Math 115 — First Midterm

August 1, 2011

Name: ______________________________

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	10	
3	10	
4	10	
5	10	
6	10	
7	8	
8	6	
9	10	
10	14	
Total	100	

1. [12 points] For each of the following statements, circle **True** if the statement is *always* true and circle **False** otherwise. No explanation is needed.

a. [2 points] Let $g(x) = f(2x)$, then the graph of $g'(x)$ can be obtained by horizontally compression the graph of $f'(x)$ by a factor of $\frac{1}{2}$.

True False

b. [2 points] If $f'(x)$ is increasing for $x < 1$, and decreasing for $x > 1$, then $f(x)$ must have an inflection point at $x = 1$.

True False

c. [2 points]

$$\lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = 4$$

True False

d. [2 points] There exists a function $f(x)$, which has both a critical point and an inflection point at $(5, 10)$.

True False

e. [2 points] The rational function

$$f(x) = \frac{2x^4 - x^5 + x^3}{x^4 - 2x^3}$$

has a horizontal asympotote $y = 2$.

True False

f. [2 points] Suppose $g(x) = f(f(x))$, then $g''(x) = f''(f(x))f'(x) + f'(f(x))f''(x)$.

True False

2. [10 points] Consider $f(x) = ax^3 + bx^2 + cx$ where a, b, and c are constants. Suppose $f(x)$ has a local maximum at $x = -1$ and an inflection point at $(1, -10)$. Find a, b, and c.

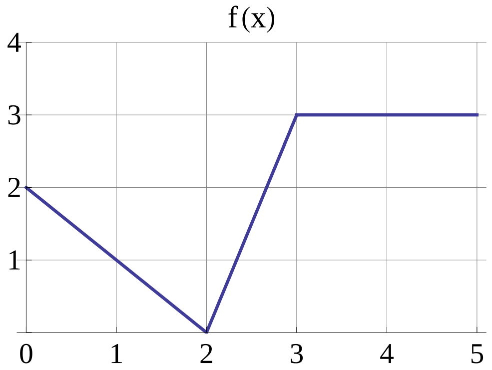
Solution: $f'(x) = 3ax^2 + 2bx + c$, $f''(x) = 6ax + 2b$
$f(x)$ has a local maximum at $x = -1$ implies $f'(-1) = 0$, so $3a - 2b + c = 0$ (equation [1])
$(1, -10)$ is on the graph of $f(x)$, so $f(1) = a + b + c = -10$ (equation [2])
$f(x)$ has an inflection point at $(1, -10)$ implies $f''(1) = 6a + 2b = 0$ (equation [3])
equation [1]-equation [2], we have $2a - 3b = 10$ (equation [4])
from equation [3], we have $b = -3a$, plug this into equation [4]
we find, $11a = 10$, so $a = 10/11$
so $b = -3a = -30/11$, and use equation [2], we have $c = -10 - a - b = -90/11$

$a = 10/11$

$b = -30/11$

$c = -90/11$

3. [10 points] Given below is a graph of a function $f(x)$ and a table for an invertible function $g(x)$.



x	1	2	3	4	5
$g(x)$	0.5	1	4	5	6
$g'(x)$	1	3	5	4	2

Give answers for the following or write "Does not exist". No partial credit will be given.

a. [2 points] $h(x) = f(g(x))$, find $h'(2)$

Solution: $h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(2) = f'(g(2)) \cdot g'(2) = f'(1) \cdot g'(2) = -1 \cdot 3 = -3$

b. [2 points] $l(x) = \frac{1}{g^{-1}(x)}$, find $l'(4)$

Solution: $l'(x) = -\frac{1}{(g^{-1}(x))^2} \cdot \frac{1}{g'(g^{-1}(x))}$, so

$$l'(4) = -\frac{1}{(g^{-1}(4))^2} \cdot \frac{1}{g'(g^{-1}(4))} = -\frac{1}{3^2} \cdot \frac{1}{5} = -\frac{1}{45}$$

c. [2 points] $t(x) = \ln(\sqrt{e^{f(x)}})$, find $t'(2.5)$

Solution: $t(x) = \frac{1}{2}\ln(e^{f(x)}) = \frac{1}{2}f(x)$, so $t'(2.5) = \frac{1}{2}f'(2.5) = \frac{1}{2} \cdot 3 = 1.5$

d. [2 points] $k(x) = f(x)g(x)$, find $k'(3)$

Solution: $k'(3) = f'(3)g(3) + f(3)g'(3)$, since $f'(3)$ doesn't exist, $k'(3)$ doesn't exist.

e. [2 points] $p(x) = 2f(x) - 3g(x)$, find $p'(4)$

Solution: $p'(4) = 2f'(4) - 3g'(4) = 2 \cdot 0 - 3 \cdot 4 = -12$

4. [10 points] Dirk just finished his last class of the week Friday afternoon at 3pm and he is very excited about the weekly 2 vs 2 free throw shooting contest scheduled later that night at 11pm. In order to prepare himself for the competition, he can either practice shooting free throws or get focused by listening to motivational music. His free throw percentage, P, is a function of how many hours, S, he spends shooting and how many hours, M, he spend listening to music and is given by

$$P = (9 + 8S - S^2)e^{0.2M}$$

How should Dirk spend his time in order to maximize his free throw percentage in the contest, and what will his free throw percentage be if he prepares optimally?

Solution: From 3pm to 11pm, there are 8 hours in total, so $S + M = 8$, we have $M = 8 - S$, and we know $0 \leq S \leq 8$. Substitute M with $8 - S$, the free throw percentage is:

$$P = (9 + 8S - S^2)e^{0.2(8-S)}$$

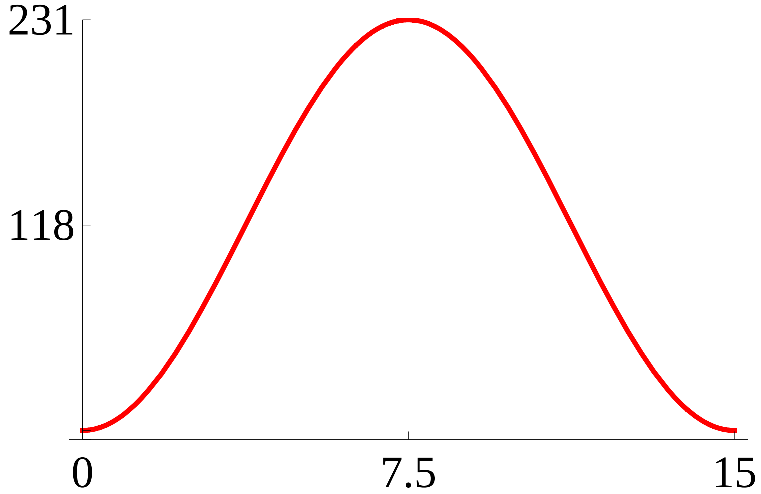
Let $P' = 0.2 \cdot e^{0.2(8-S)}(S^2 - 18S + 31) = 0$, we have $S^2 - 18S + 31 = 0$, by the quadratic formula, we find 2 solutions, $S = 5\sqrt{2}+9 \approx 16$ or $S = -5\sqrt{2}+9 \approx 2$. Only the second solution is in the domain $0 \leq S \leq 8$, so P has a unique critical point at $S = 2$.
Use the first derivative test, pick 1 from interval $(0, 2)$, $P'(1) > 0$, so $P' > 0$ on this interval; pick 3 from interval $(2, 8)$, $P'(3) < 0$, so $P' < 0$ on this interval.
By the first derivative test, we know P has a local max at $S = 2$, and since it's the unique critical point, P must have a global max there.
Therefore, when $S = 2$, $M = 8 - S = 6$, the max free throw percentage $P = (9 + 8 \cdot 2 - 2^2)e^{0.2(8-2)} = 69.7$

5. [10 points] Doug and Patti are at the Tri-County Fair and are about to ride the Giant Tornado, the fastest ferris wheel on Earth. It is so fast that it makes four full rotations in just a minute. Doug and Patti get on the Giant Tornado at the very bottom, which is 5 feet off the ground. When at the top, they will be able to see all of Bluffington from a height of 231 feet. Let $h(t)$ be the height, in feet, that Doug and Patti are off the ground t seconds after they get on.

a. [2 points] How fast does the Giant Tornado spin, in radians per second?

it makes 4 fulll rotations in 1 minute, so 8π radians in 60 seconds. So the answer is $8\pi/60$ radians per second.

b. [4 points] On the axes below, draw a sketch of $h(t)$.



c. [4 points] Write a formula for $h(t)$.

$h(t) = -113\cos(\frac{2\pi}{15}t) + 118$

6. [10 points] Consider the curve given by $x^2 + xy + y^2 = y$.

a. [5 points] Compute $\frac{dy}{dx}$. (In other words, y')

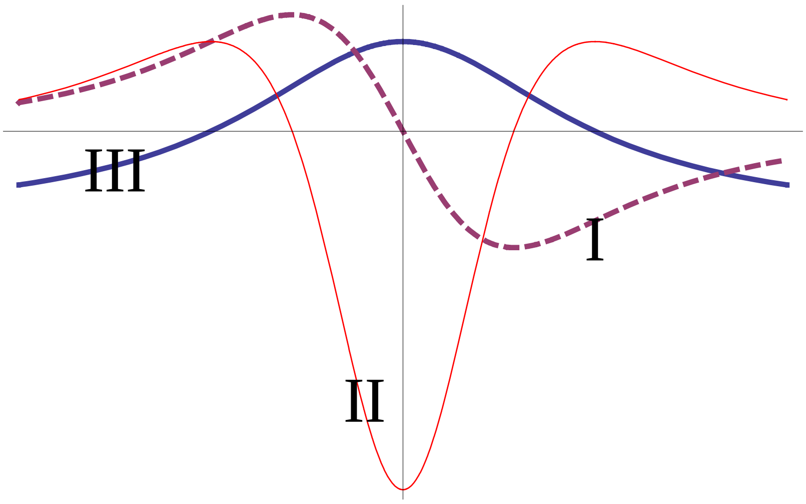
Solution: take the derivative, we have $2x+y+xy'+2yy' = y'$, so $xy'+2yy'-y' = -(2x+y)$, factor y' out on the left side, we have $y'(x + 2y - 1) = -(2x + y)$, so $y' = -\frac{2x+y}{x+2y-1}$.

b. [5 points] Find all points at which the curve has a horizontal tangent line. If none exists, write "None."

Solution: $y' = -\frac{2x+y}{x+2y-1} = 0$ at the points where the curve has a horizontal tangent line. so $2x + y = 0$, $y = -2x$. Plug this back into the original equation $x^2 + xy + y^2 = y$, we find $3x^2 = -2x$, so $x = 0$ (so $y = -2x = 0$) or $x = -\frac{2}{3}$ (so $y = -2x = \frac{4}{3}$).

Therefore there are 2 such points, one is $(0, 0)$, the other is $(-\frac{2}{3}, \frac{4}{3})$.

7. [8 points] Given below are the graphs of $f(x)$, $f'(x)$ and $f''(x)$. Find which graph corresponds to which function.



Graph of $f(x)$ is: III

Graph of $f'(x)$ is: I

Graph of $f''(x)$ is: II

8. [6 points] The Detroit Tigers host the Texas Rangers this Thursday at Comerica Park. The game begins at 2:30pm and fans start entering the stadium at 12:00pm (noon). Let $f(t)$ be the number of fans, in thousands,in the ballpark t minutes after noon on Thursday.

a. [2 points] Give a practical interpretation of the statement $f'(2) = 12$. Include units.

At 12:02 pm on Thursday, the fans are entering the stadium by approximately 12 thousands people per minute.

Or: There are about 12 thousands more fans in the stadium at 12:03pm than at 12:02pm.

b. [2 points] Interperet the statement $(f^{-1})'(25) = 2$. Include units.

When there are 25 thousands fans in the stadium, it takes about 2 minutes for 1 more thousand fans to enter the stadium.

c. [2 points] Suppose that in addition to the fans in attendence, there are an additional 1512 people at the ballpark (players, staff, security, ...). Let $g(t)$ be the total number of people in the stadium t minutes *before* the game starts. Write a formula for $g(t)$ in terms $f(t)$.

The game begins at 2:30pm, which is 150 minutes after noon, so t minutes before the game starts would be $150 - t$ minutes after noon, at that time, there are $1000f(150 - t)$ fans in the stadium. Since there are 1512 additional people, in total there are $g(t) = 1000f(150 - t) + 1512$ people.

9. [10 points] Last week, the 2011 World Snail Racing Championships were held in Norfolk, England. This years winner is Zoomer, a snail from Ann Arbor. Let $s(t)$ be the distance Zoomer has run (in inches), where t is the time that has passed (in seconds) since the start of the race, and $s(t)$ is given by

$$s(t) = e^t(7 - 4t + t^2) - 7$$

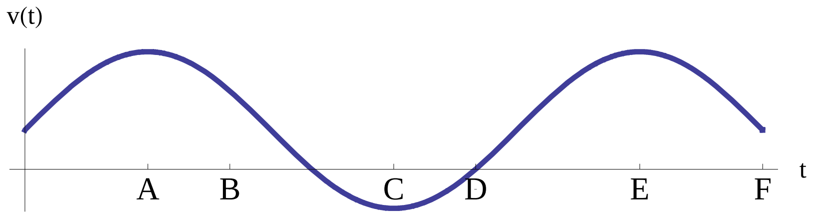
a. [2 points] Find the velocity of Zoomer at $t = 1$.
$s'(1) = e^1(3 - 2 \cdot 1 + 1^2) = 2e$ inch/s

b. [3 points] Find the linear localization (in other words, tangent line approximation) of $s(t)$ at $t = 1$.
$s'(1) = 2e$, $s(1) = 4e - 7$, so the tangent line approximation is $L(t) = 4e - 7 + 2e(t - 1) = 2et + 2e - 7$

c. [2 points] Use part (b) to approximate the distance Zoomer has travelled in the first 1.02 seconds.
$s(1.02) \approx 2e \cdot 1.02 + 2e - 7 = 4.04e - 7$ inches

d. [3 points] Is your estimate in part (c) and overestimate or underestimate? Use concavity to justify your reasoning.
$s''(t) = e^t(1 + t^2) > 0$, so graph of $s(t)$ is concave up, the estimate must be an underestimate.

10. [14 points] Kenny opens the concert in Ford Field with his recent hit "Living in Fast Forward" and spends the entire song, $0 \le t \le F$, running along the front of the stage. Beth contains her excitement enough to record Kenny's **velocity** $v(t)$ as a function of time t, shown in the graph below; when velocity is positive Kenny is running to the right. Further, let $s(t)$ represent Kenny's position right of the center of the stage at time t. Answer the following questions using the values of t labelled on the graph. If no intervals or labelled points satisfy the condition you must write "**NONE**" to receive credit. No explanation is needed.



a. [2 points] During which interval(s) is Kenny's acceleration is positive?
(0, A), (C, E)

b. [2 points] At which point(s) does Kenny turn around?
G, D

c. [2 points] During which interval(s) is the graph of $s(t)$ increasing?
(0, G), (D, F)

d. [2 points] During which interval(s) is the graph of $s(t)$ concave down?
(A, C), (E, F)

e. [2 points] At which point is Kenny running the fastest toward the left end of the stage?
C

f. [2 points] During the interval $0 \le t \le D$, at which point is Kenny farthest to the right?
G

g. [2 points] During which interval(s) are both Kenny's velocity increasing and his acceleration increasing?
(C, D)