Math 116 - Quiz 4 Name: Key

Answer the following questions showing all work. 5 points each.
For questions 1-4, consider the graph given by the parametric equations $x(t) = 2e^t - 4$ and $y(t) = \frac{1}{2}e^{2t} - t$.

1. Precisely one of the points $(2e^5 - 4, \frac{1}{2}e^5 - 5)$ and $(2e - 4, \frac{e^2}{2} - 1)$ is on the graph. Which one is?

No Yes

$2e^5 - 4 = 2e^t - 4$, $t = 5$ Not on graph

$\frac{1}{2}e^5 - 5 = \frac{1}{2}e^{2t} - t$, $t = ?$

but 5 clearly doesn't work b/c $\frac{1}{2}e^5 - 5 \neq \frac{1}{2}e^{10} - 5$

$2e - 4 = 2e^t - 4$, $t = 1$

$\frac{e^2}{2} - 1 = \frac{1}{2}e^{2t} - t$, $t = 1$

same $t \to$ ON Graph

2. Find the equation of the tangent line to the graph at the point you found in 1.

$\frac{dx}{dt} = 2e^t$

$\frac{dy}{dt} = e^{2t} - 1$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{2t} - 1}{2e^t}$

@ $t = 1$ $m = \frac{dy}{dx} = \frac{e^2 - 1}{2e}$

$$y - \left(\frac{e^2}{2} - 1\right) = \frac{e^2 - 1}{2e}\left(x - (2e - 4)\right)$$

3. Is the graph concave up or concave down at the point you found in 1?

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right] \Big/ \left(\frac{dx}{dt}\right) = \frac{\frac{e^t}{2} + \frac{e^{-t}}{2}}{2e^t} = \frac{1}{4} + \frac{e^{-2t}}{4} > 0$$ for all t. So graph is always ccu

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{e^{2t} - 1}{2e^t}\right) = \frac{d}{dt}\left(\frac{e^t}{2} - \frac{1}{2}e^{-t}\right) = \frac{e^t}{2} + \frac{e^{-t}}{2}$$

4. Find length along the graph from $\boxed{(4, \frac{1}{2})}$ to $\boxed{(8e^2 - 4, \frac{1}{2}e^4 - 1).}$ ✡ ✡

$$\text{arc length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b \sqrt{(2e^t)^2 + (e^{2t} - 1)^2}\,dt = \int_a^b \sqrt{4e^{2t} + e^{4t} - 2e^{2t} + 1}\,dt$$

$$= \int_a^b \sqrt{e^{4t} + 2e^{2t} + 1}\,dt = \int_a^b \sqrt{(e^{2t} + 1)^2}\,dt = \int_a^b e^{2t} + 1\,dt = \frac{1}{2}e^{2t} + t\Big|_a^b =$$

✡ due to sloppy editing of this quiz, neither pt is on the curve. For the soln, let a be the t corresponding to the 1st pt & b to the second.

5. Suppose $r = f(\theta)$ satisfies the differential equation $(\frac{dr}{d\theta})^2 + r^2 = \theta^2$. Compute the arc length of f from $\theta = 0$ to $\theta = 2\pi$.

arc length: $\int_0^{2\pi} \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^2}\, d\theta = \int_0^{2\pi} \theta\, d\theta = \frac{1}{2}\theta^2 \Big|_0^{2\pi} = \frac{1}{2}(2\pi)^2 - \frac{1}{2}(0)^2$

$= 2\pi^2$

$w = 3\theta \quad dw = 3\,d\theta$

6. In honor of St. Patricks Day, you decide to pick a shamrock. Unfortunately, you can't find any three leaf clovers, which makes you sad. Your friend offers to cheer you up by conjuring up one in the form of the polar equation $r = 4\cos(3\theta)$. With that in mind, compute the area of a single leaf.

Area $= \int_{\pi/6}^{\pi/2} \frac{1}{2} r^2\, d\theta = \int_{\pi/6}^{\pi/2} \frac{1}{2}(4\cos 3\theta)^2\, d\theta = \int_{\pi/6}^{\pi/2} 8\cos^2 3\theta\, d\theta = \frac{8}{3}\int_{\pi/2}^{3\pi/2} \cos^2 w\, dw$

$= \frac{8}{3}\left(\frac{\pi}{2}\right) \approx 4.1887$

7. Also, write down (but do not evaluate) a definite integral to compute the perimiter of one leaf.

leaf begins/ends when $0 = r = 4\cos 3\theta$

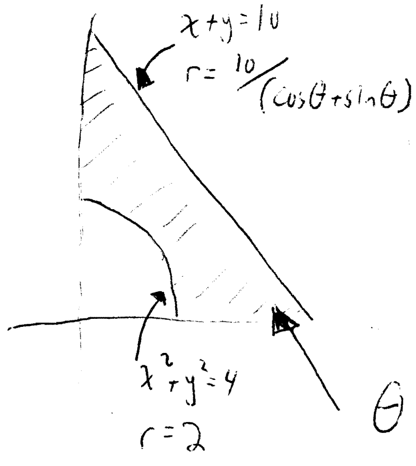
$0 = \cos 3\theta$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \ldots$

Perimeter = arc length $= \int_{\pi/6}^{\pi/2} \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta$

$= \int_{\pi/6}^{\pi/2} \sqrt{(4\cos 3\theta)^2 + (-12\sin 3\theta)^2}\, d\theta = \int_{\pi/6}^{\pi/2} \sqrt{16\cos^2 3\theta + 144\sin^2 3\theta}\, d\theta$

8. Consider the region, Ω, in the first quadrant bounded by the x-axis, the y-axis, the line $x + y = 10$, and the circle $x^2 + y^2 = 4$. Give inequalities on r and θ that describe Ω.



cartesian

$x+y=10 \quad\longleftrightarrow\quad r\cos\theta + r\sin\theta = 10$

$r(\cos\theta + \sin\theta) = 10$

$r = \frac{10}{\cos\theta + \sin\theta}$

$x^2+y^2=4 \quad\longleftrightarrow\quad r^2 = 4$

$(r=2)$

Polar

$\theta \in [0, \frac{\pi}{2}]$

$2 \le r \le \frac{10}{\cos\theta + \sin\theta}$