Math 116-101 or 102 (circle one) (Spring 2012) **Name:** ____________________________

Quiz 4: §8.7-8.8, 9.1-9.3

6/05/2012 Show all work and include units where appropriate. You have 30 minutes to

complete this quiz. (25 pts)

1. Consider the function

$$p(x) = \begin{cases} 0, & \text{if } x < 0 \\ kx^2(1-x), & \text{if } 0 \le x \le 1 \\ 0, & \text{if } x > 1 \end{cases}$$

(a) For what value of k is $p(x)$ a probability density function? (3 pts)

We have

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} p(x)dx &= \int_0^1 kx^2(1-x)dx = \int_0^1 k(x^2-x^3)dx \\ &= k\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{k}{12}. \end{aligned}$$

Thus $k = 12$.

(b) Using that value of k, find the probability that x is greater than 0.5.(2 pts)

We set up

$$\int_{0.5}^{\infty} p(x)dx = \int_{0.5}^1 12(x^2-x^3)dx = 12\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_{0.5}^1 = \frac{11}{16}.$$

(c) Find the mean.(3 pts)

We set up and calculate

$$\begin{aligned} \text{mean of } x = \int_{-\infty}^{\infty} xp(x)dx &= \int_0^1 12x(x^2-x^3)dx = \int_0^1 12(x^3-x^4)dx \\ &= 12\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{12}{20} = 0.6. \end{aligned}$$

(d) Find the median. (Hint: Use your calculator, but not do do any definite integrals.) (3 pts)

We set up the definition of the median T and obtain

$$0.5 = \int_{-\infty}^{T} p(x)dx = \int_0^T 12(x^2-x^3)dx = 12\left(\frac{T^3}{3} - \frac{T^4}{4}\right).$$

Using a calculator to solve numerically using either the root or intersect function yields $T \approx 0.61427$.

2. Use the formulas for the sums of geometric series on the following. (3 pts each)

   (a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$

   The infinite geometric series formula is $\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$. This series has $a = 1$ and $x = -\frac{1}{2}$ and so the sum is

$$\frac{1}{1-(-\frac{1}{2})} = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}$$

   (b) $\sum_{n=5}^{15} \left(\frac{2}{3}\right)^n$ The finite geometric series formula is $\sum_{i=0}^{n-1} ax^i = \frac{a(1-x^n)}{1-x}$.

$$\begin{aligned}\left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6 + \cdots + \left(\frac{2}{3}\right)^{15} &= \left(\frac{2}{3}\right)^5 \left(1 + \left(\frac{2}{3}\right) + \cdots + \left(\frac{2}{3}\right)^{10}\right) \\ &= \left(\frac{2}{3}\right)^5 \left(\frac{1-\left(\frac{2}{3}\right)^{11}}{1-\left(\frac{2}{3}\right)}\right);\end{aligned}$$

   where we used the finite geometric series formula with $a = 1$, $x = \frac{2}{3}$, and $n - 1 = 10$ once we factored out the $\left(\frac{2}{3}\right)^5$.

3. Suppose that you make monthly deposits into a savings account of \$250, with the first deposit occurring today. Every month, your account pays 4% interest. Let B_n represent the balance in your account immediately after the nth deposit.

   (a) Find B_1, B_2, and B_3.(3 pts) With each deposit, the balance increases by \$250. Moreover, the amount in the account at the time of the previous deposit has increased by 4%, therefore is multiplied by 1.04. We have

$$B_1 = 250, \qquad B_2 = 250 + B_1 \cdot 1.04 = 250 + 250(1.04)$$

$$B_3 = 250 + B_2 \cdot 1.04 = 250 + (250 + 250(1.04))1.04 = 250 + 250(1.04) + 250(1.04)^2$$

   (b) Find a closed form, explicit formula (that is, no summation signs or $+ \cdots +$) for B_n. (5 pts) We extrapolate the pattern and observe

$$B_n = 250 + 250(1.04) + 250(1.04)^2 + \cdots + 250(1.04)^{n-1}.$$

   Using the finite geometric series formula with $a = 250$ and $x = 1.04$ we obtain

$$B_n = 250\frac{(1-1.04^n)}{(1-1.04)}.$$