Solution to quiz 4

1. [6 points] Suppose $f(x) = e^{x^2}$.

(1) [2 points] Find the tangent line approximation of $f(x)$ at $x = 1$.
Solution: $f'(x) = 2xe^{x^2}$, so $f'(1) = 2e$, the tangent line approximation $L(x) = f(1) + f'(1)(x-1) = e + 2e(x-1) = 2ex - e$.
(2) [2 points] Use your answer in part (1) to estimate $e^{1.01^2}$.
Solution: $e^{1.01^2} = f(1.01) \approx L(1.01) = 2e \cdot 1.01 - e = 2.02e - e$.

(3) [2 points] Is your estimation in part (2) an overestimate or underestimate? Use concavity to explain your answer.
Solution: $f''(x) = (4x^2+2)e^{x^2} > 0$, so $f(x)$ is always concave up, so the estimation is an underestimate.

2. [6 points]Suppose that x and y satisfy the relation given by the curve

$$x^2 + xy + y^2 = 3$$

(1) [4 points] Find $\frac{dy}{dx}$ (in other words, y').
Solution: take derivative directly to both sides, we have $2x + y + xy' + 2yy' = 0$, so $(x+2y)y' = -(2x+y)$, so $y' = -\frac{2x+y}{x+2y}$.

(2) [2 points]Find all points on the curve at which the tangent line is horizontal.
Solution: The tangent line is horizontal means the slope is 0. So let $y' = -\frac{2x+y}{x+2y} = 0$, we have $2x + y = 0$, so $y = -2x$. Then plug $y = -2x$ back in the original equation, we have $x^2 + x \cdot (-2x) + (-2x)^2 = 3$, so $3x^2 = 3$, then $x = 1$ (so $y = -2$) or $x = -1$ (so $y = 2$). We find 2 points, $(1, -2)$ and $(-1, 2)$.

3. [7 points] Let $f(x) = x^2(x-1)^2$. Find all the critical points of $f(x)$ and use either the first derivative test or the second derivative test to classify these critical points as a local max, local min, or neither. Show all your steps!
Solution: $f(x) = x^4 - 2x^3 + x^2$, so $f'(x) = 4x^3 - 6x^2 + 2x = 2x(2x^2 - 3x + 1) = 2x(2x-1)(x-1)$. Let $f'(x) = 0$, we have 3 solutions, $x = 0$, $x = 0.5$ or $x = 1$. These are where the critical points locate.
Let's use the second derivative test. $f''(x) = 12x^2 - 12x + 2$, so $f''(0) = 2 > 0$, $f''(0.5) = -1 < 0$, $f''(1) = 2 > 0$. By the test, we know $f(x)$ has a local min at $x = 0$ and $x = 1$, and a local max at $x = 0.5$.
We can also use the first derivative test. Now the three critical points divide the x-axis into 4 intervals, $(-\infty, 0)$, $(0, 0.5)$, $(0.5, 1)$ and $(1, \infty)$. We pick any value on each interval to find the sign of f' on that interval.
$(-\infty, 0)$, e.g, pick -1, $f'(-1) = -12 < 0$, so $f' < 0$ on $(-\infty, 0)$;
$(0, 0.5)$, e.g, pick 0.25, $f'(0.25) = 0.1875 > 0$, so $f' > 0$ on $(-\infty, 0)$;

$(0.5, 1)$, e.g, pick 0.75, $f'(0.75) = -0.1875 < 0$, so $f' < 0$ on $(-\infty, 0)$;
$(1, \infty)$, e.g, pick 2, $f'(2) = 12 > 0$, so $f' > 0$ on $(-\infty, 0)$;
By the first derivative test, we can get the same result.