aut_0(Ω) × Λ for some discrete group Λ. Further up to conjugation
   \[ \text{Aut}_0(Ω) = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A ∈ \text{SL}_2(ℝ) \right\} \]
   and
   \[ \Lambda ≤ \left\{ \begin{bmatrix} a \cdot \text{Id}_2 & b \cdot \text{Id}_2 \\ c \cdot \text{Id}_2 & d \cdot \text{Id}_2 \end{bmatrix} : ad – bc = 1 \right\}. \]
4. p = 2, Aut_0(Ω) ≤ Aut(Ω) has finite-index and acts transitively on Ω, and up to conjugation
   \[ \text{Aut}_0(Ω) = \left\{ \begin{bmatrix} aA & bA \\ cA & dA \end{bmatrix} : A ∈ \text{SL}_2(ℝ), ad – bc = 1 \right\}. \]
5. Aut_0(Ω) is a simple Lie group with trivial center that acts transitively on Ω.

In Sections 9, 10, and 11 we use the dynamics of the action of PGL_2p(ℝ) on Gr_p(ℝ^{2p}) to show that the first four cases are impossible. Finally in Section 12 we use the classification of simple Lie groups and the representation theory of simple Lie groups to complete the proof Theorem 1.3.

Acknowledgments. We would like to thank Benson Farb and Ralf Spatzier for many helpful conversations. The first author gratefully acknowledges support from the University of Chicago while part of this work was done. The second author was partially supported by the National Science Foundation under grant number 1045119 and grant number 1400919.

2. Preliminaries

2.1. Notations. Given some object o we will let [o] be the projective equivalence class of o, for instance: if v ∈ ℝ^{d+1} \ {0} let [v] denote the image of v in P(ℝ^{d+1}); if φ ∈ GL_{d+1}(ℝ) let [φ] denote the image of φ in PGL_{d+1}(ℝ); if T ∈ Hom(ℝ^{d_1+1}, ℝ^{d_2+1}) \ {0} let [T] denote the image of T in P(Hom(ℝ^{d_1+1}, ℝ^{d_2+1})).
2.2. The Hilbert metric. The Hilbert metric is classically only defined for convex domains in real projective space, but Kobayashi [Kob77] gave a construction that works for any open connected domain in real projective space. In this subsection we recall Kobayashi’s construction.

Given four points \( a, x, y, b \in \mathbb{P}(\mathbb{R}^d) \) that are collinear, that is contained in a projective line, one can define the cross ratio by

\[
[a; x; y; b] = \log \frac{|x - b| |y - a|}{|x - a| |y - b|}.
\]

The cross ratio is \( \text{PGL}_d(\mathbb{R}) \)-invariant in the sense that

\[
[a; x; y; b] = [\varphi a; \varphi x; \varphi y; \varphi b]
\]

for any \( \varphi \in \text{PGL}_d(\mathbb{R}) \).

Next consider the interval

\( I := \{[1 : t] \in \mathbb{P}(\mathbb{R}^2) : |t| < 1 \} \)

and the function \( H_I : I \times I \to \mathbb{R}_{\geq 0} \) given by

\[
H_I(s, t) = |\log[-1; s; t; 1]|.
\]

Then \( H_I \) is a complete \( \text{Aut}(I) \)-invariant length metric on \( I \).

Now suppose that \( \Omega \subset \mathbb{P}(\mathbb{R}^d) \) is an open connected set. Let

\[
\text{Proj}(I, \Omega) \subset \mathbb{P}(\text{End}(\mathbb{R}^2, \mathbb{R}^d))
\]

be the set of projective maps \( T \) such that \( I \cap \ker T = \emptyset \) and \( T(I) \subset \Omega \). Then define a function \( \rho_\Omega : \Omega \times \Omega \to \mathbb{R} \cup \{\infty\} \) as follows:

\[
\rho_\Omega(x, y) := \inf \{H_I(s, t) : \text{there exists } f \in \text{Proj}(I, \Omega) \text{ with } f(s) = x \text{ and } f(t) = y\}.
\]

Finally, using \( \rho_\Omega \), one defines the pseudometric \( K_\Omega \) as

\[
K_\Omega(x, y) = \inf \left\{ \sum_{i=0}^{N-1} \rho_\Omega(x_i, x_{i+1}) : N > 0, x_0, \ldots, x_N \in \Omega, x_0 = x, x_N = y \right\}.
\]

Note that if \( x, y \in \Omega \) are such that the projective line through \( x \) and \( y \) has unbounded intersection with \( \Omega \), then \( K_\Omega(x, y) = 0 \). Kobayashi proved the following:

**Theorem 2.1.** [Kob77] Suppose \( \Omega \subset \mathbb{P}(\mathbb{R}^d) \) is an open connected set. Then

1. \( K_\Omega \) is a \( \text{Aut}(\Omega) \)-invariant pseudometric on \( \Omega \), i.e. \( K_\Omega \) is finite, symmetric, and satisfies the triangle inequality.
2. If \( \Omega \) is bounded in an affine chart, then \( K_\Omega \) is a metric.
3. If \( \Omega \) is convex and bounded in some affine chart, then \( K_\Omega \) coincides with the Hilbert metric.
4. \( K_\Omega \) is a complete metric if and only if \( \Omega \) is convex and bounded in some affine chart.

3. The Grassmannians

In this expository section we recall the two standard models of the Grassmannians, define affine charts, and describe the projective lines contained in the Grassmannians.
3.1. The matrix model. We can identify $\text{Gr}_p(\mathbb{R}^{p+q})$ with the quotient
$$\{ X \in M_{p+q,p}(\mathbb{R}) : \text{rank} \ X = p \}/ \text{GL}_p(\mathbb{R})$$
where $\text{GL}_p(\mathbb{R})$ acts on $M_{p+q,p}(\mathbb{R})$ by multiplication on the right and the identification with $\text{Gr}_p(\mathbb{R}^{p+q})$ is given by $X \mapsto \text{Im}(X)$. Note that in this model the action of $\text{PGL}_{p+q}(\mathbb{R})$ on $\text{Gr}_p(\mathbb{R})$ is given by the action by multiplication on the left on $M_{p+q,p}(\mathbb{R})$.

3.2. The projective model. We have a natural embedding $\text{Gr}_p(\mathbb{R}^{p+q}) \rightarrow \mathbb{P}(\wedge^p \mathbb{R}^{p+q})$ defined by
$$\text{Span}(v_1, \ldots, v_p) \rightarrow [v_1 \wedge \cdots \wedge v_p].$$

Remark 3.1. The image of $\text{Gr}_p(\mathbb{R}^{p+q})$ is a closed smooth algebraic subvariety of dimension $pq$ in $\mathbb{P}(\Lambda^p \mathbb{R}^{p+q})$, which has dimension $(p+q) - 1$. Nevertheless, if $O \subset \text{Gr}_p(\mathbb{R}^{p+q})$ is open, then the cone over the image of $O$ in $\mathbb{P}(\Lambda^p \mathbb{R}^{p+q})$ spans $\Lambda^p \mathbb{R}^{p+q}$.

Remark 3.2. The following characterization of the image will also be useful: For $x \in \Lambda^p \mathbb{R}^{p+q}$, we have that $[x]$ belongs to $\text{Gr}_p(\mathbb{R}^{p+q})$ if and only if the linear map $T_x : \mathbb{R}^{p+q} \rightarrow \Lambda^{p+1} \mathbb{R}^{p+q}$ given by $T_x(v) = v \wedge x$ has rank $q$.

It is also straightforward to describe the action of $\text{PGL}_{p+q}(\mathbb{R})$ on $\text{Gr}_p(\mathbb{R}^{p+q})$: Any element $g \in \text{PGL}_{p+q}(\mathbb{R})$ induces a natural projective linear map $\Lambda^p g$ of $\mathbb{P}(\Lambda^p \mathbb{R}^{p+q})$ defined by
$$\Lambda^p g[v_1 \wedge \cdots \wedge v_p] := [gv_1 \wedge \cdots \wedge gv_p].$$

The image of $\text{Gr}_p(\mathbb{R}^{p+q})$ in $\mathbb{P}(\Lambda^p \mathbb{R}^{p+q})$ is invariant under the action of $\text{PGL}_{p+q}(\mathbb{R})$.

3.3. Affine charts. Suppose $W_0$ is a $q$-dimensional subspace of $\mathbb{R}^{p+q}$. Then consider the set
$$M := \{ U \in \text{Gr}_p(\mathbb{R}^{p+q}) : U \cap W_0 = \{0\} \}.$$
Note that $M$ is an open dense subset of $\text{Gr}_p(\mathbb{R}^{p+q})$. We call $M$ an affine chart.

If we fix a subspace $U_0 \in M$, we can identify $M$ with the set $\text{Hom}(U_0, W_0)$ via
$$\text{Hom}(U_0, W_0) \rightarrow M \quad T \mapsto \text{Graph}(T) := \{(\text{Id} + T)u : u \in U_0\}.$$
Fixing bases of $U_0$ and $W_0$ gives an identification of $M$ with the space of $q$-by-$p$ real matrices. Notice that a different choice of bases or of $U_0$ only changes this identification by a map of the form
$$X \mapsto AXB + C$$
where $A \in \text{GL}_q(\mathbb{R})$, $B \in \text{GL}_p(\mathbb{R})$, and $C$ is a $q$-by-$p$ matrix. This observation leads to the next definition:

Definition 3.3. For an affine chart $M \subset \text{Gr}_p(\mathbb{R}^{p+q})$ let $\text{Aff}(M)$ be the transformations of $M$ that are affine maps with respect to some (and hence any) identification of $M$ with the space of $q$-by-$p$ real matrices.

We end this subsection with some basic facts about affine charts.

Observation 3.4. For an affine chart $M \subset \text{Gr}_p(\mathbb{R}^{p+q})$, the group $\text{Aff}(M) \cap \text{PGL}_{p+q}(\mathbb{R})$ coincides with the stabilizer of $M$ in $\text{PGL}_{p+q}(\mathbb{R})$. 

Proof. It is straightforward to see that
\[
\text{Aff}(\mathcal{M}) \cap \text{PGL}_{p+q}(\mathbb{R}) = \left\{ \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} : A \in \text{GL}_p(\mathbb{R}), C \in M_{q,p}(\mathbb{R}), D \in \text{GL}_q(\mathbb{R}) \right\}
\]
and that \( \text{Aff}(\mathcal{M}) \cap \text{PGL}_{p+q}(\mathbb{R}) \) stabilizes \( \mathcal{M} \). So suppose that \( g(\mathcal{M}) = \mathcal{M} \) and
\[
g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]
for some \( A \in \text{GL}_p(\mathbb{R}), B \in M_{p,q}(\mathbb{R}), C \in M_{q,p}(\mathbb{R}), \) and \( D \in \text{GL}_q(\mathbb{R}) \). Then \( \det(A + BX) \neq 0 \) for every \( X \in M_{q,p}(\mathbb{R}) \) which is only possible if \( B = 0 \). Thus \( g \in \text{Aff}(\mathcal{M}) \cap \text{PGL}_{p+q}(\mathbb{R}) \). \(\square\)

If \( \mathcal{M} \) is an affine chart then there exists \( g \in \text{PGL}_{p+q}(\mathbb{R}) \) such that
\[
g\mathcal{M} = \left\{ \begin{bmatrix} \text{Id}_p \\ X \end{bmatrix} : X \in M_{q,p}(\mathbb{R}) \right\}
\]
in the matrix model. Moreover, if \( e_1, \ldots, e_{p+q} \) is the standard basis of \( \mathbb{R}^{p+q} \) then
\[
g\mathcal{M} = \{(e_1 + v_1) \wedge \cdots \wedge (e_p + v_p) : v_1, \ldots, v_p \in \text{Span}\{e_{p+1}, \ldots, e_{p+q}\}\}
\]
in the projective model.

3.4. Projective lines in the two models. The description of an affine chart of \( \text{Gr}_p(\mathbb{R}^{p+q}) \) as a subset of \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \), given by Equation (3.2), shows that a generic line in \( \mathcal{M} \) is not contained in a projective line in \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \). However, there is a natural set of lines in \( \mathcal{M} \) which are. In this subsection we characterize these lines.

Lemma 3.5. If \( \ell \) is a projective line in \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \) contained in \( \text{Gr}_p(\mathbb{R}^{p+q}) \), then there exist \( v_1, \ldots, v_p, w \in \mathbb{R}^{p+q} \) such that
\[
\ell = \left\{ [v_1 \wedge \cdots \wedge v_{p-1} \wedge (v_p + tw)] : t \in \mathbb{R} \right\} \cup \left\{ [v_1 \wedge \cdots \wedge v_p \wedge w] \right\}
\]
Proof. Recall that for \( x \in \wedge^p \mathbb{R}^{p+q} \), we have that \( [x] \) belongs to \( \text{Gr}_p(\mathbb{R}^{p+q}) \) if and only if the linear map \( T_x : \mathbb{R}^{p+q} \to \wedge^{p+1} \mathbb{R}^{p+q} \) given by \( T_x(v) = v \wedge x \) has rank \( q \).

Now since \( \ell \) is a projective line there exist \( w_1, \ldots, w_p, v_1, \ldots, v_p \in \mathbb{R}^{p+q} \) such that
\[
\ell = \left\{ [(v_1 \wedge \cdots \wedge v_p) + t(w_1 \wedge \cdots \wedge w_p)] : t \in \mathbb{R} \right\} \cup \left\{ [w_1 \wedge \cdots \wedge w_p] \right\}
\]
Let
\[
V = \text{Span}\{v_1, \ldots, v_p\} \cap \text{Span}\{w_1, \ldots, w_p\}
\]
and \( r = \text{dim} V \). We claim that \( r = p - 1 \).

We can assume that \( v_i = w_i \) for \( 1 \leq i \leq r \) and thus \( v_1, \ldots, v_p, w_{r+1}, \ldots, w_p \) are all linearly independent. So if
\[
x_t = (v_1 \wedge \cdots \wedge v_p) + t(w_1 \wedge \cdots \wedge w_p)
\]
and \( v \wedge x_t = 0 \) then either \( v \in V \) or
\[
v \wedge v_1 \wedge \cdots \wedge v_p = -t(v \wedge w_1 \wedge \cdots \wedge w_p) \neq 0.
\]
This last case is only possible when \( r = p - 1 \) and \( v = v_p - tw_p \). Since \( \dim \ker T_{x_t} = p \) and \( \dim V = r \leq p - 1 \) this implies that \( r = p - 1 \). Then
\[
[(v_1 \wedge \cdots \wedge v_p) + t(w_1 \wedge \cdots \wedge w_p)] = [v_1 \wedge \cdots \wedge v_{p-1} \wedge (v_p + tw_p)]
\]
for all \( t \in \mathbb{R} \), which implies the lemma. \(\square\)
Corollary 3.6. Suppose \( x, y \in \text{Gr}_p(\mathbb{R}^{p+q}) \). Then the following are equivalent:

1. There exists a projective line \( \ell \) in \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \) contained in \( \text{Gr}_p(\mathbb{R}^{p+q}) \) such that \( x, y \in \ell \).
2. \( \dim(x \cap y) \geq p - 1 \).

Lemma 3.7. Suppose \( \mathcal{M} \) is an affine chart in \( \text{Gr}_p(\mathbb{R}^{p+q}) \) and we identify \( \mathcal{M} \) with the set of \( q \)-by-\( p \) matrices. Then

1. if \( \ell \) is a projective line in \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \) contained in \( \text{Gr}_p(\mathbb{R}^{p+q}) \) and \( \ell \cap \mathcal{M} \neq \emptyset \), then

\[
\ell \cap \mathcal{M} = \{ X + tS : t \in \mathbb{R} \}
\]

for some \( X, S \in \mathcal{M} \) with \( \text{rank}(S) = 1 \).

2. Conversely, if \( X, S \in \mathcal{M} \) and \( \text{rank}(S) = 1 \) then the closure of

\[
\{ X + tS : t \in \mathbb{R} \}
\]

in \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \) is a projective line contained in \( \text{Gr}_p(\mathbb{R}^{p+q}) \).

Proof. First suppose that \( \ell \) is a projective line contained in \( \text{Gr}_p(\mathbb{R}^{p+q}) \) and \( \ell \cap \mathcal{M} \neq \emptyset \). There exists some \( W_0 \in \text{Gr}_q(\mathbb{R}^{p+q}) \) such that \( \mathcal{M} = \{ U \in \text{Gr}_p(V) : U \cap W_0 = (0) \} \). By the previous Lemma we can assume

\[
\ell = \left\{ [v_1 \wedge \cdots \wedge v_{p-1} \wedge (v_p + tw) : t \in \mathbb{R}] \cup \left\{ [v_1 \wedge \cdots \wedge v_{p-1} \wedge w] \right\} \right\},
\]

for some \( w, v_1, \ldots, v_p \in \mathbb{R}^{p+q} \). By modifying these vectors we can assume that \( [v_1 \wedge \cdots \wedge v_p] \in \mathcal{M} \) and \( w \in W_0 \) (in particular \( [w \wedge v_2 \wedge \cdots \wedge v_p] \notin \mathcal{M} \) ). Let \( U_0 = \text{Span}(v_1, \ldots, v_p) \) and identify \( \mathcal{M} \) with \( \text{Hom}(U_0, W_0) \). Under this identification \( [v_1 \wedge \cdots \wedge v_{p-1} \wedge (v_p + tw)] \) corresponds to the homomorphism \( tS \) where \( S \) is the linear map

\[
S \left( \sum_{i=1}^{p} \alpha_i v_i \right) = \alpha_1 w.
\]

Then \( \ell \cap \mathcal{M} = \{ tS : t \in \mathbb{R} \} \). Then the first part of the lemma follows from the change of coordinates formula (3.1).

Next suppose that \( X, S \in \mathcal{M} \) and \( \text{rank}(S) = 1 \). There exists a basis \( v_1, \ldots, v_p \in \mathbb{R}^p \) such that \( v_1, \ldots, v_{p-1} \in \text{ker} S \) and \( Sv_p \neq 0 \). Then \( X + tS \) corresponds to the subspace

\[
\text{Span}(v_1 + X(v_1), \ldots, v_{p-1} + X(v_{p-1}), v_p + X(v_p) + tS(v_p))
\]

and hence in the projective model the line

\[
\left[ (v_1 + X(v_1)) \wedge \cdots \wedge (v_{p-1} + X(v_{p-1})) \wedge (v_p + X(v_p) + tS(v_p)) \right].
\]

So the closure of \( \{ X + tS : t \in \mathbb{R} \} \) in \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \) is a projective line. \( \square \)

Since the lines in \( \mathcal{M} \) that arise from projective lines in \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \) will play an important role, it is convenient to make the following definition.

Definition 3.8. A rank-one line is a line \( \ell \) in \( \text{Gr}_p(\mathbb{R}^{p+q}) \) of the form of Lemma 3.7, i.e. such that the image of \( \ell \) in \( \mathbb{P}(\wedge^p \mathbb{R}^{p+q}) \) is a line.
3.5. **A Trivial Example.** In this subsection we observe that an entire affine chart is an example of a convex divisible domain. Using the matrix model of $\text{Gr}_p(\mathbb{R}^{p+q})$ let

$$
\Omega = \left\{ \begin{bmatrix} \text{Id}_p \\ X \end{bmatrix} : X \in M_{q,p}(\mathbb{R}) \right\}.
$$

Then

$$
\Gamma = \left\{ \begin{bmatrix} \text{Id}_p \\ 0 \\ \text{Id}_q \end{bmatrix} : Y \in M_{q,p}(\mathbb{Z}) \right\} \leq \text{Aut}(\Omega)
$$

is a discrete group which acts freely, properly discontinuously, and cocompactly on $\Omega$. Notice that the quotient $\Gamma \backslash \Omega$ can be identified with the torus of dimension $pq$.

**Part 1. An invariant metric**

4. The metric

The purpose of this section is to extend Kobayashi’s definition of the Hilbert metric to domains in $\text{Gr}_p(\mathbb{R}^{p+q})$.

Suppose that $\Omega \subset \text{Gr}_p(\mathbb{R}^{p+q})$ is open and connected. Recall from Subsection 2.2 that $I \subset \mathbb{P}(\mathbb{R}^2)$ is the open interval

$$
I := \{ [1 : t] \in \mathbb{P}(\mathbb{R}^2) : |t| < 1 \}
$$

and $H_I$ is the Hilbert metric on $I$. Using the projective model of the Grassmannians, view $\Omega$ as a subset of $\mathbb{P}(\wedge^p \mathbb{R}^{p+q})$ and let

$$
\text{Proj}(I, \Omega) \subset \mathbb{P}(\text{End}(\mathbb{R}^2, \wedge^p \mathbb{R}^{p+q}))
$$

be the set of projective maps such that $I \cap \ker T = \emptyset$ and $T(I) \subset \Omega$. Then define a function $\rho_\Omega : \Omega \times \Omega \to \mathbb{R} \cup \{ \infty \}$ as follows:

$$
\rho_\Omega(x,y) := \inf \{ H_I(s,t) : \text{there exists } f \in \text{Proj}(I, \Omega) \text{ with } f(s) = x \text{ and } f(t) = y \}.
$$

We then define

$$
K^{(n)}_{\Omega}(x,y) := \inf \left\{ \sum_{i=0}^{n-1} \rho_\Omega(x_i, x_{i+1}) : x = x_0, x_1, \ldots, x_{n-1}, x_n = y, x \in \Omega \right\}.
$$

In particular $K^{(n)}_{\Omega}(x,y)$ is finite precisely when there is a path in $\Omega$ from $x$ to $y$ consisting of at most $n$ segments of projective lines. Further we evidently have $K^{(n)}_{\Omega} \leq K^{(n+1)}_{\Omega}$ for any $n$, so we set

$$
K_\Omega(x,y) := \lim_{n \to \infty} K^{(n)}_{\Omega}(x,y).
$$

Note that at the moment it is not clear that $K_\Omega$ is finite, but we will prove this in Proposition 4.2.1.

**Remark 4.1.** For $x, y \in \Omega$ it is possible to explicitly compute $\rho_\Omega(x,y)$:

1. If $\dim(x \cap y) < p - 1$ then $\rho_\Omega(x,y) = \infty$.
2. If $\dim(x \cap y) \geq p - 1$, let $\ell$ be the projective line in $\text{Gr}_p(\mathbb{R}^{p+q})$ containing $x$ and $y$. If $x$ and $y$ are in different connected components of $\ell \cap \Omega$, then $\rho_\Omega(x,y) = \infty$. Finally, if $x, y$ are contained in the same connected component $O$ of $\ell \cap \Omega$, then

$$
\rho_\Omega(x,y) = |\log[a; x; y; b]|$$

where $a, b$ are equations in $\ell \cap \Omega$.
where \(a,b\) are the endpoints of \(O\).

**Proposition 4.2.** If \(\Omega \subset \text{Gr}_p(\mathbb{R}^{p+q})\) is an open connected set then:

1. if \(\varphi \in \text{PGL}_{p+q}(\mathbb{R})\) then \(K_\Omega(x,y) = K_{\varphi \Omega}(\varphi x, \varphi y)\) for all \(x,y \in \Omega\),
2. \(K_\Omega(x,y) \leq K_\Omega(x,z) + K_\Omega(z,y)\) for any \(x,y,z \in \Omega\),
3. if \(\Omega_1 \subset \Omega_2\) then \(K_{\Omega_2}(x,y) \leq K_{\Omega_1}(x,y)\) for all \(x,y \in \Omega_1\),
4. for any compact set \(K \subset \Omega\) there exists \(N > 0\) such that \(K_\Omega^{(N)}(x,y) < \infty\) for every \(x,y \in K\),
5. \(K_\Omega\) is continuous.

**Proof.** Parts (1)-(3) follow from the definition of \(K_\Omega\) and the invariance of the cross-ratio.

To establish part (4) it is enough to show the following: for any \(x \in \Omega\) there exist an open neighborhood \(U\) of \(x\) and a number \(n = n(x)\) such that \(K_\Omega^{(n)}(z,y) < \infty\) for any \(z,y \in U\). Suppose that \(x = [v_1 \wedge v_2 \wedge \cdots \wedge v_p]\). Then there exists \(\epsilon > 0\) such that

\[
U := \{[w_1 \wedge w_2 \wedge \cdots \wedge w_p] : \|v_i - w_i\| < \epsilon \text{ for } 1 \leq i \leq p\} \subset \Omega.
\]

But then clearly \(K_\Omega^{(p-1)}(z,y) < \infty\) for any \(z,y \in U\).

To establish part (5), first observe that

\[
|K_\Omega(x_0,y_0) - K_\Omega(x,y)| \leq K_\Omega(x_0,x) + K_\Omega(y,y_0)
\]

so it is enough to show that the map \(x \mapsto K_\Omega(x_0,x)\) is continuous at \(x_0\). But if \(x_0 = [v_1 \wedge v_2 \wedge \cdots \wedge v_p]\) then there exists \(\epsilon > 0\) such that

\[
U := \{[w_1 \wedge w_2 \wedge \cdots \wedge w_p] : \|v_i - w_i\| < \epsilon \text{ for } 1 \leq i \leq p\} \subset \Omega.
\]

But then for \([w_1 \wedge \cdots \wedge w_p] \in U\) we have

\[
K_\Omega(x_0,[w_1 \wedge w_2 \wedge \cdots \wedge w_p]) \leq K_\Omega(x_0,[w_1 \wedge w_2 \wedge \cdots \wedge w_p]) \leq \sum_{i=2}^{p} \log \frac{\epsilon + \|v_i - w_i\|}{\epsilon - \|v_i - w_i\|}
\]

and so

\[
\lim_{x \to x_0} K_\Omega(x_0,x) = 0.
\]

The above Proposition shows that \(K_\Omega\) is an \(\text{Aut}(\Omega)\)-invariant pseudometric. We will next show that \(K_\Omega\) is a complete metric for certain convex subsets.

**Definition 4.3.**

1. Let \(\mathcal{L}\) be the space of rank one lines in \(\text{Gr}_p(\mathbb{R}^{p+q})\), that is the space of projective lines in \(\mathbb{P}(\wedge^p \mathbb{R}^{p+q})\) which are contained in \(\text{Gr}_p(\mathbb{R}^{p+q})\).
2. An open connected set \(\Omega \subset \text{Gr}_p(\mathbb{R}^{p+q})\) is called \(\mathcal{R}\)-proper if

\[
|\ell \setminus \ell \cap \Omega| > 1
\]

for all \(\ell \in \mathcal{L}\).

**Example 4.4.** If \(\mathbb{M} \subset \text{Gr}_p(\mathbb{R}^{p+q})\) is an affine chart and \(\Omega\) is a bounded subset of \(\mathbb{M}\), then \(\Omega\) is an \(\mathcal{R}\)-proper subset of \(\text{Gr}_p(\mathbb{R}^{p+q})\) (see Lemma 3.7 above).

**Theorem 4.5.** Suppose \(\mathbb{M} \subset \text{Gr}_p(\mathbb{R}^{p+q})\) is an affine chart and \(\Omega \subset \mathbb{M}\) is an open convex set. Then the following are equivalent:
14 RIGIDITY OF CONVEX DIVISIBLE DOMAINS IN FLAG MANIFOLDS

(1) $\Omega$ is $\mathcal{R}$-proper,
(2) $K_\Omega$ is a complete length metric on $\Omega$,
(3) $K_\Omega$ is a metric on $\Omega$.

Remark 4.6. The above theorem should be compared to two well-known results in
real projective geometry and several complex variables:

(1) For an open convex set $\Omega \subset \mathbb{R}^{d+1}$ the Hilbert metric is complete if and only
if $\Omega$ does not contain any real affine lines.
(2) For an open convex set $\Omega \subset \mathbb{C}^{d+1}$ the Kobayashi metric is complete if and
only if $\Omega$ does not contain any complex affine lines (Barth [Bar80]).

Proof. Clearly (2) implies (3). Moreover, if there exists a projective line $\ell \in \mathcal{L}$ such
that $|\ell \setminus \ell \cap \Omega| \leq 1$ then $\rho_\Omega(x, y) = 0$ for all $x, y \in \ell \cap \Omega$. Thus if $\Omega$ is not $\mathcal{R}$-proper then $K_\Omega$ is not
a metric. Thus (3) implies (1). The proof that (1) implies (2) can be found in
Appendix A. □

The existence of an invariant metric implies that the action of $\text{Aut}(\Omega)$ on $\Omega$ is
proper:

Proposition 4.7. Suppose $\mathcal{M} \subset \text{Gr}_p(\mathbb{R}^{p+q})$ is an affine chart and $\Omega \subset \mathcal{M}$ is an
open convex set. If $\Omega$ is $\mathcal{R}$-proper, then

(1) $\text{Aut}(\Omega)$ is a closed subgroup of $\text{PGL}_{p+q}(\mathbb{R})$,
(2) $\text{Aut}(\Omega)$ is a closed subgroup of $\text{Isom}(\Omega, K_\Omega)$, and
(3) $\text{Aut}(\Omega)$ acts properly on $\Omega$.

Proof. We first observe that $\text{Aut}(\Omega)$ is closed in $\text{PGL}_{p+q}(\mathbb{R})$. Suppose that $\varphi_n \in \text{Aut}(\Omega)$ and $\varphi_n \to \varphi$ in $\text{PGL}_{p+q}(\mathbb{R})$. Then $\varphi(\Omega) \subset \overline{\Omega}$. Since $\Omega$ is convex in an affine
chart $\text{int}(\overline{\Omega}) = \Omega$. Then since $\varphi$ induces a homeomorphism $\text{Gr}_p(\mathbb{R}^{p+q}) \to \text{Gr}_p(\mathbb{R}^{p+q})$ we must have

$$\varphi(\Omega) \subset \text{int}(\overline{\Omega}) = \Omega.$$ But the same argument implies that $\varphi^{-1}(\Omega) \subset \Omega$. So $\varphi(\Omega) = \Omega$ and $\varphi \in \text{Aut}(\Omega)$.

We next show that the action of $\text{Aut}(\Omega)$ on $\Omega$ is proper. Suppose that $\varphi_n \in \text{Aut}(\Omega)$ is a sequence of automorphisms such that

$$\varphi_n x_0 \in \{ y \in \Omega : K_\Omega(x_0, y) \leq R \}$$

for some $x_0 \in \Omega$ and $R \geq 0$. We need to show that a subsequence of $\varphi_n$ converges
in $\text{PGL}_{p+q}(\mathbb{R})$.

Since $\text{Aut}(\Omega)$ acts by isometries on the metric space $(\Omega, K_\Omega)$, by the Arzelà-
Ascoli theorem there exist an isometry $f : (\Omega, K_\Omega) \to (\Omega, K_\Omega)$ and a subsequence $n_k \to \infty$ such that

$$f(x) = \lim_{k \to \infty} \varphi_{n_k}(x)$$

for all $x \in \Omega$. Since $f$ is an isometry it is injective.

Now let $T_k \in \text{GL}(\wedge^p \mathbb{R}^{p+q})$ be representatives of $\wedge^p \varphi_{n_k} \in \text{PGL}(\wedge^p \mathbb{R}^{p+q})$. We
normalize $T_k$ such that $\|T_k\| = 1$, where $\|\cdot\|$ denotes the operator norm. By passing
to another subsequence we can suppose that $T_k \to T \in \text{End}(\wedge^p \mathbb{R}^{p+q})$ with $\|T\| = 1$. Now for $x \in \Omega \setminus \ker T$ we have

$$T(x) = \lim_{k \to \infty} \varphi_{n_k}(x) = f(x)$$

and so $T$ is injective on $\Omega \setminus \ker T$. This implies that $T \in \text{GL}(\wedge^p \mathbb{R}^{p+q})$, see Remark 3.1. Hence $\varphi_{n_k} \to \varphi$ in $\text{PGL}_{p+q}(\mathbb{R})$ for some $\varphi$ with $\wedge^p \varphi = [T]$. So $\text{Aut}(\Omega)$ acts properly.

Notice that the above argument to prove that $T = f$, also implies that $\text{Aut}(\Omega)$ is a closed subgroup of $\text{Isom}(\Omega, K_\Omega)$. \hfill $\Box$

5. Limits in the local Hausdorff topology and rescaling

Given a set $A \subset \mathbb{R}^d$, let $\mathcal{N}_\epsilon(A)$ denote the $\epsilon$-neighborhood of $A$ with respect to the Euclidean distance. The Hausdorff distance between two bounded sets $A, B$ is given by

$$d_H(A, B) = \inf \{ \epsilon > 0 : A \subset \mathcal{N}_\epsilon(B) \text{ and } B \subset \mathcal{N}_\epsilon(A) \}.$$ 

Equivalently,

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| a - b \| \right\}.$$ 

The Hausdorff distance is a complete metric on the space of compact sets in $\mathbb{R}^d$.

The space of closed sets in $\mathbb{R}^d$ can be given a topology from the local Hausdorff seminorms. For $R > 0$ and a set $A \subset \mathbb{R}^d$ let $A(R) := A \cap B_R(0)$. Then define the local Hausdorff seminorms by

$$d_H^{(R)}(A, B) := d_H(A^{(R)}, B^{(R)}).$$

Finally we say that a sequence of open convex sets $A_n$ converges in the local Hausdorff topology to an open convex set $A$ if there exists some $R_0 \geq 0$ so that $d_H^{(R)}(A_n, A) \to 0$ for all $R \geq R_0$.

**Theorem 5.1.** Let $M$ be an affine chart of $\text{Gr}_p(\mathbb{R}^{p+q})$ and suppose $\Omega_n \subset M$ is a sequence of $\mathcal{R}$-proper convex open sets converging to an $\mathcal{R}$-proper convex open set $\Omega \subset M$ in the local Hausdorff topology. Then

$$K_\Omega(x, y) = \lim_{n \to \infty} K_{\Omega_n}(x, y)$$

for all $x, y \in \Omega$ uniformly on compact sets of $\Omega \times \Omega$.

We provide the proof of Theorem 5.1 in Appendix B.

**Theorem 5.2.** Let $M$ be an affine chart of $\text{Gr}_p(\mathbb{R}^{p+q})$ and suppose $\Omega \subset M$ is an $\mathcal{R}$-proper open convex subset. Assume in addition that there exist a subgroup $H \leq \text{Aut}(\Omega)$ and a compact set $K \subset \Omega$ such that $H \cdot K = \Omega$.

If there exists a sequence $A_n \in \text{Aff}(M) \cap \text{PGL}_{p+q}(\mathbb{R})$ such that $A_n \Omega$ converges in the local Hausdorff topology to an $\mathcal{R}$-proper open convex set $\tilde{\Omega} \subset M$, then there exist $n_k \to \infty$ and $h_k \in H$ such that

$$\phi = \lim_{k \to \infty} A_{n_k} h_k$$

exists in $\text{PGL}_{p+q}(\mathbb{R})$ and $\tilde{\Omega} = \phi(\Omega)$.
Fix $y_0 \in \hat{\Omega}$. Then we have $y_0 = A_n \varphi_n k_n$. Let $T_n := A_n \varphi_n \in \text{PGL}_{p+q}(\mathbb{R})$. Then

$$\Omega_n := T_n(\hat{\Omega}) = A_n(\Omega)$$

is an $\mathcal{R}$-proper open convex subset and $T_n$ is an isometry $(\Omega, K_\Omega) \to (\Omega_n, K_{\Omega_n})$. By Theorem 5.1

$$K_{\Omega_n} \to K_{\hat{\Omega}}$$

uniformly on compact sets on $\hat{\Omega}$, so we can pass to a subsequence such that $T_n$ converges uniformly on compact sets to an isometry $T : (\hat{\Omega}, K_{\hat{\Omega}}) \to (\hat{\Omega}, K_{\hat{\Omega}})$. Since $T$ is an isometry it is injective. On the other hand since the metrics converge and closed metric balls are compact we also see that $T$ is onto.

Now we can pick a representative $\Phi_n \in \text{GL}(\wedge^P \mathbb{R}^{P+q})$ of $\wedge^P T_n \in \text{PGL}(\wedge^P \mathbb{R}^{P+q})$ such that $\|\Phi_n\| = 1$. By passing to a subsequence we can assume that $\Phi_n \to \Phi$ in $\text{End}(\wedge^P \mathbb{R}^{P+q})$. The set $\wedge^P \text{End}(\mathbb{R}^{P+q}) \subset \text{End}(\wedge^P \mathbb{R}^{P+q})$ is closed and so $\Phi = \wedge^P \phi$ for some $\phi \in \text{End}(\mathbb{R}^{P+q})$. Moreover $\Phi(x) = T(x)$ for any $x \notin \text{ker} \Phi$. Since $\text{Gr}_p(\mathbb{R}^{P+q}) \setminus \text{ker} \Phi$ is an open dense set and $\Omega$ is open, this implies that $\Phi$ is injective on $\text{Gr}_p(\mathbb{R}^{P+q}) \setminus \text{ker} \Phi$. It follows that $\Phi \in \text{GL}(\wedge^P \mathbb{R}^{P+q})$ and hence $\phi \in \text{GL}_{p+q}(\mathbb{R})$.

Finally, we have that $\phi = T$ on $\hat{\Omega}$, so that $\hat{\Omega} = \phi(\Omega)$. □

6. The geometry near the boundary

For the classical Hilbert metric on a convex divisible domain in real projective space, there are many connections between the shape of the boundary and the behavior of the metric (see e.g. Ben04, Ben03b, KN02). In a similar spirit, we will prove some basic results connecting the geometry of $K_\Omega$ with the geometry of $\partial \Omega$.

As before, let $\mathcal{L}$ be the set of projective lines $\ell \subset \mathbb{P}(\wedge^P \mathbb{R}^{P+q})$ which are contained in $\text{Gr}_p(\mathbb{R}^{P+q})$.

**Definition 6.1.** Suppose $\Omega \subset \text{Gr}_p(\mathbb{R}^{P+q})$ is an open connected set.

1. Two points $x, y \in \partial \Omega$ are adjacent, denoted $x \sim y$, if either $x = y$ or there exists a projective line $\ell \in \mathcal{L}$ such that $x, y$ are contained in a connected component of the interior of $\ell \cap \partial \Omega$ in $\ell$.
2. The $\mathcal{R}$-face of $x \in \partial \Omega$, denoted $\mathcal{R} F(x)$, is the set of points $y \in \partial \Omega$ where there exists a sequence $x = y_0, y_1, \ldots, y_k = y$ with $y_i \sim y_{i+1}$.
3. A point $x \in \partial \Omega$ is called an $\mathcal{R}$-extreme point if $\mathcal{R} F(x) = \{x\}$.
4. Let $\text{Ext}_\mathcal{R}(\Omega) \subset \partial \Omega$ denote the set of $\mathcal{R}$-extreme points of $\Omega$.

As the next two results show this relation on the boundary is connected with the asymptotic geometry of the intrinsic metric.

**Proposition 6.2.** Suppose $M \subset \text{Gr}_p(\mathbb{R}^{P+q})$ is an affine chart and $\Omega \subset M$ is an $\mathcal{R}$-proper open convex set. If $x_n, y_n \in \Omega$ are sequences such that $x_n \to x \in \partial \Omega$ and $y_n \to y \in \partial \Omega$, and there exists $N \geq 0$ such that

$$\liminf_{n \to \infty} K_\Omega^{(N)}(x_n, y_n) < \infty,$$

then $\mathcal{R} F(x) = \mathcal{R} F(y)$. 
Proof. For each $n$, choose a sequence $x_n = x_n^{(0)}, x_n^{(1)}, \ldots, x_n^{(N)} = y_n$ with

$$\liminf_{n \to \infty} \sum_{0 \leq i \leq N-1} \rho \Omega(x_n^{(i)}, x_n^{(i+1)}) < \infty.$$ 

By passing to subsequences, we can assume that $x_n^{(i)} \to x^{(i)}$ for each $1 \leq i \leq N-1$. By inducting on $N$, it therefore suffices to consider the case $N = 1$ and $y = x^{(1)}$ so that

$$\lim_{n \to \infty} K^{(1)}_\Omega(x_n, y_n) = \lim_{n \to \infty} \rho \Omega(x_n, y_n) < \infty$$

and $x \neq y$. For each $n$ let $\ell_n$ be the projective line in $\mathbb{P}(\wedge \mathbb{R}^{p+q})$ containing $x_n$ and $y_n$. Also let $\{a_n, b_n\} = \ell_n \cap \partial \Omega$ with labeling such that the ordering of the points along $\ell_n$ is given by $a_n, x_n, y_n, b_n$. Then

$$\rho \Omega(x_n, y_n) = \log \frac{|x_n - b_n|}{|y_n - a_n|}$$

By passing to a subsequence we can suppose that $a_n \to a$ and $b_n \to b$. Then by the hypothesis we must have that $a \neq x$ and $b \neq y$. So $x \sim y$. \qed

Corollary 6.3. Suppose $M \subset Gr_p(\mathbb{R}^{p+q})$ is an affine chart, $\Omega \subset M$ is an $\mathcal{R}$-proper open convex set, and $\text{Aut}(\Omega)$ acts cocompactly on $\Omega$. If $x_n, y_n \in \Omega$ are sequences such that $x_n \to x \in \partial \Omega$, $y_n \to y \in \partial \Omega$, and

$$\liminf_{n \to \infty} K_\Omega(x_n, y_n) < \infty,$$

then $\mathcal{R} F(x) = \mathcal{R} F(y)$.

Proof. By passing to a subsequence we can suppose that

$$M = \sup_{n \in \mathbb{N}} K_\Omega(x_n, y_n) < \infty.$$ 

For $R \geq 0$ and $x \in \Omega$, let $B_R(x)$ denote the ball of radius $R$ and center $x$ with respect to the metric $K_\Omega$. Since $\text{Aut}(\Omega)$ acts cocompactly on $\Omega$ there exists $R \geq 0$ such that

$$\text{Aut}(\Omega) \cdot B_R(x_0) = \Omega.$$ 

Let $B := B_{R+M}(x_0)$ be the ball with center $x_0$ and radius $R+M$. By compactness of $B$ and Proposition 4.2 we know there exists $N > 0$ such that

$$\sup_{x, y \in B} K^{(N)}_\Omega(x, y) < \infty$$

for all $x, y \in B$. But this implies that

$$\sup_{n \in \mathbb{N}} K^{(N)}_\Omega(x_n, y_n) < \infty$$

because for any $n \in \mathbb{N}$ there exists some $\varphi \in \text{Aut}(\Omega)$ such that $\varphi x_n, \varphi y_n \in B$. \qed
Part 2. The automorphism group is non-discrete

7. Extreme points and symmetry

7.1. The geometry of extreme points. In this subsection we provide a number of characterizations of $\mathcal{R}$-extreme points for domains $\Omega \subset \text{Gr}_p(\mathbb{R}^{2p})$ where Aut($\Omega$) acts cocompactly. But first a few definitions.

Suppose $\Omega$ is a convex set in a vector space and $x \in \partial \Omega$, then the tangent cone of $\Omega$ at $x$ is the set
\[ \mathcal{T}C_x \Omega := x + \bigcup_{t > 0} t(\Omega - x). \]
Notice that the sets $x + t(\Omega - x)$ converge to $\mathcal{T}C_x \Omega$ in the local Hausdorff topology as $t \to \infty$.

We will also define natural hypersurfaces in $\text{Gr}_p(\mathbb{R}^{p+q})$.

**Definition 7.1.** Given $\xi \in \text{Gr}_q(\mathbb{R}^{p+q})$ define the hypersurface
\[ Z_\xi := \{ x \in \text{Gr}_p(\mathbb{R}^{p+q}) : x \cap \xi \neq (0) \}. \]

**Remark 7.2.** In the case in which $p = 1$, then $Z_\xi \subset \mathbb{P}(\mathbb{R}^{q+1}) = \text{Gr}_1(\mathbb{R}^{q+1})$ is the image of $\xi$ in $\mathbb{P}(\mathbb{R}^{q+1})$. In particular, if a set $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is convex and bounded in an affine chart then for any $x \in \partial \Omega$ there exists $\xi \in \text{Gr}_{d-1}(\mathbb{R}^d)$ such that $x \in Z_\xi$ and $Z_\xi \cap \Omega = \emptyset$.

In [Zim15], the second author proved that symmetry also implies the existence of such “supporting hypersurfaces.”

**Theorem 7.3.** [Zim15, Theorem 1.7] If $\Omega \subset \text{Gr}_p(\mathbb{R}^{p+q})$ is a bounded connected open subset of some affine chart and Aut($\Omega$) acts cocompactly on $\Omega$, then for all $x \in \partial \Omega$ there exists $\xi \in \text{Gr}_{q-1}(\mathbb{R}^{p+q})$ such that $x \in Z_\xi$ and $Z_\xi \cap \Omega = \emptyset$.

Henceforth we will only consider the case $p = q$. With these notations we will prove the following:

**Theorem 7.4.** Suppose $p > 1$ and $\mathcal{M} \subset \text{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, $\Omega$ is a bounded open convex subset of $\mathcal{M}$, and Aut($\Omega$) acts cocompactly on $\Omega$. If $e \in \partial \Omega$, then the following are equivalent:

1. $e \in \partial \Omega$ is an $\mathcal{R}$-extreme point,
2. $Z_e \cap \Omega = \emptyset$,
3. $\mathcal{T}C_e \Omega$ is an $\mathcal{R}$-proper cone,
4. there exist $\varphi_n \in \text{Aut}(\Omega)$ and representatives $\hat{\varphi}_n \in \text{GL}(\wedge^p \mathbb{R}^{2p})$ such that $\hat{\varphi}_n \to S$ in $\text{End}(\wedge^p \mathbb{R}^{2p})$ and $\text{Im}(S) = e$.

**Remark 7.5.** The implication (4)$\Rightarrow$(1) fails for convex divisible domains in real projective space. In particular by a result of Benoist [Ben06]: if $\Omega \subset \mathbb{P}(\mathbb{R}^4)$ is a convex divisible domain and $x \in \partial \Omega$, then there exist $\varphi_n \in \text{Aut}(\Omega)$ and representatives $\hat{\varphi}_n \in \text{GL}_4(\mathbb{R})$ such that $\hat{\varphi}_n \to S$ in $\text{End}(\mathbb{R}^4)$ and $\text{Im}(S) = x$. However, there are examples of convex divisible domains in $\mathbb{P}(\mathbb{R}^4)$ whose boundary contains non-extreme points (see [Ben06] and [BDL15]).

**Proof.** We first show that (1) $\Rightarrow$ (4). Suppose that $e \in \partial \Omega$ is an $\mathcal{R}$-extreme point. Pick a sequence $x_n \in \Omega$ such that $x_n \to e$. Since Aut($\Omega$) acts cocompactly on $\Omega$ we
can find \( R \geq 0 \) and \( \varphi_n \in \text{Aut}(\Omega) \) such that
\[
K_\Omega(x_n, \varphi_n x_0) \leq R
\]
for all \( n \geq 0 \). Now for any \( x \in \Omega \) we have
\[
K_\Omega(\varphi_n x, x_0) \leq K_\Omega(\varphi_n x, \varphi_n x_0) + R = K_\Omega(x, x_0) + R
\]
and so by Corollary 6.3 we see that \( \varphi_n x \to e \). Pick representatives \( \hat{\varphi}_n \in \text{GL}(\wedge^p \mathbb{R}^{2p}) \) of \( \wedge^p \varphi_n \) such that \( \|\hat{\varphi}_n\| = 1 \). By passing to a subsequence we can suppose that \( \hat{\varphi}_n \to S \) in \( \text{End}(\wedge^p \mathbb{R}^{2p}) \). Now if \( x \in \mathcal{O} := \text{Gr}_p(\mathbb{R}^{2p}) \setminus \ker S \) then \( S(x) = \lim_{n \to \infty} \varphi_n x \).

Since \( \mathcal{O} \) is open and dense, we see that \( \Omega \cap \mathcal{O} \) is dense in \( \Omega \). In particular \( \Omega \cap \mathcal{O} \) contains a basis of \( \wedge^p \mathbb{R}^{2p} \). However for every \( x \in \Omega \cap \mathcal{O} \) we have \( S(x) = e \). So \( \text{Im}(S) = e \). So \( 1 \) \( \Rightarrow \) \( 4 \).

We next show that \( 4 \Rightarrow 2 \). So suppose there exist \( \varphi_n \in \text{Aut}(\Omega) \) and representatives \( \hat{\varphi}_n \in \text{GL}(\wedge^p \mathbb{R}^{2p}) \) such that \( \hat{\varphi}_n \to S \) in \( \text{End}(\wedge^p \mathbb{R}^{2p}) \) and \( \text{Im}(S) = e \). Notice that if \( x \in \mathcal{O} := \text{Gr}_p(\mathbb{R}^{2p}) \setminus \ker S \) then \( S(x) = \lim_{n \to \infty} \varphi_n x \). Now similar to the case of properly convex sets in projective space, we can consider the dual of \( \Omega \)
\[
\Omega^* := \{ \xi \in \text{Gr}_p(\mathbb{R}^{2p}) : Z_\xi \cap \Omega = \emptyset \}.
\]
Note that, unlike the case of domains in projective space, \( \Omega \) and \( \Omega^* \) are both subsets of \( \text{Gr}_p(\mathbb{R}^{2p}) \). Since \( \Omega \) is open, \( \Omega^* \) is compact. Moreover since \( \Omega \) is bounded in an affine chart \( \Omega^* \) has non-empty interior: \( \mathcal{M} = \text{Gr}_p(\mathbb{R}^{2p}) \setminus Z_\xi \) for some \( \xi \) and since \( \Omega \) is bounded in \( \mathcal{M} \) we see that \( \Omega^* \) contains an open neighborhood of \( \xi \). In particular, \( \Omega^* \cap \mathcal{O} \) is non-empty. But then for \( \eta \in \Omega^* \cap \mathcal{O} \) we have \( e = S(\eta) = \lim_{n \to \infty} \varphi_n(\eta) \).

Since \( \Omega^* \) is \( \text{Aut}(\Omega) \)-invariant we then see that \( e \in \Omega^* \). So \( 4 \Rightarrow 2 \).

We next show that \( 2 \Rightarrow 3 \). So suppose that \( e \in \partial \Omega \) and \( Z_e \cap \Omega = \emptyset \). We can assume that
\[
\Omega \subset \mathcal{M} := \left\{ \begin{bmatrix} I_p & X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}
\]
and \( e = 0 \) in \( \mathcal{M} \). Then since \( Z_e \cap \Omega = \emptyset \) we see that
\[
\Omega \subset \left\{ \begin{bmatrix} I_p & X \end{bmatrix} : \det(X) \neq 0 \right\}.
\]
Since \( \Omega \) is connected, by making an affine transformation, we may assume that
\[
\Omega \subset \left\{ \begin{bmatrix} I_p & X \end{bmatrix} : \det(X) > 0 \right\}.
\]
Then, since \( \mathcal{T} \mathcal{C}_0 \Omega \) is open, we see that
\[
\mathcal{T} \mathcal{C}_0 \Omega \subset \left\{ \begin{bmatrix} I_p & X \end{bmatrix} : \det(X) > 0 \right\}.
\]
Now suppose for a contradiction that \( \mathcal{T} \mathcal{C}_0 \Omega \) is not \( \mathcal{R} \)-proper. Then by Lemma 3.7 and convexity there exists a rank-one endomorphism \( S \) such that
\[
\left\{ \begin{bmatrix} I_p & T + tS \end{bmatrix}^t : t \in \mathbb{R} \right\} \subset \mathcal{T} \mathcal{C}_0 \Omega
\]
whenever \( [I_p \ T]^t \in \mathcal{T} \mathcal{C}_0 \Omega \). So
\[
\det(T + tS) > 0
\]
for any \( [I_p \ T]^t \in \mathcal{T} \mathcal{C}_0 \Omega \) and \( t \in \mathbb{R} \). Now
\[
\det(T + tS) = \det(T) \det(I_p + tT^{-1}S) = \det(T)(1 + t \text{tr}(T^{-1}S))
\]
Suppose Corollary 7.6. \( R \) not and open this implies that \( TC \) when \( t \) since 20 RIGIDITY OF CONVEX DIVISIBLE DOMAINS IN FLAG MANIFOLDS 7.7 is closed. open convex subset of \( M \) extreme point then \( \text{Aut}(\Omega) \) and \( \text{Ext}(\Omega) \) are examples of convex divisible domains in \( \partial \) divisible domain then the extreme points of \( \Omega \) are dense in \( \Omega \). This corollary fails for convex divisible domains in real projective space. In particular by a result of Benoist [Ben06]: if \( \Omega \) \( P \) are examples of convex divisible domains in \( \partial \) divisible domain then the extreme points of \( \Omega \) are dense in \( \Omega \). However, there are examples of convex divisible domains in \( \mathbb{F}(\mathbb{R}^4) \) whose boundary contains non-extreme points (see [Ben06] and [BDL15]).

Proof. By the above proposition, the set of extreme points coincides with
\[
\{ e \in \partial \Omega : Z_e \cap \Omega = \emptyset \}
\]
which is obviously closed. \( \square \)

7.2. Constructing extreme points.

Proposition 7.8. Suppose \( M \subset \text{Gr}_p(\mathbb{R}^{p,q}) \) is an affine chart and \( \Omega \subset M \) is an open bounded convex set. Then \( \text{Ext}_R(\Omega) \cap \partial \Omega \) spans \( \Lambda^p \mathbb{R}^{p,q} \).

Proof. Identify \( M \) with \( M_{q,p}(\mathbb{R}) \). For \( x \in \partial \Omega \) let
\[
V_x = x + \text{Span}\{ v \in M_{q,p}(\mathbb{R}) : v + x \text{ is adjacent to } x \} \subset M_{q,p}(\mathbb{R}).
\]
See Definition 6.1 for the notion of adjacency. Notice that \( x \in \partial \Omega \) is an \( R \)-extreme point if and only if \( \dim V_x = 0 \).

Now, since rank-one lines in \( M_{q,p}(\mathbb{R}) \) are mapped to projective lines in \( \mathbb{P}(\Lambda^p \mathbb{R}^{p,q}) \), we have the following: if \( v \) is a rank-one matrix, \( t < 0 < s \), and \( a, b, c \in \mathbb{P}(\Lambda^p \mathbb{R}^{p,q}) \) are the images of \( x + tv, x, x + sv \in M_{q,p}(\mathbb{R}) \) respectively, then the line \( b \) is contained in the span of the lines \( a \) and \( c \). Thus is enough to show: for any \( x \in \partial \Omega \) with \( \dim V_x > 0 \) there exist a rank-one matrix \( v \in M_{q,p}(\mathbb{R}) \) and \( t < 0 < s \) such that \( x + tv, x + sv \in \partial \Omega \) and \( \dim V_x + tv, V_x + sv < \dim V_x \).

Let \( F_x = \partial \Omega \cap V_x \). This is a convex set which has non-empty interior in \( V_x \).

We claim that \( V_y \subset V_x \) for \( y \in F_x \). Suppose that \( v + y \in V_y \), that is \( v \in M_{q,p}(\mathbb{R}) \) and \( v + y \) is adjacent to \( y \). Then there exists \( \epsilon > 0 \) such that \( tv + y \in \partial \Omega \) for \( t \in (-\epsilon, 1 + \epsilon) \). Moreover, since \( y \in \partial \Omega \cap V_x \) there exists \( \delta > 0 \) such that \( \lambda x + (1 - \lambda)y \in \partial \Omega \) for \( \lambda \in [0, 1 + \delta] \). Then by convexity, there exists \( \epsilon_1 > 0 \) such that \( x + tv \in \partial \Omega \) for \( t \in (-\epsilon_1, \epsilon_1) \). Thus \( x + v \in V_x \). Since \( y \in V_x \) we then see that \( x + v + (y - x) = x + y \in V_x \).

Since \( v + y \in V_y \) was arbitrary, we then see that \( V_y \subset V_x \).

Notice that the above claim implies that if \( y \in \partial F_x \), then \( \dim V_y < \dim V_x \).
So for \( x \in \partial \Omega \) and \( \dim V_x > 0 \), pick a rank-one matrix \( v \) such that \( x + \mathbb{R}v \subset V_x \). Then if 
\[
\{ x + sv, x + tv \} = \partial F_x \cap (x + \mathbb{R}v)
\]
we have
\[
\dim V_{x+tv}, V_{x+sv} < \dim V_x
\]
and the proof is complete by the remarks above.

Suppose that \( \varphi \in \operatorname{PGL}_d(\mathbb{R}) \). Let \( \overline{\varphi} \in \operatorname{GL}_d(\mathbb{R}) \) be a representative of \( \varphi \) with \( \det(\overline{\varphi}) = \pm 1 \). Next let
\[
\lambda_1(\varphi) \geq \lambda_2(\varphi) \geq \cdots \geq \lambda_d(\varphi)
\]
be the absolute values of the eigenvalues (counted with multiplicity) of \( \overline{\varphi} \) (notice that this does not depend on the choice of \( \overline{\varphi} \)). Let \( m^+(\varphi) \) be the size of the largest Jordan block of \( \overline{\varphi} \) whose corresponding eigenvalue has absolute value \( \lambda_1(\varphi) \).

Next let \( E^+_{\mathbb{C}}(\varphi) \subset \mathbb{C}^d \) be the span of the eigenvectors of \( \overline{\varphi} \) whose eigenvalue have absolute value \( \lambda_1(\varphi) \) and are part of a Jordan block with size \( m^+(\varphi) \). Then let \( E^+(\varphi) = E^+_{\mathbb{C}}(\varphi) \cap \mathbb{R}^d \). Since \( \varphi \) is a real matrix, the non-real eigenvalues come in conjugate pairs and so we always have
\[
E^+_\mathbb{C}(\varphi) = E^+(\varphi) + iE^+(\varphi).
\]
Also define \( E^-(\varphi) = E^+(\varphi^{-1}) \).

Given \( y \in \mathbb{P}(\mathbb{R}^d) \) let \( L(\varphi, y) \subset \mathbb{P}(\mathbb{R}^d) \) denote the limit points of the sequence \( \{ \varphi^n y \}_{n \in \mathbb{N}} \). With this notation we have the following observation:

**Proposition 7.9.** Suppose \( \varphi \in \operatorname{PGL}_d(\mathbb{R}) \) and \( \{ \varphi^n \}_{n \in \mathbb{N}} \subset \operatorname{PGL}_d(\mathbb{R}) \) is unbounded, then there exists a proper projective subspace \( P \subset \mathbb{P}(\mathbb{R}^d) \) such that \( L(\varphi, y) \subset [E^+(\varphi)] \) for all \( y \in \mathbb{P}(\mathbb{R}^d) \) \( \setminus \) \( P \).

**Proof.** We can write \( \overline{\varphi} = gJg^{-1} \) where \( g \in \operatorname{GL}_d(\mathbb{C}) \) and \( J \) is a Jordan matrix. We can further assume that
\[
J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}
\]
where \( J_1 \) consists of the blocks of \( J \) whose eigenvalues have absolute value \( \lambda_1(\varphi) \) and have size \( m^+(\varphi) \). Then let
\[
V = \mathbb{R}^d \cap \left( g \ker \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix} \right)
\]
and \( P = [V] \subset \mathbb{P}(\mathbb{R}^d) \). A straightforward calculation then shows that \( L(\varphi, y) \subset [E^+(\varphi)] \) for all \( y \in \mathbb{P}(\mathbb{R}^d) \) \( \setminus \) \( P \).

**Corollary 7.10.** Suppose \( \Omega \) is an open connected set of \( \operatorname{Gr}_p(\mathbb{R}^{2p}) \), there exists an affine chart which contains \( \partial \Omega \) as a bounded convex set, and \( \operatorname{Aut}(\Omega) \) acts cocompactly on \( \Omega \). If \( \varphi \in \operatorname{Aut}(\Omega) \) and \( \{ \varphi^n \}_{n \in \mathbb{N}} \subset \operatorname{PGL}_2p(\mathbb{R}) \) is unbounded, then \( E^+(\wedge^p \varphi) \cap \partial \Omega \) is non-empty and contains an \( \mathcal{R} \)-extreme point.

**Proof.** Let \( P \subset \mathbb{P}(\wedge^p \mathbb{R}^{2p}) \) be as in the above proposition for \( \wedge^p \varphi \). Since the set of \( \mathcal{R} \)-extreme points of \( \partial \Omega \) spans \( \wedge^p \mathbb{R}^{2p} \), there exists an \( \mathcal{R} \)-extreme point \( e \in \partial \Omega \) such that \( e \notin P \). Then any limit point of \( \varphi^n e \) belongs to \( E^+(\wedge^p \varphi) \) and is also an \( \mathcal{R} \)-extreme point by Corollary 7.6.

\[ \square \]
7.3. Finding symmetry. Our goal is now to use Theorems 5.2 and Theorem 7.4 to show that for suitable domains $\Omega$, the group $\text{Aut}(\Omega)$ is not discrete.

**Corollary 7.11.** Suppose $\Omega \subset \text{Gr}_p(\mathbb{R}^{2p})$ is an $R$-proper open convex set in the affine chart

$$M = \left\{ \begin{bmatrix} \text{Id}_p \\ X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}$$

and $H \leq \text{Aut}(\Omega)$ acts cocompactly on $\Omega$. If $e = \begin{bmatrix} \text{Id}_p \\ X_0 \end{bmatrix} \in \partial \Omega$ is an $R$-extreme point, then there exist $h_n \in H$ and $t_n \to \infty$ such that

$$\varphi = \lim_{n \to \infty} \begin{bmatrix} \text{Id}_p \\ (1 - e^{t_n})X_0 \end{bmatrix} e^{t_n} h_n$$

exists in $\text{PGL}_{2p}(\mathbb{R})$ and $\varphi(\Omega) = TC_e \Omega$. In particular, $\Omega$ is invariant under the one-parameter group

$$\varphi^{-1} \left\{ \begin{bmatrix} \text{Id}_p \\ (1 - e^t)X_0 \end{bmatrix} : t \in \mathbb{R} \right\} \varphi.$$

**Proof.** Let

$$A_t = \begin{bmatrix} \text{Id}_p \\ (1 - e^t)X_0 \end{bmatrix} e^t \text{Id}_p$$

then

$$A_t \cdot \begin{bmatrix} \text{Id}_p \\ X \end{bmatrix} = \begin{bmatrix} \text{Id}_p \\ e^t(X - X_0) + X_0 \end{bmatrix}.$$

So $A_t \in \text{Aff}(M) \cap \text{PGL}_{2p}(\mathbb{R})$ and $A_t \Omega$ converges in the local Hausdorff topology to $TC_e \Omega$ as $t \to \infty$. So the corollary follows from Theorem 5.2 and Theorem 7.4. □

**Part 3. The automorphism group is simple**

8. Initial reduction

For the rest of this section suppose $p > 1$ and $\mathcal{M} \subset \text{Gr}_p(\mathbb{R}^{2p})$ is an affine chart, $\Omega \subset \mathcal{M}$ is a bounded convex open subset of $\mathcal{M}$, and there exists a discrete group $\Gamma \leq \text{Aut}(\Omega)$ such that $\Gamma$ acts cocompactly on $\Omega$.

Set $G := \text{Aut}(\Omega)$ and let $G^0$ be the connected component of the identity of $G$. By Corollary 7.11, we know that $G^0 \neq 1$. The goal of this section is to use the fact that $G^0 \neq 1$ to obtain that either $G^0$ is simple and acts transitively on $\Omega$, or we are in one of four very constrained situations (Cases (1)-(4) in Theorem 8.1 below). In Sections 9-11, we will prove that Cases (1)-(4) cannot occur.

**Theorem 8.1.** With the notation above, at least one of the following holds:

1. A finite-index subgroup of $\Gamma$ has non-trivial centralizer in $\text{PGL}_{2p}(\mathbb{R})$.
2. There exists a nontrivial abelian normal unipotent group $U \leq G$ such that $\Gamma \cap U$ is a cocompact lattice in $U$.
3. $p = 2$ and there exists a finite-index subgroup $G'$ of $G$ such that $G' = G^0 \times \Lambda$ for some discrete group $\Lambda$. Further up to conjugation

$$G^0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \text{SL}_2(\mathbb{R}) \right\}$$
and
\[ \Lambda \leq \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\}. \]

(4) \( p = 2 \), \( G^0 \leq G \) has finite-index and acts transitively on \( \Omega \), and up to conjugation
\[ G^0 = \left\{ \begin{bmatrix} aA & bA \\ cA & dA \end{bmatrix} : A \in \text{SL}_2(\mathbb{R}), ad - bc = 1 \right\}. \]

(5) \( G^0 \) is a simple Lie group with trivial center that acts transitively on \( \Omega \).

Since the statement of Theorem 8.1 may seem unmotivated at first, let us sketch the argument. First suppose that \( G^0 \) is not semisimple. Let \( G^\text{sol} \leq G^0 \) be the solvable radical of \( G^0 \) (that is, the maximal connected, closed, normal, solvable subgroup of \( G^0 \)), and let \( N \) be the nilpotent radical of \( G^\text{sol} \) (that is, the maximal connected normal closed nilpotent subgroup of \( G^\text{sol} \)).

Note that \( N \) contains the unipotent radical \( R_u(G^0) \) of \( G^0 \) (i.e. all unipotent elements of \( G^\text{sol} \)), and hence is an extension
\[ 1 \to R_u(G^0) \to N \to N/R_u(G^0) \to 1. \]

The group \( N/R_u(G^0) \) is the subgroup of \( G^\text{sol}/R_u(G^0) \) whose action on the Lie algebra \( r_u(g) \) is unipotent. Let \( Z \) be the center of \( N \). We distinguish two cases depending on whether \( Z \) is contained in \( R_u(G) \):

1. If \( Z \) only consists of unipotent elements, we will show that \( \Gamma \) intersects some normal unipotent subgroup in a lattice. This corresponds to Case (2) in Theorem 8.1.
2. Otherwise, we show that a finite-index subgroup of \( \Gamma \) centralizes some semisimple torus in the Zariski closure of \( Z \). This corresponds to Case (1) in Theorem 8.1.

Suppose now that \( G^0 \) is semisimple. We want to show \( G^0 \) actually has to be simple and acts transitively on \( \Omega \). We do this by using the virtual cohomological dimension \( \text{vcd}(\Gamma) \) of \( \Gamma \) (see below for more information). We know that \( \text{vcd}(\Gamma) = \dim(\Omega) = p^2 \). Then we relate \( \text{vcd}(\Gamma) \) to the structure of \( G^0 \) to show that \( G \) has to have finitely many components, and \( G^0 \) is simple. This latter argument only fails if \( p = 2 \), in which case we obtain very specific information on the structure of \( G^0 \) and its action on \( \Omega \) (Cases (3) and (4) in the above Theorem 8.1).

We start with the following lemma.

**Lemma 8.2.** \( \Gamma \) is a cocompact lattice in \( G \) and \( \Gamma_0 := \Gamma \cap G^0 \) is a cocompact lattice in \( G^0 \).

**Proof.** Since \( \Gamma \) acts cocompactly on \( \Omega \) and \( G \) acts properly on \( \Omega \) (see Proposition 4.7) we see that \( \Gamma \leq G \) is a cocompact lattice. Since \( G^0 \leq G \) is a connected component the set \( \Gamma \cdot G^0 \) is closed in \( G \). So \( \Gamma_0 \cdot G^0 \) is closed in \( \Gamma \cdot G \). Then since \( \Gamma \cdot G \) is compact so is \( \Gamma_0 \cdot G^0 \). \(\square\)

The rest of this section will be devoted to the proof of Theorem 8.1. We will assume that Cases (1), (2), (3), and (4) do not hold and show that Case (5) occurs.

**Lemma 8.3.** \( \Gamma_0 \cap Z \) is a cocompact lattice in \( Z \).
Proof. Let $G^{ss} \leq G$ be a semisimple subgroup such that $G^0 = G^{ss}G^{sol}$ is a Levi-Malcev decomposition of $G^0$. Then let $\sigma : G^{ss} \to \text{Aut}(G^{sol})$ be the action of $G^{ss}$ by conjugation on $G^{sol}$. If $\ker \sigma$ has no compact factors in its identity component, then $\Gamma^0 \cap N$ is a cocompact lattice in $N$ (see [Gen15 Theorem 1.3.(i)]). In this case, $\Gamma^0 \cap Z \leq Z$ is a cocompact lattice by [Rag72 Proposition 2.17]. Therefore it suffices to show $\ker \sigma$ contains no compact factors. Since $\ker \sigma \leq G^{ss}$ is a normal subgroup, we see that $\ker \sigma$ is semisimple. So there is a unique maximal connected compact normal subgroup $K_0$ in $\ker \sigma$. Assume for a contradiction that $\dim K_0 > 0$. Then $K_0$ is also a connected normal subgroup of $G^{ss}$ and hence of $G^0$ which is impossible by an argument of Farb and Weinberger [FW08 Claim II]. Let us sketch this proof for completeness.

Let $K$ be a maximal compact factor of $G^0$. Since $\dim K_0 > 0$ we see that $\dim K > 0$. Consider the natural quotient map $\Omega \to \Omega/K$. Since $\Gamma$ permutes the maximal compact factors of $G^0$, we see that a finite-index subgroup of $\Gamma$ normalizes $K$. Then it is not hard to see that there is a continuous quasi-isometric inverse $\Omega/K \to \Omega$ to this quotient map. Consider the maps induced by the composition $\Omega \to \Omega/K \to \Omega$ on locally finite simplicial homology. On the one hand, since this composition is a bounded distance from the identity map, the induced map on locally finite simplicial homology is the identity map. On the other hand, since $\Omega$ is the universal cover of a closed aspherical manifold, there is a fundamental class in top degree. But since $\dim K > 0$, the image of this fundamental class in $H_*(\Omega/K)$ vanishes. This is a contradiction. For full details, see the proof of Claim II in [FW08]. □

Lemma 8.4. $G^0$ is semisimple.

Proof. As above let $N$ be the nilpotent radical of $G^{sol}$ and $Z$ the center of $N$. If $N = 1$, then $G^0$ is semisimple. So suppose for a contradiction that $N \neq 1$. Then $Z \neq 1$. Next let $C$ be the Zariski closure of $Z$ in $\text{PSL}_{2p}(\mathbb{R})$ and let $C^0$ be the connected component of the identity in $C$. Since $G$ normalizes $Z$ it also normalizes $C$ and $C^0$.

Since $Z$ is abelian so is $C^0$. Then since $C^0$ is an abelian real algebraic group, we can write

\[ C^0 = C_{ss}C_u \]

where $C_{ss}$ is the subset of semisimple elements in $C^0$ and $C_u$ is the subset of unipotent elements of $C^0$ (see e.g. [Bor91 Theorem 4.7]). By [Bor91 Corollary 4.4] both $C_{ss}$ and $C_u$ are actually groups. Since $G$ normalizes $C^0$ it also normalizes $C_{ss}$ and $C_u$.

If $C_{ss} = 1$, then each element of $C^0$ is unipotent and thus each element of $Z$ is unipotent. Thus we are in Case (2), which is a contradiction. Therefore we have $C_{ss} \neq 1$. But the normalizer of any semisimple torus $T$ in $\text{PGL}_{2p}(\mathbb{R})$ contains the centralizer of $T$ with finite-index [Bor91 Corollary 8.10.2], so we know that a finite-index subgroup of $G$ centralizes $C_{ss}$. Hence we are in Case (1) which contradicts our initial assumption. Thus $G^0$ is semisimple. □

Lemma 8.5. $G^0$ has trivial center.

Proof. Let $Z$ be the center of $G^0$. First, we observe that $Z$ is finite. Indeed, the center of any connected semisimple linear group is finite (see e.g. [OV94 p. 146]).
We already know that $G^0$ is connected and semisimple, and $G^0$ is linear because it is a subgroup of the linear group $\text{PGL}_{2p}(\mathbb{R})$.

Next we show $Z$ is trivial. Since $G$ normalizes $G^0$, $G$ also normalizes $Z$. Since $Z$ is finite, a finite-index subgroup of $G$ centralizes $Z$. Thus if $Z \neq 1$ we are in Case (1), which has been excluded by assumption. $\square$

Next we use an argument of Farb and Weinberger to deduce:

**Lemma 8.6.** [FW08, Proposition 3.1] $G$ has a finite-index subgroup $G'$ such that $G' \cong G^0 \times \Lambda$ for some discrete group $\Lambda$ and $\Gamma$ has a finite-index subgroup $\Gamma'$ such that $\Gamma' \cong \Gamma_0 \times \Lambda$. Moreover, by possibly passing to a finite-index subgroup of $G'$ we may assume that $\Lambda$ is either trivial or infinite.

**Remark 8.7.** The above Lemma follows from the “triviality of the extension” part of the proof of Proposition 3.1 in [FW08]. This part of their proof only involves the groups and not the Riemannian metric in the statement of Proposition 3.1. In particular, this part of the argument adapts to our situation verbatim.

Now let $\text{SL}_{2p}^\pm(\mathbb{R}) = \{g \in \text{GL}_{2p}(\mathbb{R}) : \det g = \pm 1\}$.

Then let $\hat{G}$ be the inverse image of $G$ under the map $\pi : \text{SL}_{2p}^\pm(\mathbb{R}) \to \text{PGL}_{2p}(\mathbb{R})$ and let $\hat{G}^0$ be the connected component of the identity of $\hat{G}$.

Decompose the representation $\hat{G}^0 \curvearrowright \mathbb{R}^{2p}$ as a direct sum of irreducible representations of the semisimple group $\hat{G}^0$:

$$\mathbb{R}^{2p} \cong \bigoplus_{\rho} V^{n_{\rho}}_{\rho}.$$

Here the direct sum is over nonisomorphic irreducible representations $\rho$ of $\hat{G}^0$ and $n_{\rho} \geq 0$ is the multiplicity of $\rho$. Now since $\hat{G}$ normalizes $\hat{G}^0$ we see that $\hat{G}$ preserves each $V^{n_{\rho}}_{\rho}$.

First let us consider the situation that multiple irreducible representations contribute, say $\rho_1, \ldots, \rho_k$ where $k > 1$. Consider the 1-parameter group $\{b_t : t \in \mathbb{R}\}$ where $b_t$ acts by $e^t$ on the $V^{n_{\rho_1}}_{\rho_1}$ factor and by the identity on all other factors. Then $b_t$ is not a scalar matrix, and centralizes $G$, so we are in Case (1).

Therefore there is only one irreducible representation and $\mathbb{R}^{2p} \cong V^n_{\rho}$ for some irreducible representation $\rho$ and some $n$.

**Lemma 8.8.** $n = 1$.

**Proof.** Suppose for a contradiction that $n > 1$. We first claim that $p = 2$. Let us now consider the virtual cohomological dimension $\text{vcd}(\Gamma)$ of $\Gamma$. Recall that the cohomological dimension $\text{cd}(\Gamma)$ of $\Gamma$ is the supremum of all numbers $m$ such that $H^m(\Gamma, M) \neq 0$ for some $\Gamma$-module $M$ (see for instance [Bro94, Chapter VIII] for more information). We will only need the following properties of $\text{cd}(\Gamma)$:

1. $\text{cd}(\Gamma) > 0$ if $\Gamma \neq 1$.
2. If $\Gamma$ acts freely and properly discontinuously on a contractible $CW$-complex $X$, then $\text{cd}(\Gamma) \leq \text{dim}(X)$, with equality if and only if $X/\Gamma$ is compact.
3. If $\Delta \subseteq \Gamma$, then $\text{cd}(\Delta) \leq \text{cd}(\Gamma)$.
4. If $\Gamma = \Gamma_0 \times \Gamma_1$, then $\text{cd}(\Gamma) \leq \text{cd}(\Gamma_0) + \text{cd}(\Gamma_1)$. 


The virtual cohomological dimension of $\Gamma$ is then the infimum of $\text{cd}(\Delta)$ as $\Delta$ ranges over finite-index subgroups of $\Gamma$.

Now write $\dim V_\rho = d$. Since $\Gamma_0$ can be identified with a discrete subgroup of $\text{PGL}(V_\rho)$, we have by Property (2) above

$$\text{vcd}(\Gamma_0) \leq \dim \text{SL}_d(\mathbb{R})/\text{SO}(d) = \frac{d(d+1)}{2} - 1.$$  \hspace{1cm} (8.2)

Further, since $\Lambda$ commutes with $G^0$ and $\rho$ is an irreducible representation of $\hat{G}^0$, we can identify $\Lambda$ with a discrete subgroup of $\text{PGL}_n(\mathbb{R})$. Therefore

$$\text{vcd}(\Lambda) \leq \dim \text{SL}_n(\mathbb{R})/\text{SO}(n) = \frac{n(n+1)}{2} - 1.$$  \hspace{1cm} (8.3)

On the other hand $\text{vcd}(\Gamma) = \dim \Omega = p^2$ by Property (2) above. Combining this with Property (4) and Equations (8.2) and (8.3), we have

$$2p^2 = 2\text{vcd}(\Gamma) \leq 2(\text{vcd}(\Gamma_0) + \text{vcd}(\Lambda)) \leq d(d+1) - 2 + n(n+1) - 2 = d^2 + d + n^2 + n - 4.$$

Using that $2p = dn$ (from the dimension count in $\mathbb{R}^{2p} \cong V_\rho^n$), we find that

$$2p^2 \leq \frac{4p^2}{n^2} + \frac{2p}{n} + n^2 + n - 4.$$  \hspace{1cm} \text{The right-hand side is a convex function of $n$, so that on the interval $[2, p]$, it is maximal at one of the endpoints. At either endpoint the inequality reduces to}

$$p^2 - p - 2 \leq 0,$$

which is only possible if $p = 2$.

Then $(n, d) \in \{(2, 2), (1, 4), (4, 1)\}$. We assumed that $n > 1$ and since the representation $\hat{G}^0 \hookrightarrow \text{SL}(V_\rho)$ is injective we must have $d > 1$. So $n = d = 2$.

Thus $\hat{G}^0$ is a semisimple Lie group which has a faithful irreducible representation into $\text{SL}_2(\mathbb{R})$. Thus $\hat{G}^0$ has to be isomorphic to $\text{SL}_2(\mathbb{R})$ and $\rho = \text{Id}$. With respect to the decomposition $\mathbb{R}^4 = V \oplus V$ we have

$$\hat{G}^0 = \{(\varphi, \varphi) \in \text{SL}(V) \times \text{SL}(V)\}$$

and hence we are in Case (3) which is a contradiction. \hspace{1cm} \square

Since $n = 1$, we have that $\hat{G}^0 \cap \mathbb{R}^{2p}$ is an irreducible representation. Note that $\Lambda$ centralizes $G^0$ in $\text{PGL}_{2p}(\mathbb{R})$, and hence any element of $\text{GL}_{2p}(\mathbb{R})$ lying over $\Lambda$ has to be scalar by Schur’s Lemma. It follows that $\Lambda$ is trivial, so that $G' = G^0$ and thus $G^0$ has finite-index in $G$. Then $\Gamma_0$ has finite-index in $\Gamma$ and hence acts cocompactly on $\Omega$. Thus $\text{vcd}(\Gamma_0) = \dim(\Omega) = p^2$.

**Lemma 8.9.** $G^0$ acts transitively on $\Omega$.

**Proof.** Let $x \in \Omega$ be any point and let $K_x$ denote its stabilizer in $G^0$. Then $K_x$ is a compact subgroup of $G^0$ by Proposition 4.7 and the $G^0$-orbit $X$ of $x$ is diffeomorphic to $G^0/K_x$. Now let $K$ be a maximal compact subgroup of $G^0$ containing $K_x$. Then $\Gamma_0 \backslash G^0/K$ is a closed aspherical manifold with fundamental group $\Gamma_0$ so by Property
of cohomological dimension we have \( \text{vcd}(\Gamma_0) = \dim(G^0/K) \). On the other hand since \( K_x \leq K \) and \( G^0/K_x \cong X \subset \Omega \)

\[
\text{vcd}(\Gamma_0) = \dim(G^0/K) \leq \dim(G^0/K_x) = \dim(X) \leq \dim(\Omega) = \text{vcd}(\Gamma_0).
\]

We conclude that any \( \dim(X) = \dim(\Omega) \), so that \( X \) is a codimension 0 closed submanifold of \( \Omega \). Connectedness of \( \Omega \) then implies that \( X = \Omega \), as desired. \( \square \)

Remark 8.10. The above proof shows that the stabilizer of any point \( x \in \Omega \) has finite-index in a maximal compact subgroup of \( \text{Aut}(\Omega) \).

Lemma 8.11. \( G^0 \) is simple.

Proof. Since \( G^0 \) has trivial center either \( G^0 \) is simple or \( G^0 \cong G_1 \times G_2 \) for some semisimple nontrivial Lie groups \( G_1 \) and \( G_2 \).

So suppose that \( G^0 \cong G_1 \times G_2 \). Let \( \hat{G}_1 \) be the inverse image of \( G_1 \times \{\text{Id}\} \) under the map \( \text{SL}_{2p}(\mathbb{R}) \to \text{PSL}_{2p}(\mathbb{R}) \). Next decompose the representation \( \hat{G}_1 \rightleftharpoons \mathbb{R}^{2p} \) as a direct sum of irreducible representations of the semisimple group \( \hat{G}_1 \):

\[
\mathbb{R}^{2p} \cong \bigoplus_{\tau} V^*_\tau.
\]

Here the direct sum is over nonisomorphic irreducible representations \( \tau \) of \( \hat{G}_1 \), and \( n_\tau \geq 0 \) is the multiplicity of \( \tau \). Using the fact that \( \hat{G}_2 \) centralizes \( \hat{G}_1 \) and arguing as in Lemma 8.8 we see that \( p = 2 \) and \( \mathbb{R}^4 = V^*_2 \) for some irreducible representation \( \tau \) of \( \hat{G}_1 \). So \( \dim V^*_2 = 2 \) and thus \( \hat{G}_1 \) is isomorphic to \( \text{SL}_2(\mathbb{R}) \). Applying the same argument to \( \hat{G}_2 \) shows that \( \hat{G}_2 \) is also isomorphic to \( \text{SL}_2(\mathbb{R}) \). Up to conjugation, we have

\[
\hat{G}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in \text{SL}_2(\mathbb{R}) \right\}.
\]

An easy computation shows that the centralizer of \( \hat{G}_1 \) is exactly

\[
\left\{ \begin{pmatrix} a\text{Id} & b\text{Id} \\ c\text{Id} & d\text{Id} \end{pmatrix} : ad - bc = 1 \right\} \cong \text{SL}_2(\mathbb{R}).
\]

Since \( \hat{G}_2 \) centralizes \( \hat{G}_1 \) and is isomorphic to \( \text{SL}_2(\mathbb{R}) \), we must have that

\[
\hat{G}_2 = \left\{ \begin{pmatrix} a\text{Id} & b\text{Id} \\ c\text{Id} & d\text{Id} \end{pmatrix} : ad - bc = 1 \right\}.
\]

Hence we are in Case (4), which is a contradiction. \( \square \)

9. THE CENTRALIZER

In this section we prove that Case (1) in Theorem 8.1 is impossible. For a subgroup \( H \leq \text{PGL}_{p+q}(\mathbb{R}) \), let

1. \( \hat{H} = \{ h \in \text{GL}_{p+q}(\mathbb{R}) : [h] \in H, \det h = \pm 1 \} \),

2. \( C_H = \{ c \in \text{End}(\mathbb{R}^{p+q}) : ch = hc \text{ for all } h \in \hat{H} \} \), and

3. let \( C^0_H \) be the connected component of \( \text{Id}_{p+q} \) in \( C_H \cap \text{GL}_{p+q}(\mathbb{R}) \).
Remark 9.1. Note that $C_H$ is the centralizer in $\text{End}(\mathbb{R}^{p+q})$, and hence is a subalgebra of $\text{End}(\mathbb{R}^{p+q})$, whereas $C^0_H$ is a subgroup of $\text{GL}_{p+q}(\mathbb{R})$.

With this notation we will prove the following:

**Theorem 9.2.** Suppose $\Omega \subset \text{Gr}_p(\mathbb{R}^{2p})$ is an open set which is convex and bounded in some affine chart. If $\Gamma \leq \text{Aut}(\Omega)$ is a discrete group that acts cocompactly on $\Omega$, then $C^0_H = \mathbb{R} > 0 \text{Id}_{2p}$.

9.1. **The centralizer in the general case.** We begin by proving the following (which holds for any Grassmannian):

**Theorem 9.3.** Suppose $\Omega \subset \text{Gr}_p(\mathbb{R}^{p+q})$ is an open $\mathcal{R}$-proper set that is convex in some affine chart. If $H \leq \text{Aut}(\Omega)$ acts cocompactly on $\Omega$, then $C^0_H \leq \text{Aut}(\Omega)$ and there is a decomposition $\mathbb{R}^{p+q} = \bigoplus_{i=1}^m V_i$ such that $C_H = \bigoplus_{i=1}^m \mathbb{R} \cdot \text{Id}_{V_i}$.

**Remark 9.4.** In the special case where $p = 1$ the above Theorem is due to Vey [Vey70, Theorem 5].

For the rest of this subsection assume that $\Omega \subset \text{Gr}_p(\mathbb{R}^{p+q})$ and $H \leq \text{Aut}(\Omega)$ satisfy the hypothesis of Theorem 9.3.

**Lemma 9.5.** With the notation above, $C^0_H \leq \text{Aut}(\Omega)$

**Proof.** Fix a compact set $K \subset \Omega$ such that $H \cdot K = \Omega$. Then there exists a neighborhood $\mathcal{O}$ of $\text{Id}_{p+q}$ in $C^0_H$ such that $\mathcal{O}$ generates $C^0_H$ and $u \cdot K \subset \Omega$ for all $u \in \mathcal{O}$. Without loss of generality, we can assume that $\mathcal{O}$ is symmetric, i.e. for any $u \in \mathcal{O}$, we have $u^{-1} \in \mathcal{O}$. Then for $u \in \mathcal{O}$, we have $u \cdot \Omega = u \cdot H \cdot K = H \cdot u \cdot K \subset H \cdot \Omega = \Omega$.

Since $\mathcal{O}$ is symmetric we also see that $u^{-1} \cdot \Omega \subset \Omega$. Thus $u$ restricts to a diffeomorphism $\Omega \to \Omega$ and $u \in \text{Aut}(\Omega)$. Since $\mathcal{O}$ generates $C^0_H$ we then see that $C^0_H \leq \text{Aut}(\Omega)$.

**Lemma 9.6.** With the notation above, if $c \in C^0_H$ then $\sup_{x \in \Omega} K^*_\Omega(cx, x) < \infty$.

**Proof.** Fix some $x_0 \in \Omega$. Since $H$ acts cocompactly on $\Omega$, there exists $R > 0$ such that $\bigcup_{h \in H} B_R(hx_0) = \Omega$.

If $x \in \Omega$ pick $h \in H$ such that $K^*_\Omega(x, hx_0) \leq R$. Then

$$K^*_\Omega(cx, x) \leq K^*_\Omega(cx, chx_0) + K^*_\Omega(chx_0, hx_0) + K^*_\Omega(hx_0, x)$$
$$\leq K^*_\Omega(x, hx_0) + K^*_\Omega(cx_0, x_0) + R$$
$$\leq 2R + K^*_\Omega(cx_0, x_0).$$

**Lemma 9.7.** With the notation above, if $c \in C^0_H$ then $c$ fixes every $\mathcal{R}$-extreme point of $\Omega$. 

Proof. For an $R$-extreme point $x \in \partial \Omega$, choose points $p_n \in \Omega$ with $p_n \to x$. By Lemma 9.6 we have
\[ \lim_{n \to \infty} d_\Omega(c p_n, p_n) < \infty. \]
Then by Corollary 6.3, we have $c p_n \to x$. Since $c$ acts continuously on $Gr_p(\mathbb{R}^{2p})$ and $p_n \to x$, we must have that $c x = x$. \hfill \Box

We will need the following elementary facts.

**Lemma 9.8.** Let $p, q > 0$. Then the homomorphism $\wedge^p : GL_{p+q}(\mathbb{R}) \to GL(\wedge^p \mathbb{R}^{p+q})$

(i) maps unipotents to unipotents and semisimple elements to semisimple elements, and

(ii) has kernel given by $\{ \text{Id}_{p+q} \}$ if $p$ is odd and $\{ \text{Id}_{p+q} \}$ if $p$ is even.

**Proof.** Assertion (i) is obvious from the definition of $\wedge^p$. To see (ii), consider some $g \in GL_{p+q}(\mathbb{R})$ with $\wedge^p g = 1$. Let $\lambda_1, \ldots, \lambda_{p+q}$ be the eigenvalues of $g$ (listed with multiplicity). Then the eigenvalues of $\wedge^p g$ are exactly given by the product of $p$ eigenvalues of $g$, i.e. $\lambda_1 \cdots \lambda_p$ for any choice of $1 \leq i_1 < \cdots < i_p \leq p + q$. We claim that $\lambda_1 = \lambda_2 = \cdots = \lambda_{p+q}$. To see this fix $1 \leq i, j \leq p + q$ distinct and then fix some $i_1, \ldots, i_{p-1}$ such that $i, j, i_1, \ldots, i_{p-1}$ are all distinct. Since $\wedge^p g = 1$, we have
\[ \lambda_1 \lambda_{i_1} \cdots \lambda_{i_{p-1}} = 1 = \lambda_j \lambda_{i_1} \cdots \lambda_{i_{p-1}} \]
so that $\lambda_i = \lambda_j$. Since $i, j$ were arbitrary, we then have $\lambda_1 = \lambda_2 = \cdots = \lambda_{p+q}$. So $\lambda_1 = \lambda_2 = \cdots = \lambda_p = 1$.

In addition, $\lambda_1$ is real, so it follows that $\lambda_1 \in \{-1, 1\}$. We conclude that $g = \pm \text{Id}_{p+q}$. \hfill \Box

**Lemma 9.9.** With the notation above, every $c \in C_H^0$ is semisimple and $C_H^0$ is abelian.

**Proof.** Fix a basis $v_1, \ldots, v_D$ of $\wedge^p \mathbb{R}^{p+q}$ such that each $[v_i]$ is an $R$-extreme point of $\Omega$ (this is possible by Proposition 7.8). Then for any $c \in C_H^0$, each $v_i$ is an eigenvector of $\wedge^p c$ and so $\wedge^p c$ is diagonalizable with respect to the basis $v_1, \ldots, v_D$ of $\wedge^p \mathbb{R}^{p+q}$. Hence $\wedge^p C_H^0$ is an abelian group.

Now, since $\wedge^p C_H^0$ is an abelian group, we see that $\wedge^p[C_H^0, C_H^0] = 1$. Then, since $\ker \wedge^p \subset \{ \pm \text{Id}_{p+q} \}$, we see that $[C_H^0, C_H^0] \subset \{ \pm \text{Id}_{p+q} \}$. But since $C_H^0$ is connected, $[C_H^0, C_H^0]$ is connected and hence must be trivial. We conclude that $C_H^0$ is abelian.

Next, we claim that any $c \in C_H^0$ is semisimple. If $c = su$ is the Jordan decomposition of $c$ then $\wedge^p c = (\wedge^p s)(\wedge^p u)$ and by uniqueness this is the Jordan decomposition of $\wedge^p c$. It follows that $\wedge^p u = 1$, and hence $u = 1$. We conclude that $c = s$ is semisimple. \hfill \Box

**Lemma 9.10.** With the notation above, every $c \in C_H^0$ has all real eigenvalues.

Let us comment briefly on the strategy of the proof of Lemma 9.10 before carrying out the algebraic manipulations. Notice that the proof of Lemma 9.9 implies that if $c \in C_H^0$, then $\wedge^p c$ has all real eigenvalues. Therefore the product of any $p$ distinct eigenvalues of $c$ (counted with multiplicity) is real. Unfortunately this does not directly imply that the eigenvalues of $c$ are real; for example if $g \in GL_4(\mathbb{R})$ has eigenvalues $\pm i$, each with multiplicity $2$, then $\wedge^2 g$ has eigenvalues $\pm 1$. The
strategy in the proof of Lemma 9.10 is to argue by contradiction, i.e. assume there exists some element \( c \in C_H^0 \) which has a non-real eigenvalue and then use \( c \) to construct some other \( c' \in C_H^0 \) where \( \wedge c' \) has a non-real eigenvalue.

**Proof.** For \( n \in \mathbb{N}, \lambda > 0, \) and \( \theta \in [0, 2\pi) \) let \( E_n(\lambda, \theta) \) be the \( 2n \times 2n \) block diagonal matrix whose blocks are

\[
\begin{pmatrix}
\lambda \cos \theta & -\lambda \sin \theta \\
\lambda \sin \theta & \lambda \cos \theta
\end{pmatrix}.
\]

Now suppose for a contradiction that there exists some \( c \in C_0 H \) with a non-real eigenvalue. Then there exist \( g \in \text{SL}_{p+q}(\mathbb{R}); n_1, \ldots, n_k \in \mathbb{N}; \lambda_1, \ldots, \lambda_r > 0; \theta_1, \ldots, \theta_r \in [0, 2\pi); \) and \( \mu_{r+1}, \ldots, \mu_k \in \mathbb{R} \) such that

\[
c = \begin{pmatrix}
E_{n_1}(\lambda_1, \theta_1) & & \\
& \ddots & \\
& & E_{n_r}(\lambda_r; \theta_r)
\end{pmatrix}
\begin{pmatrix}
\mu_{r+1} \text{Id}_{n_{r+1}} & & \\
& \ddots & \\
& & \mu_k \text{Id}_{n_k}
\end{pmatrix} g^{-1}.
\]

We can further assume that the pairs \((\lambda_i, \theta_i)\) are all distinct and the \( \mu_i \) are all distinct. Then we have

\[
\hat{H} \leq \left\{ g \begin{pmatrix}
A_1 & & \\
& \ddots & \\
& & A_k
\end{pmatrix} g^{-1} : A_i \in \text{GL}_{n_i}(\mathbb{R}) \right\}.
\]

Which implies that

\[
\left\{ g \begin{pmatrix}
E_{n_1}(\lambda, \theta) & & \\
& \ddots & \\
& & \text{Id}_{n_2 + \ldots + n_k}
\end{pmatrix} g^{-1} : \lambda, \theta \in \mathbb{R} \right\} \leq C_H^0.
\]

Then it is easy to construct some \( c' \in C_H^0 \) such that \( \wedge c' \) has a non-real eigenvalue. So we have a contradiction. \( \square \)

**Lemma 9.11.** With the notation above, there is a decomposition \( \mathbb{R}^{p+q} = \bigoplus_{i=1}^m V_i \) such that

\[
C_H = \bigoplus_{i=1}^m \mathbb{R} \cdot \text{Id}_{V_i}.
\]

**Proof.** Since \( C_H^0 \) is abelian and every element in \( C_H^0 \) is semisimple with all real eigenvalues, there exist some \( g \in \text{SL}_{p+q}(\mathbb{R}) \) and \( n_1, \ldots, n_k \in \mathbb{N} \) such that

\[
C_H^0 \leq \left\{ g \begin{pmatrix}
\mu_1 \text{Id}_{n_1} & & \\
& \ddots & \\
& & \mu_k \text{Id}_{n_k}
\end{pmatrix} g^{-1} : \mu_1, \ldots, \mu_k > 0 \right\}.
\]

We may further assume that for every \( 1 \leq i < j \leq k \) there exists \( c \in C_H^0 \) such that

\[
c = g \begin{pmatrix}
\mu_1 \text{Id}_{n_1} & & \\
& \ddots & \\
& & \mu_k \text{Id}_{n_k}
\end{pmatrix} g^{-1}.
\]
and $\mu_i \neq \mu_j$. Then we have
\[
\hat{H} \leq \left\{ g \begin{pmatrix} A_1 & \cdots & A_k \end{pmatrix} g^{-1} : A_i \in \text{GL}_{n_i}(\mathbb{R}) \right\}
\]
and hence
\[
C_H^0 = \left\{ g \begin{pmatrix} \mu_1 \text{Id}_{n_1} & \cdots & \mu_k \text{Id}_{n_k} \end{pmatrix} g^{-1} : \mu_1, \ldots, \mu_k > 0 \right\}.
\]
Now if $X \in C_H$, then there exists some $t \in \mathbb{R}$ so that $\text{Id}_{p+q} + tX \in C_H^0$. Hence
\[
C_H = \left\{ g \begin{pmatrix} \mu_1 \text{Id}_{n_1} & \cdots & \mu_k \text{Id}_{n_k} \end{pmatrix} g^{-1} : \mu_1, \ldots, \mu_k \in \mathbb{R} \right\}
\]
which implies the lemma.

9.2. The centralizer in $\text{Gr}_p(\mathbb{R}^{2p})$. We now specialize to the case in which $p = q$ and prove Theorem 9.2. We begin by showing that we can assume that $\Omega$ is a cone in some affine chart.

**Proposition 9.12.** Suppose $\Omega \subset \text{Gr}_p(\mathbb{R}^{p+q})$ is an open set which is convex and bounded in some affine chart. If $H \leq \text{Aut}(\Omega)$ acts cocompactly on $\Omega$ and $C_H^0 \neq \mathbb{R} > \text{Id}_{2p}$, then there exists $\varphi \in \text{PGL}_2(\mathbb{R})$ so that
\[
\varphi \Omega \subset \mathcal{M} = \left\{ \begin{bmatrix} \text{Id}_p & 0 \\ X \\ \text{Id}_p \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}
\]
and $\varphi \Omega$ is a convex cone in $\mathcal{M}$ based at 0. Moreover we can select $\varphi$ so that either
\[
C_{\varphi H \varphi^{-1}}^0 = \left\{ \begin{pmatrix} e^s \text{Id}_p & 0 \\ 0 & e^s \text{Id}_p \end{pmatrix} : s, t \in \mathbb{R} \right\}.
\]
or $C_{\varphi H \varphi^{-1}}^0$ contains the subgroup
\[
\left\{ \begin{pmatrix} e^s \text{Id}_{p+\ell} & 0 \\ 0 & e^s \text{Id}_{p-\ell} \end{pmatrix} : s, t \in \mathbb{R} \right\}
\]
for some $0 < \ell < p$.

**Remark 9.13.** By Corollary 7.11, we can find some $\varphi \in \text{GL}_{2p}(\mathbb{R})$ so that $\varphi \Omega \subset \mathcal{M}$ and $\varphi \Omega$ is a convex cone in $\mathcal{M}$ based at 0. The key part of the proposition is that we can pick $\varphi$ so that the centralizer $C_{\varphi H \varphi^{-1}}$ has a subgroup of a particularly nice form.

**Proof.** We can assume that $\Omega$ is a convex bounded subset of $\mathcal{M}$. Through out the argument we will replace $\Omega$ by translates of the form
\[
\begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \Omega.
\]
This transformation preserves the affine chart $\mathcal{M}$ and acts on $\mathcal{M}$ by affine transformations.
By Theorem 9.3, there exists $g_0 \in \text{GL}_{2p}(\mathbb{R})$ and $0 \leq \ell < p$ so that

$$A := \left\{ g_0 \begin{pmatrix} e^t \text{Id}_{p+\ell} & 0 \\ 0 & e^s \text{Id}_{p-\ell} \end{pmatrix} g_0^{-1} : s, t \in \mathbb{R} \right\} \subseteq C^0_H.$$ 

Notice that we can choose $\ell > 0$ except when $C^0_H = \{ g_0(\text{Id}_{p+\ell} \ 0) g_0^{-1} : s, t \in \mathbb{R} \}$. So in the case when $\ell = 0$ we can also assume that $C^0_H = A$.

Now let $W := g_0 \text{Span}\{e_1, \ldots, e_{p+\ell}\}$. Notice that $hW = W$ for all $h \in H$. We claim that there exists an $R$-extreme point $e$ of $\Omega$ in $\text{Gr}_p(W)$. Consider some $c = g_0 \begin{pmatrix} e^t \text{Id}_p & 0 \\ 0 & e^s \text{Id}_p \end{pmatrix} g_0^{-1} \in A$ with $e^t > e^s$. Then $E^+(\wedge^\ell c) \cap \text{Gr}_p(\mathbb{R}^{2p}) \subset \text{Gr}_p(W)$ and by Corollary 7.10 there is an $R$-extreme point $e$ of $\Omega$ in $E^+(\wedge^\ell c) \cap \partial \Omega \subset \text{Gr}_p(W)$.

Now by replacing $\Omega$ with an affine translate we can assume that $e = \text{Id}_p$. By construction, if $a \in A$ then $a|_W = e^t \text{Id}_W$ for some $t \in \mathbb{R}$. So any $a \in A$ can be written as

$$a = \begin{pmatrix} e^t \text{Id}_p & B \\ 0 & C \end{pmatrix}$$

for some $t \in \mathbb{R}$ and $B, C \in \text{GL}_p(\mathbb{R})$.

Since $e$ is an extreme point, by Corollary 7.11, there exists $t_n \to \infty$ and $h_n \in H$ so that

$$\phi = \lim_{n \to \infty} \begin{pmatrix} \text{Id}_p & 0 \\ 0 & e^{tn} \text{Id}_p \end{pmatrix} h_n$$

in $\text{PGL}_{2p}(\mathbb{R})$ and $\phi(\Omega) = \mathcal{C}_0 \Omega$. Let $\tilde{\phi} \in \text{GL}_{2p}(\mathbb{R})$ be a representative of $\phi$ and for each $n \in \mathbb{N}$ pick $\tilde{h}_n \in \text{GL}_{2p}(\mathbb{R})$ a representative of $h_n$ so that

$$\tilde{\phi} = \lim_{n \to \infty} \begin{pmatrix} \text{Id}_p & 0 \\ 0 & e^{tn} \text{Id}_p \end{pmatrix} \tilde{h}_n$$

in $\text{GL}_{2p}(\mathbb{R})$.

Then if

$$a = \begin{pmatrix} e^t \text{Id}_p & B \\ 0 & C \end{pmatrix} \in A$$

we have

$$\tilde{\phi}a \tilde{\phi}^{-1} = \lim_{n \to \infty} \begin{pmatrix} \text{Id}_p & 0 \\ 0 & e^{tn} \text{Id}_p \end{pmatrix} \tilde{h}_n \begin{pmatrix} e^t \text{Id}_p & B \\ 0 & C \end{pmatrix} \tilde{h}_n^{-1} \begin{pmatrix} \text{Id}_p & 0 \\ 0 & e^{-tn} \text{Id}_p \end{pmatrix}$$

$$= \lim_{n \to \infty} \begin{pmatrix} \text{Id}_p & 0 \\ 0 & e^{tn} \text{Id}_p \end{pmatrix} \begin{pmatrix} e^t \text{Id}_p & B \\ 0 & C \end{pmatrix} \begin{pmatrix} \text{Id}_p & 0 \\ 0 & e^{-tn} \text{Id}_p \end{pmatrix}$$

$$= \begin{pmatrix} e^t \text{Id}_p & 0 \\ 0 & C \end{pmatrix}.$$ 

In the second equality we used that $a \in C^0_H$. 


Then since $A$ is abelian, we can find some $g_0 \in \text{GL}_p(\mathbb{R})$ so that if
\[
g = \begin{pmatrix} \text{Id}_p & 0 \\ 0 & g_0 \end{pmatrix},
\]
then
\[
g\hat{\varphi}A(g\hat{\varphi})^{-1} = \left\{ \begin{pmatrix} e^t \text{Id}_{p+\ell} & 0 \\ 0 & e^s \text{Id}_{p-\ell} \end{pmatrix} : s, t \in \mathbb{R} \right\}.
\]
So replacing $\varphi$ by $g\varphi$ (and hence replacing $\hat{\varphi}$ by $g\hat{\varphi}$) we can assume
\[
\hat{\varphi}A\hat{\varphi}^{-1} = \left\{ \begin{pmatrix} e^t \text{Id}_{p+\ell} & 0 \\ 0 & e^s \text{Id}_{p-\ell} \end{pmatrix} : s, t \in \mathbb{R} \right\}.
\]
Since $\hat{\varphi}A\hat{\varphi}^{-1} \leq C_0^0_{\nu H}^{-1}$ this completes the proof. $\square$

**Proof of Theorem 9.2** By Proposition 9.12 we can assume that
\[
\Omega \subset M = \left\{ \begin{pmatrix} X \end{pmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}
\]
is a convex cone in $M$ based at 0, and that $C_0^p$ contains the subgroup
\[
\left\{ \begin{pmatrix} e^t \text{Id}_{p+\ell} & 0 \\ 0 & e^s \text{Id}_{p-\ell} \end{pmatrix} : s, t \in \mathbb{R} \right\}
\]
for some $0 \leq \ell < p$. Then
\[
\Gamma \leq \left\{ \begin{pmatrix} A \\ 0 \\ B \end{pmatrix} : A \in \text{GL}_{p+\ell}(\mathbb{R}), B \in \text{GL}_{p-\ell}(\mathbb{R}) \right\}.
\]

Throughout the argument we will write a matrix $X \in M_{p,p}(\mathbb{R})$ as
\[
X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}
\]
where $X_1 \in M_{\ell,p}(\mathbb{R})$ and $X_2 \in M_{p-\ell,p}(\mathbb{R})$. Let
\[
\Omega_2 = \left\{ \begin{pmatrix} \text{Id}_p \\ 0 \\ X_2 \end{pmatrix} : \text{there exists } X_1 \text{ such that } \begin{pmatrix} \text{Id}_p \\ X_1 \\ X_2 \end{pmatrix} \in \Omega \right\}.
\]

**Lemma 9.14.** $\Omega_2$ is a proper convex cone in $M$, i.e. $\Omega_2$ does not contain any affine lines.

**Proof.** Since $\Omega_2$ is open and convex, it is easy to see that
\[
\{ x + tv : t \in \mathbb{R} \} \subset \Omega_2 \text{ for some } x \in \Omega_2 \iff \{ x + tv : t \in \mathbb{R} \} \subset \Omega_2 \text{ for all } x \in \Omega_2.
\]
Hence $\Omega_2$ contains an affine line if and only if there exists some nonzero $v \in M$ such that
\[
\begin{pmatrix} \text{Id}_p & 0 \\ 0 & \text{Id}_\ell \\ v & 0 & \text{Id}_{p-\ell} \end{pmatrix} \in \text{Aut}(\Omega).
\]
Thus to complete the proof, it suffices to show that
\[
\{ \text{Id}_{2p} \} = \left\{ \begin{pmatrix} \text{Id}_{p+\ell} & 0 \\ Y & \text{Id}_{p-\ell} \end{pmatrix} : Y \in M_{p-\ell,p+\ell}(\mathbb{R}) \right\} \cap \text{Aut}(\Omega).
So suppose that
\[ g := \left[ \begin{array}{cc} \text{Id}_{p+\ell} & 0 \\ Y & \text{Id}_{p-\ell} \end{array} \right] \in \text{Aut}(\Omega) \]
for some \( Y \in M_{p-\ell,p+\ell}(\mathbb{R}) \). Since \( \Gamma \) is a cocompact lattice in \( \text{Aut}(\Omega) \), there exist
\[ \gamma_n := \left[ \begin{array}{cc} A_n & 0 \\ 0 & B_n \end{array} \right] \in \Gamma \]
such that \( \{ \gamma_n g^n \}_n \) is bounded in \( \text{PGL}_2(p) \). By picking representatives of \( \gamma_n \) and \( g^n \) in \( \text{GL}_2(p) \) correctly we can assume that
\[ \left( \begin{array}{cc} A_n & 0 \\ 0 & B_n \end{array} \right) \left( \begin{array}{cc} \text{Id}_{p+\ell} & 0 \\ nY & \text{Id}_{p-\ell} \end{array} \right) = \left( \begin{array}{cc} A_n & 0 \\ nB_n Y & B_n \end{array} \right) \]
is a bounded sequence in \( \text{GL}_2(p) \). This implies \( \{ B_n \}_n \) and \( \{ nB_n Y \}_n \) are bounded sequences in \( \text{GL}_{p-\ell}(\mathbb{R}) \) and \( \text{M}_{p-\ell,p+\ell}(\mathbb{R}) \) respectively. Therefore we must have \( Y = 0 \), as desired. \( \square \)

Since Proposition \ref{prop:rigidity} yields different conclusions depending on whether \( \ell = 0 \) or \( \ell > 0 \), we will consider these two situations separately below.

**Case 1:** First suppose that \( \ell = 0 \). Then \( \Omega = \Omega_2 \) is a proper convex cone and by Proposition \ref{prop:rigidity} we may assume that
\[ C^0_\Gamma = \left\{ \left( \begin{array}{cc} e^s \text{Id}_p & 0 \\ 0 & e^t \text{Id}_p \end{array} \right) : s, t \in \mathbb{R} \right\}. \]
Then
\[ \Gamma \leq \left\{ \left[ \begin{array}{cc} A & 0 \\ 0 & B \end{array} \right] : A, B \in \text{GL}_p(\mathbb{R}) \right\}. \]
So \( \Gamma \) acts by linear transformations on \( \Omega \). We will now use the theory of linear automorphisms of a proper convex cone to establish a contradiction.

Define a homomorphism
\[ \Phi : \left\{ \left[ \begin{array}{cc} A & 0 \\ 0 & B \end{array} \right] \in \text{PGL}_2(p) : A, B \in \text{GL}_p(\mathbb{R}) \right\} \to \text{GL}(\mathcal{M}) \]
by
\[ \Phi \left( \left[ \begin{array}{cc} A & 0 \\ 0 & B \end{array} \right] \right) \cdot X = BXA^{-1}. \]
Notice that \( \Phi \) is injective and well-defined.

Then \( \Lambda := \Phi(\Gamma) \) acts cocompactly on \( \Omega \subset \mathcal{M} \). Let \( \Gamma^Z \) be the Zariski closure of \( \Gamma \) in \( \text{PGL}_2(p) \) and \( \overline{\Lambda}^Z \) the Zariski closure of \( \Lambda \) in \( \text{GL}(\mathcal{M}) \). Then
\[ \Phi \left( \Gamma^Z \right) = \overline{\Lambda}^Z. \]
By possibly passing to a finite-index subgroup we can assume that \( \Gamma^Z \) is connected in the Zariski topology.

Recall that a convex cone \( C \subset V \) in a real finite-dimensional vector space \( V \) is called *reducible* if there exist a pair of proper subspaces \( V_i \subset V \) and convex cones \( C_i \subset V_i \) for \( i = 1, 2 \) such that \( V = V_1 \oplus V_2 \) and \( C = C_1 + C_2 \). A convex cone \( C \subset V \) is called *irreducible* if it is not reducible.
Let $C_\Lambda \leq \text{GL}(M)$ denote the centralizer of $\Lambda$ in $\text{GL}(M)$. By a result of Vey [Vey70, Theorem 5] either $\Omega$ is an irreducible cone and $C_\Lambda = \mathbb{R}^* \cdot \text{Id}_M$ or $\dim C_\Lambda > 1$. By [Ben03a, Theorem 1.1], we see that $C_\Lambda \leq \text{GL}(M)$. Now if $[C^0_\Gamma]$ is the image of $C^0_\Gamma$ in $\text{PGL}_{2p}(\mathbb{R})$ we see that $\Phi^{-1}(C_\Lambda) \subset [C^0_\Gamma]$.

Since $\dim [C^0_\Gamma] = 1$, so we see that $\dim C_\Lambda = 1$. Thus $\Omega$ is an irreducible cone. Then by [Vey70, Theorem 3] (see also [Ben03a]) there exists a simple group $H \leq \text{GL}(M)$ such that $\Lambda Z = (\mathbb{R}^* \cdot \text{Id}) \cdot H$.

So $\Gamma^Z \cong \mathbb{R}^* \times H$.

Now consider the projections

$$
\pi_1, \pi_2 : \Gamma^Z \to \text{PGL}_p(\mathbb{R})
$$

given by

$$
\pi_1 \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = A \text{ and } \pi_2 \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = B.
$$

Since $H$ is simple, we see that $\ker \pi_i = \Gamma^Z$ or

$$
\ker \pi_i = \left\{ \begin{bmatrix} e^t \text{Id}_p & 0 \\ 0 & e^s \text{Id}_p \end{bmatrix} \in \text{PGL}_{2p}(\mathbb{R}) : s, t \in \mathbb{R} \right\}.
$$

Since

$$
\ker(\pi_1 \times \pi_2) = \left\{ \begin{bmatrix} e^t \text{Id}_p & 0 \\ 0 & e^s \text{Id}_p \end{bmatrix} \in \text{PGL}_{2p}(\mathbb{R}) : s, t \in \mathbb{R} \right\}
$$

we must have that $\ker \pi_i \neq \Gamma^Z$ for some $i \in \{1, 2\}$. Then we see that

$$
\pi_i \circ \Phi^{-1} : H \to \text{PGL}_p(\mathbb{R})
$$

is an injection and thus we obtain an injective homomorphism

$$
\Gamma^Z \hookrightarrow \mathbb{R} \times \text{PGL}_p(\mathbb{R}).
$$

But then

$$
p^2 = \dim(\Omega) = \text{vcd}(\Gamma) \leq 1 + \dim(\text{SL}_p(\mathbb{R}) / \text{SO}(p)) = \frac{p(p + 1)}{2} = \frac{1}{2}p^2 + \frac{1}{2}p < p^2
$$

which is a contradiction.

**Case 2:** Suppose that $C^0_\Gamma$ contains the subgroup

$$
\left\{ \begin{bmatrix} e^t \text{Id}_p + \ell & 0 \\ 0 & e^s \text{Id}_p - \ell \end{bmatrix} : s, t \in \mathbb{R} \right\}
$$

for some $0 < \ell < p$.

Let

$$
\Omega_1 = \left\{ \begin{bmatrix} \text{Id}_p \\ X_1 \\ 0 \end{bmatrix} : \text{ there exists } X_2 \text{ such that } \begin{bmatrix} \text{Id}_p \\ X_1 \\ X_2 \end{bmatrix} \in \Omega \right\}.
$$

**Lemma 9.15.** $\Omega = \Omega_1 + \Omega_2$. 
Proof. By construction
\[ \overline{\Omega} \subset \overline{\Omega}_1 + \overline{\Omega}_2. \]

Now
\[ \left( \begin{array}{cc} \text{Id}_{p+\ell} & 0 \\ 0 & e^{s} \text{Id}_{p-\ell} \end{array} \right) \cdot \left[ \begin{array}{c} \text{Id}_p \\ X_1 \\ X_2 \end{array} \right] = \left[ \begin{array}{c} \text{Id}_p \\ X_1 \\ e^{s}X_2 \end{array} \right]. \]

So by sending \( s \to -\infty \) we see that \( \overline{\Omega} \supset \overline{\Omega}_1 \).

On the other hand,
\[ \left( \begin{array}{cc} \text{Id}_p & 0 \\ 0 & e^{-s} \text{Id}_p \end{array} \right) \cdot \left[ \begin{array}{c} \text{Id}_p \\ X_1 \\ X_2 \end{array} \right] = \left[ \begin{array}{c} \text{Id}_p \\ e^{-s}X_1 \\ X_2 \end{array} \right]. \]

So sending \( s \to \infty \) we see that \( \overline{\Omega} \supset \overline{\Omega}_2 \).

Then if \( X_1 \in \overline{\Omega}_1 \) and \( X_2 \in \overline{\Omega}_2 \) we have
\[ X_1 + X_2 = \frac{1}{2}(2X_1) + \frac{1}{2}(2X_2) \in \overline{\Omega}. \]
Thus \( \overline{\Omega} = \overline{\Omega}_1 + \overline{\Omega}_2 \) which by convexity implies that
\[ \Omega = \Omega_1 + \Omega_2. \]

Now if \( \gamma \in \Gamma \) then we can write
\[ \gamma = \left[ \begin{array}{ccc} A_1 & A_2 & 0 \\ A_3 & A_4 & 0 \\ 0 & 0 & B \end{array} \right] \]
for some \( A_1 \in M_{p,p}(\mathbb{R}) \), \( A_2 \in M_{p,\ell}(\mathbb{R}) \), \( A_3 \in M_{\ell,p}(\mathbb{R}) \), \( A_4 \in M_{\ell,\ell}(\mathbb{R}) \), and \( B \in \text{GL}_{p-\ell}(\mathbb{R}) \). With this decomposition
\[ \left[ \begin{array}{cc} A_1 & 0 \\ A_3 & 0 \\ 0 & B \end{array} \right] \cdot \left[ \begin{array}{c} \text{Id}_p \\ X_1 \\ X_2 \end{array} \right] = \left[ \begin{array}{c} \text{Id}_p \\ (A_3 + A_4X_1)(A_1 + A_2X_1)^{-1}X_1 \\ BX_2(A_1 + A_2X_1)^{-1} \end{array} \right]. \]

Now by identifying \( M_{p-\ell,p}(\mathbb{R}) \) with \( \mathbb{R}^{(p-\ell)p} \) we can view \( \Omega_2 \) as a convex subset of \( \mathbb{P}(\mathbb{R}^{(p-\ell)p+1}) \). Let \( e \) be an extreme point of \( \Omega_2 \) in \( \mathbb{P}(\mathbb{R}^{(p-\ell)p+1}) \setminus \mathbb{P}(\mathbb{R}^{(p-\ell)p+1}) \). Fix a sequence of points \( y_n \in \Omega_2 \) which converges to \( e \) in \( \mathbb{P}(\mathbb{R}^{(p-\ell)p+1}) \).

Next fix some \( x_0 \in \Omega_1 \) and consider the sequence
\[ z_n = \left[ \begin{array}{c} \text{Id}_p \\ x_0 \\ y_n \end{array} \right] \in \Omega. \]

where we view \( x_0 \in M_{\ell,p}(\mathbb{R}) \) and \( y_n \in M_{p-\ell,p}(\mathbb{R}) \).

Since \( \Gamma \) acts cocompactly on \( \Omega \), there exist \( \gamma_n \in \Gamma \) and a compact subset \( K \) of \( \Omega \) such that
\[ \gamma_n^{-1}z_n \in K. \]
Suppose
\[ \gamma_n = \begin{bmatrix} A_1^{(n)} & A_2^{(n)} & 0 \\ A_3^{(n)} & A_4^{(n)} & 0 \\ 0 & 0 & B^{(n)} \end{bmatrix}. \]

Now let
\[ \text{GL}(\Omega_2) = \{ T \in \text{GL}(M_{p-\ell,p}(\mathbb{R})) : T(\Omega_2) = \Omega_2 \}. \]

Since \( \Omega_2 \subset M_{p-\ell,p}(\mathbb{R}) \) is a proper convex cone, the Hilbert metric \( H_{\Omega_2} \) is a complete \( \text{GL}(\Omega_2) \)-invariant metric on \( \Omega_2 \). Moreover, since \( \Omega = \Omega_1 + \Omega_2 \) we see that the linear map
\[ T_n(X) = B^{(n)} X (A_1^{(n)} + A_2^{(n)} x_0)^{-1} \]
is in \( \text{GL}(\Omega_2) \) for all \( n \geq 0 \), where we again view \( x_0 \in M_{p-\ell,p}(\mathbb{R}) \). So there exists \( R \geq 0 \) such that
\[ H_{\Omega_2}(y_n, B^{(n)} y_0 (A_1^{(n)} + A_2^{(n)} x_0)^{-1}) \leq R \]
for all \( n \geq 0 \). Since \( y_n \) converges to an extreme point of \( \Omega_2 \) we see that \([T_n] \in \mathcal{P}(\text{End}(M_{p-\ell,p}(\mathbb{R})))\) converges to some \( T_\infty \in \mathcal{P}(\text{End}(M_{p-\ell,p}(\mathbb{R}))) \) and \( \text{rank} T_\infty = 1 \) (see either Vey [Vey70] Lemma 4 or Theorem 7.4 above).

Now if \( \sigma_1^{(n)} \geq \cdots \geq \sigma_{p-\ell}^{(n)} \) are the singular values of \( B^{(n)} \) and \( \mu_1^{(n)} \geq \cdots \geq \mu_p^{(n)} \) are the singular values of \( (A_1^{(n)} + A_2^{(n)} x_0)^{-1} \) then \( T_n \) has singular values
\[ \{ \sigma_i^{(n)} \mu_j^{(n)} : 1 \leq i \leq p-\ell, 1 \leq j \leq p \}. \]
Then since \([T_n] \to T_\infty \) and \( \text{rank} T_\infty = 1 \) we must have
\[ \lim_{n \to \infty} \frac{\sigma_i^{(n)} \mu_j^{(n)}}{\sigma_i^{(n)} \mu_j^{(n)}} = \infty \]
for all \( 1 \leq i \leq p-\ell, 1 \leq j \leq p \) with \((i,j) \neq (1,1)\).

In particular,
\[ \lim_{n \to \infty} \frac{\mu_1^{(n)}}{\mu_2^{(n)}} = \infty. \]

So we will finish the proof by establishing the following:

**Lemma 9.16.**
\[ \lim_{n \to \infty} \sup \frac{\mu_1^{(n)}}{\mu_2^{(n)}} < \infty \]

**Proof.** Now view \( \Omega_1 \) as an open subset of \( \text{Gr}_\ell(V) \) where \( V = \text{Span}\{e_1, \ldots, e_{p-\ell}\} \).

By construction \( \Omega_1 \) is an \( \mathcal{R} \)-proper convex open subset of some affine chart of \( \text{Gr}_p(V) \). Thus \( K_{\Omega_1} \) is a proper metric and there exists \( R_1 \geq 0 \) such that
\[ K_{\Omega_1}(x_0, (A_3^{(n)} + A_4^{(n)} x_0)(A_1^{(n)} + A_2^{(n)} x_0)^{-1}) \leq R_1. \]

So the set
\[ \left\{ \begin{bmatrix} A_1^{(n)} & A_2^{(n)} \\ A_3^{(n)} & A_4^{(n)} \end{bmatrix} : n \in \mathbb{N} \right\} \subset \text{PGL}(V) \]
is relatively compact in $\text{PGL}(V)$. So we can pass to a subsequence and pick representatives such that

$$
\begin{pmatrix}
A_1^{(n)} & A_2^{(n)} \\
A_3^{(n)} & A_4^{(n)}
\end{pmatrix} \to
\begin{pmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{pmatrix}
$$

in $\text{GL}(V)$. Now we claim that $(A_1 + A_2x_n)$ is an invertible matrix. Suppose this is not the case. Then for each $n$ we can find a unit eigenvector $v_n \in \mathbb{C}^p$ such that

$$(A_1^{(n)} + A_2^{(n)}x_n)v_n \to 0$$

Since $(A_3^{(n)} + A_4^{(n)}x_0)(A_1^{(n)} + A_2^{(n)}x_0)^{-1}$ stays within a compact subset of $\Omega_2$, we must have that $(A_3^{(n)} + A_4^{(n)}x_0)v_n \to 0$. Then we can pass to a subsequence such that $v_n \to v$. We have

$$0 = \lim_{n \to \infty} \begin{pmatrix} A_1^{(n)} & A_2^{(n)} \\ A_3^{(n)} & A_4^{(n)} \end{pmatrix} \begin{pmatrix} v_n \\ x_0 v_n \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} v \\ q_0 v \end{pmatrix},$$

which contradicts the fact that $(A_1 + A_2q_0) \in \text{GL}_{p+\ell}(\mathbb{R})$.

So $(A_1 + A_2q_0)$ is an invertible matrix. But this implies that there exists $C > 0$ such that

$$\{\mu_i^{(n)}: 1 \leq i \leq p\} \subset [1/C, C]$$

which implies Equation (9.1). □

10. Unipotent subgroups

In this section we show that Case (2) of Theorem 8.1 is impossible.

**Theorem 10.1.** Suppose $\Omega \subset \text{Gr}_p(\mathbb{R}^{2p})$ is an open set which is bounded and convex in some affine chart. If $\Gamma \leq \text{Aut}(\Omega)$ is a discrete group which acts cocompactly on $\Omega$, then there does not exist a nontrivial abelian normal unipotent group $U \leq \text{Aut}(\Omega)$ such that $\Gamma \cap U$ is a cocompact lattice in $U$.

For the rest of the section suppose $\Omega \subset \text{Gr}_p(\mathbb{R}^{2p})$ and $\Gamma \leq \text{Aut}(\Omega)$ satisfy the hypothesis of Theorem 10.1. Assume for a contradiction that there exists a nontrivial abelian normal unipotent group $U \leq \text{Aut}(\Omega)$ such that $\Gamma \cap U$ is a cocompact lattice in $U$.

Since $\Gamma$ is finitely generated, by passing to a finite-index subgroup we can assume that $\Gamma$ is torsion-free. Then, since $\Gamma$ acts properly on $\Omega$, we see that $\Gamma$ acts freely on $\Omega$. Then, using the fact that $\Gamma \setminus \Omega$ is compact, we see that

$$\inf_{\gamma \in \Gamma, x \in \Omega} K_{\Omega}(\gamma x, x) > 0. \quad (10.1)$$

The basic idea of the following argument is that if $u \in U \cap \Gamma$, then the translation distance

$$\inf_{x \in \Omega} K_{\Omega}(ux, x)$$

should be zero, which then implies that $U \cap \Gamma = 1$. This approach is motivated by Lemma 2.8 in [Ben06] and Proposition 2.13 in [CLT15].
The group $\wedge^p U \leq \text{PGL}(\wedge^p \mathbb{R}^{2p})$ is also unipotent so the set

$$E_1 = \{ v \in \mathbb{P}(\wedge^p \mathbb{R}^{2p}) : (\wedge^p u)v = v \text{ for all } u \in U \}$$

is non-empty. Note that $\wedge^p U$ can be conjugated so that it is upper triangular. Since $U \cap \Gamma$ is a lattice in $U$, we can choose $u_0 \in U \cap \Gamma$ such that its Jordan decomposition is generic among elements of $U$, that is to say

$$E_1 = \{ v \in \mathbb{P}(\wedge^p \mathbb{R}^{2p}) : (\wedge^p u_0)v = v \}.$$

Then with the notation of Proposition 7.9

$$E^+(\wedge^p u_0) \subset E_1$$

and by Corollary 7.10 there exists an $\mathcal{R}$-extreme point $e \in E^+(\wedge^p u_0) \cap \partial \Omega$.

Now suppose that $\Omega$ is a bounded convex open set in the affine chart

$$\mathcal{M} = \left\{ \begin{bmatrix} \text{Id}_p & X \end{bmatrix} : X \in M_{p,p}(\mathbb{R}) \right\}.$$

Without loss of generality we can assume $e = 0$ in this affine chart. Then by Corollary 7.11 there exist $\gamma_n \in \Gamma$ and $t_n \to \infty$ such that

$$\varphi = \lim_{n \to \infty} \begin{bmatrix} \text{Id}_p & 0 \\ 0 & e^{t_n} \text{Id}_p \end{bmatrix} \gamma_n$$

exists in $\text{PGL}_{2p}(\mathbb{R})$ and $\varphi \Omega \subset \mathcal{M}$ is an $\mathcal{R}$-proper convex open cone based at 0. In particular, $\text{Aut}(\varphi \Omega)$ contains the one-parameter subgroup

$$a_t := \begin{bmatrix} \text{Id}_p & 0 \\ 0 & e^t \text{Id}_p \end{bmatrix}.$$

Now if

$$\varphi_n := \begin{bmatrix} \text{Id}_p & 0 \\ 0 & e^{t_n} \text{Id}_p \end{bmatrix} \gamma_n$$

then

$$\varphi_n^{-1}(e) = \gamma_n^{-1}(e) \in \gamma_n^{-1}E_1 \cap \gamma_n^{-1}E^+(\wedge^p u_0) = E_1 \cap E^+(\wedge^p \gamma_n^{-1}u_0 \gamma_n)$$

so

$$\varphi_n^{-1}(e) \in E_1 \cap \left( \cup_{u \in U} E^+(\wedge^p u) \right)$$

so sending $n \to \infty$ we see that

$$\varphi^{-1}(e) \in E_1 \cap \overline{\cup_{u \in U} E^+(\wedge^p u)}.$$

And thus

$$e \in \varphi(E_1) \cap \overline{\cup_{u \in \varphi \varphi^{-1}E^+(\wedge^p u)}}.$$

In particular, since $e = \text{Span}\{e_1, \ldots, e_p\} \subset \varphi(E_1)$, we have

$$\varphi U \varphi^{-1} \leq \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} : A, B, C \in M_{p,p}(\mathbb{R}) \right\}.$$

**Lemma 10.2.** If

$$\begin{bmatrix} \text{Id}_p & X \\ 0 & \text{Id}_p \end{bmatrix} \in \varphi U \varphi^{-1}$$

then $X = 0$. 

Proof. Suppose for a contradiction that there exists $u = \begin{bmatrix} \text{Id}_p & X \\ 0 & \text{Id}_p \end{bmatrix} \in \varphi U \varphi^{-1}$ with $X \neq 0$. We claim there exist $n_k \to \infty$ and $\gamma_k \in \varphi(\Gamma \cap U)\varphi^{-1}$ such that $\gamma_k^{-1} u^{n_k} \to \text{Id}_{2p}$. Indeed, consider the group $\Lambda := \langle \varphi(\Gamma \cap U)\varphi^{-1}, u \rangle$. If $\Lambda$ is discrete, some power of $u$ belongs to $\varphi(\Gamma \cap U)\varphi^{-1}$, in which case the claim obviously holds. If $\Lambda$ is not discrete, we can find $\gamma_k^{-1} u^{n_k} = \lambda_k \in \Lambda$ such that $\lambda_k \to \text{Id}_{2p}$. Further it is clear that $n_k \to \infty$, for otherwise $\lambda_k$ lie in a union of finitely many translates of $\varphi(\Gamma \cap U)\varphi^{-1}$, which is a discrete set. This proves the claim.

So let $\gamma_k \in \varphi(\Gamma \cap U)\varphi^{-1}$ and $n_k \to \infty$ such that $\gamma_k^{-1} u^{n_k} \to \text{Id}_{2p}$. By picking representatives correctly we can assume that

$$\gamma_k = \begin{bmatrix} A_k & B_k \\ 0 & C_k \end{bmatrix}$$

and

$$\begin{bmatrix} A_k^{-1} & -A_k^{-1} B_k \\ 0 & C_k^{-1} \end{bmatrix} \begin{bmatrix} \text{Id}_p & n_k X \\ 0 & \text{Id}_p \end{bmatrix} = \begin{bmatrix} A_k^{-1} & n_k A_k^{-1} X - A_k^{-1} B_k \\ 0 & C_k^{-1} \end{bmatrix} \to \begin{bmatrix} \text{Id}_p & 0 \\ 0 & \text{Id}_p \end{bmatrix}$$

in $\text{GL}_{2p}(\mathbb{R})$. So $A_k \to \text{Id}_p$ and $C_k \to \text{Id}_p$. But then there exist $t_k \to \infty$ such that $a_{t_k} \gamma_k a_{-t_k} \to \text{Id}_{2p}$. But then for any $p \in \varphi \Omega$

$$\lim_{k \to \infty} K_{\varphi \Omega}(a_{t_k} a_{-t_k} p, a_{-t_k} p) = \lim_{k \to \infty} K_{\varphi \Omega}(a_{t_k} \gamma_k a_{-t_k} p, p) = 0$$

which contradicts Equation (10.1). □

Lemma 10.3.

$$\varphi U \varphi^{-1} \leq \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \text{GL}_p(\mathbb{R}) \right\}.$$

Proof. Suppose for a contradiction that there exists $u = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \varphi U \varphi^{-1}$ with $C \neq 0$.

Then

$$u' = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \lim_{t \to \infty} a_t u a_{-t} \in \varphi U \varphi^{-1}$$

and so

$$\begin{bmatrix} \text{Id}_p & A^{-1} C \\ 0 & \text{Id}_p \end{bmatrix} = (u')^{-1} u \in \varphi U \varphi^{-1}$$

which we just showed is impossible. □

Lemma 10.4. If $u \in \varphi U \varphi^{-1}$ is non-trivial then

$$E^+ (\wedge^p u) \cap \text{Gr}_p(\mathbb{R}^{2p}) \subset \text{Gr}_p(\mathbb{R}^{2p}) \setminus M.$$
Since both $B$ and $A$ are unipotent, for a generic $X \in M_{p,p}(\mathbb{R})$ we have

$$\lim_{m \to \infty} \|B^m X A^{-m}\| = \infty.$$  

Which implies that $E^+ (\wedge^P u) \cap \text{Gr}_p(\mathbb{R}^{2p}) \subset \text{Gr}_p(\mathbb{R}^{2p}) \setminus \mathbb{M}$.

Now we have a contradiction because

$$e \in \text{Gr}_p(\mathbb{R}^{2p}) \cap \bigcup_{u \in \mathcal{U}} u^{-1} E^+ (\wedge^P u) \subset \text{Gr}_p(\mathbb{R}^{2p}) \setminus \mathbb{M}$$

and $e \in \mathbb{M}$.

11. When $p = 2$

In this section we show that Cases (3) and (4) of Theorem 8.1 are impossible.

**Theorem 11.1.** Suppose $\Omega \subset \text{Gr}_2(\mathbb{R}^4)$ is a bounded convex open subset of some affine chart of $\text{Gr}_2(\mathbb{R}^4)$ and there exists a discrete group $\Gamma \leq \text{Aut}(\Omega)$ such that $\Gamma \setminus \Omega$ is compact. Then the connected component of the identity in $\text{Aut}(\Omega)$ is a simple Lie group with trivial center that acts transitively on $\Omega$.

For the rest of the section let $\Omega \subset \text{Gr}_2(\mathbb{R}^4)$ and $\Gamma \leq \text{Aut}(\Omega)$ be as in the hypothesis of Theorem 11.1. As in Section 8, let $G := \text{Aut}(\Omega)$ and let $G^0$ be the connected component of the identity of $G$.

Define the subgroups

$$G_1 := \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \text{SL}_2(\mathbb{R}) \right\}$$

and

$$G_2 := \left\{ \begin{bmatrix} a \text{Id}_2 & b \text{Id}_2 \\ c \text{Id}_2 & d \text{Id}_2 \end{bmatrix} : ad - bc = 1 \right\}.$$

By Theorem 8.1 we may assume that either

1. $G^0$ is a simple Lie group with trivial center that acts transitively on $\Omega$, or
2. there exists a cocompact lattice $\Lambda \leq G_2$ such that $G_1 \times \Lambda$ has finite-index in $\text{Aut}(\Omega)$, or
3. $G_1 \times G_2$ has finite-index in $\text{Aut}(\Omega)$ and acts transitively on $\Omega$.

**Lemma 11.2.** With the notation above, Case (2) cannot occur.

**Proof.** Suppose not. Then there exists a cocompact lattice $\Lambda \leq G_2$ such that $G_1 \times \Lambda$ has finite-index in $\text{Aut}(\Omega)$. By possibly changing $\Gamma$, we may also assume that $\Gamma = \Gamma_1 \times \Lambda$ for some cocompact lattice $\Gamma_1 \leq G_1$.

For a subgroup $H \leq \text{Aut}(\Omega)$ let $\mathcal{L}(H)$ denote the set of points $x \in \partial \Omega$ where there exist some $y \in \Omega$ and sequence $h_n \in H$ such that $h_n y \to x$. Recall that $\text{Ext}_R(\Omega) \subset \partial \Omega$ is the set of $R$-extreme points of $\Omega$. Then define

$$\text{Ext}_R(H) := \mathcal{L}(H) \cap \text{Ext}_R(\Omega).$$

Let $e_1, \ldots, e_4$ be the standard basis of $\mathbb{R}^4$. Then a direct computation (using Part (4) of Theorem 7.4) shows that

$$\text{Ext}_R(G_1) = \{ [(\alpha e_1 + \beta e_2) \wedge (\alpha e_3 + \beta e_4)] : \alpha, \beta \in \mathbb{R} \}$$

and

$$\text{Ext}_R(\Lambda) \subset \{ [(\alpha e_1 + \beta e_3) \wedge (\alpha e_2 + \beta e_4)] : \alpha, \beta \in \mathbb{R} \}.$$
This description implies that $\text{Ext}_R(G_1)$ and $\text{Ext}_R(\Lambda)$ are disjoint and $\Gamma$-invariant sets. Moreover since $\Lambda \leq G_2$ is a cocompact lattice there exists some $\lambda \in \Lambda$ such that $\wedge^2 \lambda$ has a unique eigenvalue of maximum absolute value (see [Pra94]). Then part (4) of Theorem 7.4 implies that $\text{Ext}_R(\Lambda) \neq \emptyset$. So suppose that $e \in \text{Ext}_R(\Lambda)$.

Now up to a projective isomorphism we can assume that $\Omega$ is a convex subset of the affine chart

$$\mathbb{M} = \left\{ \begin{bmatrix} \text{Id}_2 & 0 \\ X & 0 \end{bmatrix} : X \in M_{2,2}(\mathbb{R}) \right\}.$$ 

and $e = \begin{bmatrix} \text{Id}_2 & 0 \\ 0 & e^t \end{bmatrix} \in \partial \Omega$. Then by Corollary 7.11 there exist $\gamma_n \in \Gamma$ and $t_n \to \infty$ such that

$$\varphi_n = \lim_{n \to \infty} \begin{bmatrix} \text{Id}_2 & 0 \\ 0 & e^{t_n} \text{Id}_2 \end{bmatrix} \gamma_n$$

exists in $\text{PGL}_4(\mathbb{R})$ and $\varphi_n(\Omega) = \mathcal{T}C_e\Omega$. In particular, $\Omega$ is invariant under the one-parameter group

$$\varphi^{-1} \left\{ \begin{bmatrix} \text{Id}_p & 0 \\ 0 & e^t \text{Id}_p \end{bmatrix} : t \in \mathbb{R} \right\} \varphi.$$ 

This implies that $\varphi^{-1}(e) \in \text{Ext}_R(G_1)$. But

$$\gamma_n^{-1} \begin{bmatrix} \text{Id}_2 & 0 \\ 0 & e^{-t_n} \text{Id}_2 \end{bmatrix} e = \gamma_n^{-1} e \subset \text{Ext}_R(\Lambda)$$

and thus

$$\varphi^{-1}(e) \in \text{Ext}_R(G_1) \cap \text{Ext}_R(\Lambda).$$

This is a contradiction. $\square$

We rule out Case (3) above by proving the following:

**Lemma 11.3.** With the notation above, Case (3) cannot occur.

**Proof.** Suppose not, then $G_1 \times G_2$ has finite-index in $\text{Aut}(\Omega)$. By possibly changing $\Gamma$, we may assume that $\Gamma = \Gamma_1 \times \Gamma_2$ for some cocompact lattices $\Gamma_1 \leq G_1$ and $\Gamma_2 \leq G_2$.

Define the subgroups

$$K_1 = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \text{SO}(2) \right\}$$

and

$$K_2 = \left\{ \begin{bmatrix} a \text{Id}_2 & b \text{Id}_2 \\ c \text{Id}_2 & d \text{Id}_2 \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SO}(2) \right\}.$$ 

Then $K_1 \times K_2 \leq G_1 \times G_2$ is a maximal compact connected subgroup. Moreover the action of $K_1 \times K_2$ on $\text{Gr}_2(\mathbb{R}^4)$ has no fixed points.

Next let $K_x \leq \text{Aut}(\Omega)$ be the connected component of the stabilizer of some $x \in \Omega$. Since $\text{Aut}(\Omega)$ acts properly on $\Omega$ (see Proposition 4.7), $K_x$ is a compact subgroup. Moreover, since $G^0 = G_1 \times G_2$, we see that $K_x \leq G_1 \times G_2$. Thus, since maximal compact subgroups are conjugate in semisimple Lie groups, there exists some $g \in G_1 \times G_2$ such that

$$gK_xg^{-1} \leq K_1 \times K_2.$$
RIGIDITY OF CONVEX DIVISIBLE DOMAINS IN FLAG MANIFOLDS

But \( \dim(K_1 \times K_2) = 2 \). Moreover
\[
6 - \dim(K_x) = \dim(G_1 \times G_2/K_x) \leq \dim(\Omega) = 4
\]
so \( \dim K_x \geq 2 \). Thus \( gK_xg^{-1} = K_1 \times K_2 \). This contradicts the fact that \( K_1 \times K_2 \) has no fixed points in \( \text{Gr}_2(\mathbb{R}^4) \). \( \square \)

12. Finishing the proof of Theorem 1.3

Theorem 8.1, Theorem 9.2, Theorem 10.1, and Theorem 11.1 reduce the proof of Theorem 1.3 to the following:

Theorem 12.1. Suppose \( p > 1 \) and \( \Omega \subset \text{Gr}_p(\mathbb{R}^{2p}) \) is a bounded convex open subset of some affine chart of \( \text{Gr}_p(\mathbb{R}^{2p}) \). If the connected component of the identity of \( \text{Aut}(\Omega) \) is a simple Lie group with trivial center which acts transitively on \( \Omega \), then \( \Omega \) is projectively isomorphic to \( B_{p,p} \).

For the rest of the section suppose that \( \Omega \) satisfies the hypothesis of Theorem 12.1. As in Section 8 let \( G := \text{Aut}(\Omega) \) and let \( G^0 \) be the connected component of the identity of \( G \). Also let \( e_1, \ldots, e_{2p} \in \mathbb{R}^{2p} \) be the standard basis.

Throughout the argument we will replace \( \Omega \) with translates \( g\Omega \) for some \( g \in \text{PGL}_{2p}(\mathbb{R}) \). This will have the effect of replacing \( G \) with \( gGg^{-1} \).

Fix some \( x_0 \in \Omega \) and let \( K \leq G^0 \) be the identity component of the stabilizer of \( x_0 \). By Remark 8.10, \( K \) is a finite-index subgroup of some maximal compact subgroup of \( G^0 \). Moreover, since \( K \) is compact, by translating \( \Omega \) we may assume that \( K \leq \text{PSO}(2p) \). Then since \( \text{PSO}(2p) \) acts transitively on \( \text{Gr}_p(\mathbb{R}^{2p}) \) we can translate \( \Omega \) and assume that \( x_0 = [e_1 \wedge \cdots \wedge e_p] \). Then using the fact that \( K \) is connected
\[
K \leq \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \text{SO}(p) \right\}.
\]
In particular, \( \dim(K) \leq p(p-1) \).

Now let \( \text{rank}_\mathbb{R}(G^0) \) be the real rank of \( G^0 \).

Lemma 12.2. With the notation above, \( \text{rank}_\mathbb{R}(G^0) \geq p \).

Proof. Using the Cartan decomposition, we see there exists a connected abelian group \( A \leq G^0 \) such that \( \dim(A) = \text{rank}_\mathbb{R}(G^0) \) and \( KAK = G^0 \). In particular, in the matrix model of \( \text{Gr}_p(\mathbb{R}^{2p}) \)
\[
\Omega = KAK \cdot \begin{bmatrix} \text{Id}_p & 0 \\ 0 & \text{Id}_p \end{bmatrix}^t = KA \cdot \begin{bmatrix} \text{Id}_p & 0 \\ 0 & \text{Id}_p \end{bmatrix}^t.
\]
Thus we must have
\[
(12.1) \quad \text{dim}(K) + \text{dim}(A) \geq \text{dim}(\Omega) = p^2.
\]
Since \( \dim(K) \leq p(p-1) \) we then have
\[
\text{rank}_\mathbb{R}(G^0) = \text{dim}(A) \geq p.
\]
\( \square \)

Lemma 12.3. With the notation above, \( G^0 \) is isomorphic to \( \text{PSO}(p,p) \).
Proof. Now
\[
\dim \left( \frac{G^0}{K} \right) = \dim(\Omega) = p^2
\]
and
\[
\text{rank}_R(G^0) \geq p.
\]
In particular
\[
\text{rank}_R(G^0) \geq \sqrt{\dim \left( \frac{G^0}{K} \right)}.
\]
The only two simple Lie groups of non-compact type and with trivial center with this property are \(\text{PSL}_{d+1}(\mathbb{R})\) for \(d \geq 3\) and \(\text{PSO}(d,d)\) for \(d \geq 2\) (see the classification of simple Lie groups in [Hel78, Chapter X]).

If \(G^0\) is isomorphic to \(\text{PSL}_{d+1}(\mathbb{R})\) then \(K\) is isomorphic to \(\text{PSO}(d+1)\). In particular \(K\) is a simple Lie group and
\[
\dim K = \frac{d(d+1)}{2}
\]
Next consider the natural projections,
\[
\pi_1, \pi_2 : K \to \text{PSO}(p)
\]
given by
\[
\pi_1 \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = A \text{ and } \pi_2 \left( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right) = B
\]
Now since \(K\) is simple either \((\pi_1 \times \pi_2) : K \to \text{PSO}(p) \times \text{PSO}(p)\) is trivial or injective. But
\[
\ker(\pi_1 \times \pi_2) \leq \left\{ \text{Id}_{2p}, \begin{bmatrix} \text{Id}_p & -\text{Id}_p \\ \text{Id}_p & \text{Id}_p \end{bmatrix}, \begin{bmatrix} -\text{Id}_p & \text{Id}_p \\ \text{Id}_p & \text{Id}_p \end{bmatrix} \right\}
\]
so \(\pi_1 \times \pi_2\) is injective. Thus at least one \(\pi_i\) has non-trivial image. Then by the simplicity of \(K\) we see that \(K \cong \pi_i(K) \leq \text{PSO}(p)\). So
\[
\dim K \leq \frac{p(p-1)}{2}
\]
and so
\[
d(d+1) \leq p(p-1).
\]
Thus \(d \leq p+1\). But then we have a contradiction, because by Equation (12.1), we have
\[
p^2 \leq \text{rank}(G^0) + \dim(K) \leq d + \frac{p(p-1)}{2} \leq p+1 + \frac{p(p-1)}{2} = \frac{p^2 + p}{2} + 1
\]
which is only true when \(p = 2\). Then \(d = p+1 = 3\), but
\[
\dim \text{PSL}_4(\mathbb{R})/\text{PSO}(4) = 9 \neq 4 = \dim \Omega
\]
so this case is impossible.

Thus we must have that \(G^0\) is isomorphic to \(\text{PSO}(p,p)\). \(\square\)

Now the inclusion \(G^0 \leq \text{PGL}_{2p}(\mathbb{R})\) induces a representation \(\phi : \text{PSO}(p,p) \to \text{PGL}_{2p}(\mathbb{R})\). Notice that replacing \(\Omega\) with \(g\Omega\) for some \(g \in \text{PGL}_{2p}(\mathbb{R})\) has the effect of replacing \(\phi\) with \(\text{Ad}(g) \circ \phi\).
At this point there is a number of ways to deduce that this representation is conjugate to the standard inclusion, but we will use the representation theory of $\text{SO}(2p, \mathbb{C})$ because it is appears explicitly in standard references (for instance [FH91]).

Now since $K$ has finite-index in a maximal compact subgroup of $G^0 \cong \text{PSO}(p, p)$ and
\[ K \leq \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \text{SO}(p) \right\}, \]
so we see that
\[ K = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A, B \in \text{SO}(p) \right\}. \]
Then since maximal compact subgroups are conjugate in $G^0$ we may translate $\Omega$ to assume that
\[ \phi(P(\text{SO}(p) \times \text{SO}(p))) = P(\text{SO}(p) \times \text{SO}(p)). \]
Now if $K_1 = P(\text{SO}(p) \times \{\text{Id}_p\})$ and $K_2 = P(\text{SO}(p) \times \{\text{Id}_p\})$ then, using the simplicity of $K_1, K_2$ and the fact that $\phi(K_1), \phi(K_2)$ commute, we see that
\[ \{\phi(K_1), \phi(K_2)\} = \{K_1, K_2\}. \]
So by translating $\Omega$ we may assume that $\phi(K_1) = K_1$ and $\phi(K_2) = K_2$. Now each $K_i$ is isomorphic to $\text{SO}(p)$.

It is well-known that any automorphism of $\text{SO}(p)$ is given by conjugation by some element of $\text{O}(p)$ (e.g. because such an automorphism is determined by the automorphism of the Dynkin diagram). Therefore by translating $\Omega$, we can assume that $\phi(k) = k$ for all $k \in K_1 \cup K_2$.

Now let $d(\phi) : \mathfrak{so}(p, p) \to \mathfrak{sl}_{2p}(\mathbb{R})$ be the corresponding Lie algebra representation. We can complexify to obtain a representation $d(\phi) : \mathfrak{so}(2p, \mathbb{C}) \to \mathfrak{sl}_{2p}(\mathbb{C})$. But then by the classification of irreducible representations of $\text{SO}(2p, \mathbb{C})$ (see for instance [FH91, Chapter 19]) we see that there exists $g \in \text{SL}_{2p}(\mathbb{C})$ such that
\[ \text{Ad}(g)d(\phi) = \iota \]
where $\iota : \mathfrak{so}(2p, \mathbb{C}) \to \mathfrak{sl}_{2p}(\mathbb{R})$ is the standard inclusion representation. Since
\[ g^{-1} \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} g = d(\phi) \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \]
for all $X_1, X_2 \in \mathfrak{so}(p)$ it is easy to see that
\[ g = \begin{pmatrix} \alpha \text{Id}_p & 0 \\ 0 & \alpha^{-1} \text{Id}_p \end{pmatrix} \]
for some $\alpha \in \mathbb{C}^\times$. Now
\[ g \begin{pmatrix} A & B \\ C & D \end{pmatrix} g^{-1} = \begin{pmatrix} A & \alpha^2 B \\ \alpha^{-2} C & D \end{pmatrix} \]
and $gd(\phi)(\mathfrak{so}(p, p))g^{-1} = \mathfrak{so}(p, p)$. So $\alpha^2 \in \mathbb{R}$. So either $\alpha \in \mathbb{R}^\times$ or $\alpha = \lambda i$ for some $\lambda \in \mathbb{R}^\times$. In the latter case, we also have
\[ \text{Ad}(-ig)d(\phi) = \text{Ad}(g)d(\phi) = \iota. \]
So by possibly replacing $g$ by $-ig$ we can assume that $g \in \text{SL}_{2p}(\mathbb{R})$. 

Then if we replace $\Omega$ by $g\Omega$, then $\phi : \text{PSO}(p,p) \hookrightarrow \text{PGL}_{2p}(\mathbb{R})$ is the standard inclusion representation and so $G^0 = \text{PSO}(p,p)$.

Finally
\[ \Omega = G^0 \cdot x_0 = \text{PSO}(p,p) \cdot \begin{bmatrix} \text{Id}_p \\ 0 \end{bmatrix} = \mathcal{B}_{p,p} \]

and so Theorem 12.1 is proven.

**Part 4. Appendices**

**Appendix A. Proof of Theorem 4.5**

In this section we prove that (1) implies (2) in Theorem 4.5:

**Theorem A.1.** Suppose $M \subset \text{Grp}(\mathbb{R}^{p+q})$ is an affine chart and $\Omega \subset M$ is an open convex set. If $\Omega$ is $\mathcal{R}$-proper, then $K_\Omega$ is a complete length metric on $\Omega$.

We will use some basic properties of the Hilbert metric $H_C$ on a convex set $C \subset \mathbb{R}^d$. In particular we will use:

1. (equivariance) If $A \in \text{Aff}(\mathbb{R}^d)$ then $H_{AC}(Ax,Ay) = H_C(x,y)$.
2. (properness) If $x \in \partial C$ and $x_n \in C$ is a sequence with $x_n \to x$, then $H_C(x_0,x_n) \to \infty$.
3. (completeness) If $C$ contains no affine lines then $H_C$ is a complete metric,
4. If $C = \mathbb{R}^d \times C'$, then $H_C((x_1,y_1),(x_2,y_2)) = H_{C'}(y_1,y_2)$.

All these properties follow immediately from the cross ratio definition of the Hilbert metric.

**Proof.** Identify $M$ with the set of $q$-by-$p$ matrices and let $M_1 \subset M$ be the subset of rank-one matrices. Define a function $\delta_\Omega : \Omega \times M_1 \to \mathbb{R}_{\geq 0}$ by
\[ \delta_\Omega(x;v) = \inf \{ \|y-x\| : y \in \partial \Omega \cap (x + \mathbb{R}v) \}. \]

Since $\Omega$ is $\mathcal{R}$-proper, we must have that $\delta_\Omega(x;v) < \infty$ for all $x \in \Omega$ and $v \in M_1$. Moreover, since $\Omega$ is convex, $\delta_\Omega$ is a continuous function.

We will first show that $K_\Omega$ is a metric, using Proposition 4.2 we only need show that $K_\Omega(x,y) > 0$ for $x,y \in \Omega$ distinct. Now we can find $\epsilon > 0$ such that the closed Euclidean ball
\[ B_\epsilon(x) = \{ z \in M : \|x-z\| \leq \epsilon \} \]

is contained in $\Omega$ but $y \notin B_\epsilon(x)$. Since $\delta_\Omega$ is continuous, there exists $M > 0$ such that
\[ \delta_\Omega(z;v) \leq M \]

for all $z \in B_\epsilon(x)$ and $v \in M_1$.

We claim that if $[z_1,z_2] \subset B_\epsilon(x)$, then $\rho_\Omega(z_1,z_2) \geq \|z_1-z_2\|/(\epsilon + M)$. If $z_2 - z_1 \notin M_1$, then $\rho_\Omega(z_1,z_2) = \infty$. So we may assume that $z_2 - z_1 \in M_1$. Then
let \( (a, b) = \overline{z_1z_2} \cap \Omega \) labeled such that \( a, z_1, z_2, b \) is the ordering along the line. By relabeling we may assume that \( \|a - z_1\| = \delta_\Omega(z_1, z_2) \leq M \). Then

\[
\rho_\Omega(z_1, z_2) = \left\lfloor \log \frac{\|z_1 - a\| \|z_2 - b\|}{\|z_1 - b\| \|z_2 - a\|} \right\rfloor \geq \log \frac{\|z_2 - a\|}{\|z_1 - a\|} = \int_{\|z_1 - a\|}^{\|z_2 - a\|} \frac{dt}{t} \geq \frac{1}{\|z_2 - a\|} (\|z_2 - a\| - \|z_1 - a\|).
\]

Since \( z_1, z_2, a \) are all collinear and \( \|z_1 - z_2\| \leq \epsilon \) we then have

\[
\rho_\Omega(z_1, z_2) \geq \frac{1}{M + \epsilon} \|z_1 - z_2\|.
\]

Now we wish to show that \( K_\Omega(x, y) > 0 \). We claim that

\[
\rho_\Omega(x, a_1) + \sum_{i=1}^{n-1} \rho_\Omega(a_i, a_{i+1}) + \rho_\Omega(a_n, y) \geq \frac{\epsilon}{M + \epsilon},
\]

for any \( a_1, \ldots, a_n \in \Omega \). This will imply that \( d_\Omega(x, y) > 0 \). Now by definition if \( a, b \in M \) and \( c \in [a, b] \) then

\[
\rho_\Omega(a, b) + \rho_\Omega(b, c) = \rho_\Omega(a, c).
\]

So without loss of generality there exists \( 1 \leq \ell < n \) such that \( a_1, \ldots, a_\ell \in B_\epsilon(x) \) and \( a_{\ell+1} \in \partial B_\epsilon(x) \). Then by the above calculation

\[
\rho_\Omega(x, a_1) + \ell \rho_\Omega(a_i, a_{i+1}) \geq \frac{1}{M + \epsilon} (\|x - a_1\| + \ell \|a_i - a_{i+1}\|) \geq \frac{\epsilon}{M + \epsilon}.
\]

This shows that \( K_\Omega \) is a metric.

We will next show that \( K_\Omega \) is a length metric. This follows from the fact that if \( x, y \in \Omega \) and \( x - y \in M_1 \), then

\[
\rho_\Omega(x, y) = \rho_\Omega(x, z) + \rho_\Omega(z, y)
\]

for any \( z \in [x, y] \). Thus when \( x - y \in M_1 \), there is a curve of length at most \( \rho_\Omega(x, y) \) joining \( x \) to \( y \). Then by definition for any \( x, y \in \Omega \) there exists a sequence of curves \( \sigma_n \) joining \( x \) to \( y \) and whose length converges to \( K_\Omega(x, y) \).

Next we show that \( K_\Omega \) is proper, that is for any \( x_0 \in \Omega \) and \( R \geq 0 \) the closed metric ball \( B = \{ x \in \Omega : K_\Omega(x, x_0) \leq R \} \) is compact. Let \( x_n \in B \) be a subsequence, we will show that a subsequence of \( x_n \) converges in \( B \). By passing to a subsequence we can suppose that \( x_n \to x \in M \) or \( x_n \to \infty \) (that is, \( x_n \) leaves every compact subset of \( M \)).

First suppose that \( x_n \to x \in M \). If \( x \in \Omega \), then \( x \in B \) by part (5) of Proposition 4.2. Otherwise \( x \in \partial \Omega \). Let \( H_\Omega \) be the Hilbert metric on \( \Omega \), then \( H_\Omega \leq K_\Omega \) by Kobayashi’s construction of the Hilbert metric (described in Subsection 2.2). So

\[
K_\Omega(x_0, x_n) \geq H_\Omega(x_0, x_n) \to \infty
\]

which is a contradiction.

Finally suppose that the sequence \( x_n \) leaves every compact subset of \( \Omega \). If \( \Omega \) contains no affine lines, then \( H_\Omega \) is a proper metric and so

\[
K_\Omega(x_0, x_n) \geq H_\Omega(x_0, x_n) \to \infty.
\]

If \( \Omega \) is not proper, then we can identify \( M \) with \( \mathbb{R}^D \) where \( D = pq \) and find an affine map \( \Phi \in \text{Aff}(\mathbb{R}^D) \) such that \( \Phi \Omega = \mathbb{R}^d \times \Omega' \) where \( \Omega' \) is a proper convex set and
d \leq D$. Notice that $H_\Omega(z_1, z_2) = H_{\Phi\Omega}(\Phi z_1, \Phi z_2)$ for all $z_1, z_2 \in \Omega$, but the metrics $K_{\Phi\Omega}$ and $K_\Omega$ have no clear relationship because $\Phi$ will in general not preserve the rank-one lines. Since $\Omega$ is $\mathcal{R}$-proper we must have that $d < D$. Let $\pi : \mathbb{R}^D \to \mathbb{R}^{D-d}$ be the projection onto the second factor. Next let $\sigma_n : [0, 1] \to \Omega$ be a curve joining $x_0$ to $x_n$ with $K_\Omega$-length less than $R + \epsilon$.

We claim that the set $\{\pi(\Phi\sigma_n(t)) : n \in \mathbb{N}, t \in [0, 1]\}$ is a compact subset of $\Omega'$. This follows from the fact that

$$R + \epsilon \geq K_\Omega(x_0, \sigma_n(t)) \geq H_\Omega(x_0, \sigma_n(t)) = H_{\Phi\Omega}(\Phi x_0, \Phi\sigma_n(t)) = H_{\Omega'}(\pi(\Phi x_0), \pi(\Phi\sigma_n(t)))$$

and the fact that $H_{\Omega'}$ is a proper metric on $\Omega'$. So if $x_n = \Phi^{-1}(y_n, z_n)$, we must have $y_n \to \infty$. But then notice that

$$\delta_\Omega(x + a; v) = \delta_\Omega(x; v)$$

for all $a \in \Phi^{-1}(\mathbb{R}^d \times \{0\})$ and $v \in \mathcal{M}_1$. And so there exists $M \geq 0$ such that

$$\delta_\Omega(\sigma_n(t); v) \leq M$$

for all $n \in \mathbb{N}$, all $t \in [0, 1]$, and $v \in \mathcal{M}_1$. But then arguing as before we see that

$$\text{length}(\sigma_n) \geq \frac{1}{M} \|x_0 - x_n\|.$$

Since $x_n$ leaves every compact subset of $\Omega$ and $\text{length}(\sigma_n) < R + \epsilon$ we have a contradiction.

Finally we observe that $K_\Omega$ is a complete metric on $\Omega$. If $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\Omega, K_\Omega)$, then we can pass to a subsequence such that

$$\sum_{n=1}^\infty \|x_n - x_{n+1}\| = R < \infty.$$

But then $x_n \in \{x \in \Omega : K_\Omega(x_1, x) \leq R\}$ which is a compact subset of $\Omega$. \qed

**APPENDIX B. PROOF OF THEOREM 5.1**

In this section we prove Theorem [5.1]

**Theorem B.1.** Let $\mathcal{M}$ be an affine chart of $\text{Gr}_p(\mathbb{R}^{p+q})$ and suppose $\Omega_n \subset \mathcal{M}$ is a sequence of $\mathcal{R}$-proper convex open sets converging to an $\mathcal{R}$-proper convex open set $\Omega \subset \mathcal{M}$ in the local Hausdorff topology. Then

$$K_\Omega(x, y) = \lim_{n \to \infty} K_{\Omega_n}(x, y)$$

for all $x, y \in \Omega$ uniformly on compact sets of $\Omega \times \Omega$.

It will be helpful to introduce an infinitesimal version of $\rho_\Omega$. As in the proof of Theorem [A.1] identify $\mathcal{M}$ with the vector space of $q$-by-$p$ matrices and let $\mathcal{M}_1 \subset \mathcal{M}$ be the space of rank one matrices. Next for a $\mathcal{R}$-proper convex open set $\Omega \subset \mathcal{M}$, define a function $k_\Omega : \Omega \times \mathcal{M}_1 \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$k_\Omega(x; v) = \frac{1}{t^+} + \frac{1}{t^-}$$

where $t^+, t^- \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfy $x + t^+ v, x + t^-(v) \in \partial \Omega$ and we define $1/\infty = 0$. Notice, that by definition $k_\Omega(x; \lambda v) = |\lambda| k_\Omega(x; v)$ for any $\lambda \in \mathbb{R}$.
Now if \( x, x + tv \in \Omega, \ v \in \mathbb{M}_1, \) and \( t > 0, \) then it is easy to show that
\[
\rho_\Omega(x, x + tv) = \int_0^t k_\Omega(x + sv; v)ds. \tag{B.1}
\]

The following lemma is a simple consequence of this formulation of \( \rho_\Omega. \)

**Lemma B.2.** With the notation in Theorem B.1, for any compact subset \( K \subset \Omega \) and \( \epsilon > 0 \) there exists \( N > 0 \) such that
\[
(1 - \epsilon)\rho_{\Omega_n}(x, y) \leq \rho_\Omega(x, y) \leq (1 + \epsilon)\rho_{\Omega_n}(x, y)
\]
for all \( x, y \in K \) and \( n \geq N. \)

**Proof.** By possibly increasing \( K, \) we can assume that \( K \) is convex. We first claim that there exists \( N > 0 \) so that
\[
(1 - \epsilon)k_{\Omega_n}(x; v) \leq k_\Omega(x; v) \leq (1 + \epsilon)k_{\Omega_n}(x; v)
\]
for all \( n > N, \ x \in K, \) and \( v \in \mathbb{M}_1. \) Suppose not, then there exists \( n_k \to \infty, \)
\( x_{n_k} \in K, \) and \( v_{n_k} \in \mathbb{M}_1 \) so that
\[
\frac{k_\Omega(x_{n_k}; v_{n_k})}{k_{\Omega_n}(x_{n_k}; v_{n_k})} \notin [1 - \epsilon, 1 + \epsilon].
\]
We can assume that \( \|v_{n_k}\| = 1 \) (where \( \|\cdot\| \) is the operator norm). Then we can pass to a subsequence and assume that \( x_{n_k} \to x \in K \) and \( v_{n_k} \to v \in \mathbb{M}_1. \) But using the fact that \( \Omega_n \) converges to \( \Omega \) in the local Hausdorff topology we have
\[
\lim_{k \to \infty} \frac{k_\Omega(x_{n_k}; v_{n_k})}{k_{\Omega_n}(x_{n_k}; v_{n_k})} = \frac{k_\Omega(x; v)}{k_{\Omega_n}(x; v)} = 1.
\]
So we have a contradiction.

Then the Lemma follows from Equation B.1. \( \square \)

**Proof of Theorem B.3.** Now suppose that \( K \subset \Omega \) is compact. Then we can pick \( R > 0 \) and \( x_0 \in \Omega \) such that \( K \subset \{ x \in \Omega : d_\Omega(x, x_0) \leq R \}. \) Let
\[
K' = \{ x \in \Omega : K_\Omega(x, x_0) \leq (1 + \epsilon)^2(R + 1) + R + \epsilon \}.
\]
Next pick \( N > 0 \) such that
\[
(1 - \epsilon)\rho_{\Omega_n}(x, y) \leq \rho_\Omega(x, y) \leq (1 + \epsilon)\rho_{\Omega_n}(x, y)
\]
for all \( x, y \in K' \) and \( n \geq N. \) Now we claim that
\[
K_{\Omega_n}(x, y) \leq (1 + \epsilon)K_\Omega(x, y)
\]
for \( x, y \in K \) and \( n \geq N. \) For \( x, y \in K \) and \( \delta \in (0, 1) \) pick \( x = a_0, a_1, \ldots, a_m = y \)
such that
\[
\rho_\Omega(x, a_1) + \rho_\Omega(a_1, a_2) + \cdots + \rho_\Omega(a_{m-1}, y) \leq K_\Omega(x, y) + \delta.
\]
Then \( a_0, \ldots, a_m \in K' \) and so
\[
\rho_{\Omega_n}(x, a_1) + \rho_{\Omega_n}(a_1, a_2) + \cdots + \rho_{\Omega_n}(a_{m-1}, y) \leq (1 + \epsilon)(K_\Omega(x, y) + \delta)
\]
for \( n \geq N. \) Since \( \delta > 0 \) was arbitrary we see that
\[
K_{\Omega_n}(x, y) \leq (1 + \epsilon)K_\Omega(x, y)
\]
for \( x, y \in K \) and \( n \geq N. \)
Now suppose $n \geq N$, $x, y \in K$, $\delta \in (0, 1)$, and $x = a_0, a_1, \ldots, a_m = y \in \Omega_n$ such that
\[
\rho_{\Omega_n}(x, a_1) + \rho_{\Omega_n}(a_1, a_2) + \cdots + \rho_{\Omega_n}(a_{m-1}, y) \leq K_{\Omega_n}(x, y) + \delta.
\]
If $a_0, a_1, \ldots, a_m \in K'$ then we immediately see that
\[
K_{\Omega}(x, y) \leq \rho_{\Omega}(x, a_1) + \rho_{\Omega}(a_1, a_2) + \cdots + \rho_{\Omega}(a_{m-1}, y) \leq (1 + \epsilon)(K_{\Omega_n}(x, y) + \delta)
\]
Otherwise we can assume that there is some $a_\ell$ such that $a_\ell \in \partial K'$. Then $K_{\Omega}(a_\ell, x_0) = (1 + \epsilon)^2(R + 1) + R + \epsilon$ and so
\[
(1 + \epsilon)^2(R + 1) + \epsilon \leq K_{\Omega}(x_0, a_\ell) - K_{\Omega}(x_0, x) \leq K_{\Omega}(x, a_\ell)
\]
\[
\leq \rho_{\Omega}(x, a_1) + \rho_{\Omega}(a_1, a_2) + \cdots + \rho_{\Omega}(a_{\ell-1}, a_\ell)
\]
\[
\leq (1 + \epsilon)(K_{\Omega_n}(x, y) + \delta) \leq (1 + \epsilon)(1 + \epsilon)K_{\Omega}(x, y) + 1
\]
\[
\leq (1 + \epsilon)^2(R + 1)
\]
which is a contradiction. Thus $a_0, a_1, \ldots, a_m \in K'$ and
\[
K_{\Omega}(x, y) \leq (1 + \epsilon)(K_{\Omega_n}(x, y) + \delta).
\]
Since $\delta \in (0, 1)$ was arbitrary we see that
\[
K_{\Omega}(x, y) \leq (1 + \epsilon)K_{\Omega_n}(x, y).
\]

\[\square\]

References


