Name:

Show all work and include units where appropriate. You have 30 minutes to complete this quiz. (25 pts)

- 1. Circle "True" if the statement is always true. Otherwise, circle "False". Include a brief explanation for your answer. (3 pts each)
 - (a) If f(x) is a continuous, decreasing function for $x \ge 1$, then $\int_1^\infty f(x) dx$ is convergent.

f(x) = 1/x is continuous and decreasing for $x \ge 1$, but we know $\int_1^\infty \frac{1}{x} dx$ diverges, so the statement must be false.

False

False

(b) The point given in Cartesian coordinates by $\left(-\frac{\sqrt{3}}{4}, -\frac{1}{4}\right)$ can be expressed in polar coordinates as $r = \frac{1}{2}, \ \theta = \frac{\pi}{6}$.

True

The point is clearly in the third quadrant, but $r = \frac{1}{2}, \theta = \frac{\pi}{6}$ corresponds to a point in the first quadrant. Therefore, the statement must be false. (Recall that using $\theta = \arctan\left(\frac{y}{x}\right)$ is only valid for points (x, y) that are in the first and fourth quadrants. For second and third quadrant points, you can take $\theta = \arctan\left(\frac{y}{x}\right) + \pi$.)

(c) The parametric equations $x(t) = 3 + t^2$, $y(t) = 4 + 2t^2$ describe a parabola.

False

We can eliminate the parameter by observing that $t^2 = x - 3$ and so y = 4 + 2(x - 3), which we know is the equation for a line.

2. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{2x}{x^2+1} - \frac{C}{2x+1}\right) dx$$

converges. Evaluate the integral for this value of C. (Hint: You will find the properties of logarithms very useful.) (7 pts)

We begin by rewriting the improper integral as a limit, and proceeding with antiderivatives. This yields

$$\begin{split} \int_{0}^{\infty} \left(\frac{2x}{x^{2}+1} - \frac{C}{2x+1} \right) dx &= \lim_{b \to \infty} \int_{0}^{b} \left(\frac{2x}{x^{2}+1} - \frac{C}{2x+1} \right) dx \\ &= \lim_{b \to \infty} \left(\ln |x^{2}+1| - \left(\frac{C}{2} \right) \ln |2x+1| \right) \Big|_{0}^{b} \\ &= \lim_{b \to \infty} \left(\ln |x^{2}+1| - \ln(|2x+1|^{C/2}) \right) \\ &= \lim_{b \to \infty} \ln \left(\frac{x^{2}+1}{(2x+1)^{C/2}} \right) \Big|_{0}^{b} \\ &= \lim_{b \to \infty} \left(\ln \left(\frac{b^{2}+1}{(2b+1)^{C/2}} \right) - \ln \left(\frac{0^{2}+1}{(2\cdot0+1)^{C/2}} \right) \right) \\ &= \lim_{b \to \infty} \ln \left(\frac{b^{2}+1}{(2b+1)^{C/2}} \right) \end{split}$$

where we used the properties of logarithms and the fact that $\ln(1) = 0$.

Now, we see that if C/2 > 2, then the expression inside the logarithm will go to zero, and since $\ln 0$ is undefined, the limit will not exist. Similarly, if C/2 < 2, the expression inside the logarithm will go to infinity, and so the limit will not exist. Therefore, the only way the limit could exist is when C/2 = 2, which means C = 4. In that case we have

$$\lim_{b \to \infty} \ln\left(\frac{b^2 + 1}{(2b+1)^{C/2}}\right) = \lim_{b \to \infty} \ln\left(\frac{b^2 + 1}{(2b+1)^2}\right)$$
$$= \lim_{b \to \infty} \ln\left(\frac{b^2 + 1}{4b^2 + 4b + 1}\right)$$
$$= \ln\left(\frac{1}{4}\right).$$

Therefore, C = 4 is the only value of C for which the integral converges, and it converges to $\ln(1/4)$ in that case.

3. Consider the region below bounded by the curves $y = \sqrt{x}$ and $y = x^3$.



Set up, but DO NOT EVALUATE, definite integrals giving the values of the quantities indicated. Circle or box your final answer. (3 pts each)

(a) The volume of the solid obtained by revolving the region about the y-axis. We obtain washer shaped slices with thickness Δy , outer radius the x-coordinate on the curve $y = x^3$, and the inner radius the x-coordinate on the curve $y = \sqrt{x}$. Therefore, the volume of the slice is $\pi \left((y^{1/3})^2 - (y^2)^2 \right) \Delta y$, giving us the integral

$$\int_0^1 (y^{2/3} - y^4) dy.$$

(b) The volume of the solid obtained by revolving the region about the line y = 2. We obtain washer shaped slices with thickness Δx , outer radius the vertical distance between the curve $y = x^3$ and y = 2, and inner radius the vertical distance between the curve $y = \sqrt{x}$ and y = 2. Therefore, the volume of the slice is $\pi ((2 - x^3)^2 - (2 - \sqrt{x})^2) \Delta x$, leading to the integral

$$\int_0^1 \left((2 - x^3)^2 - (2 - \sqrt{x})^2 \right) dx.$$

(c) The perimeter of the region. We need to find the arc length of each of the curves and add the results together. For $y = x^3$, we have $y'(x) = 3x^2$, and for $y = \sqrt{x}$, we have $y'(x) = \frac{1}{2\sqrt{x}}$, and so we obtain

$$\int_0^1 \sqrt{1 + (3x^2)^2} dx + \int_0^1 \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx$$