Math 116 - Midterm May 29, 2012

Name: SOLUTIONS

Instructor:

1. Do not open this exam until you are told to begin.

- 2. This exam has 11 pages including this cover. There are 10 questions.
- 3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
- 4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
- 5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
- 6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
- 7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
- 8. Please turn off all cellphones and remove all headphones.

PROBLEM	POINTS	SCORE
1	12	
2	10	
3	9	
4	11	
5	10	
6	9	
7	12	
8	12	
9	6	
10	9	
TOTAL	100	

1. (3 pts each) Circle whether each statement is true or false. Give a brief explanation of why your answer is correct.

(a) If f(x) is continuous and positive for x > 1, and if $\lim_{x \to \infty} f(x) = \infty$, then $\int_{1}^{\infty} \frac{1}{f(x)} dx$ converges.



It is possible for the integral to converge or diverge. If f(x) = x, the integral diverges, while if $f(x) = x^2$, the integral converges.

(b) For constants a, b, and c, $\int_{-5}^{5} (ax^2 + bx + c)dx = 2 \int_{0}^{5} (ax^2 + c)dx$.



We have that $\int_{-5}^{5} (ax^2 + bx + c)dx = \int_{-5}^{5} (ax^2 + c)dx + \int_{-5}^{5} bx \, dx$. bx is an odd function, and so the second integral is 0. $ax^2 + c$ is an even function, and so the first integral is equal to $2\int_{0}^{5} (ax^2 + c)dx$.

(c) If F(x) is an antiderivative of f(x), and G(x) is an antiderivative of g(x), then $F(x) \cdot G(x)$ is an antiderivative of $f(x) \cdot g(x)$.



The derivative of $F(x) \cdot G(x)$ will be f(x)G(x) + F(x)g(x), which will not in general be equal to f(x)g(x).

(d) $\int_0^2 (x-x^3) dx$ represents the area under the curve $y = x - x^3$ from x = 0 to x = 2.



Definite integrals represent the net signed area between the curve and the x-axis (that is, area above the x-axis counts positive while area below the x-axis counts negative). In this case the definite integral is negative, and since area is always positive the result must be false.

2. You and your fellow Math 116 students are running from a horde of zombies. Suppose that the positions of the zombie horde and of the group of students are given by parametric equations. The zombie horde has position

$$x_z = 4 + t^2, \qquad y_z = t^2 - t^3,$$

and the students have position

$$x_s = -t^2 + 6t + 4, \qquad y_s = -t^2 - 9.$$

(a) Write parametric equations for the tangent line to the students' path at t = 1. (2 pts)

We have $x'_s(t) = -2t + 6$, $y'_s(t) = -2t$ so that $x_s(1) = 9$, $x'_s(1) = 4$, $y_s(1) = -10$, $y'_s(1) = -2$, and so we can take

$$x = 9 + 4t, \qquad y = -10 - 2t$$

(b) Which group is moving faster at time t = 2? The zombies or the students? (2 pts) The speed of the zombies at t = 2 is given by

$$\sqrt{x'_z(2)^2 + y'_z(2)^2} = \sqrt{4^2 + (-8)^2} = \sqrt{80},$$

while the speed of the students is

$$\sqrt{x'_s(2)^2 + y'_s(2)^2} = \sqrt{2^2 + (-4)^2} = \sqrt{20},$$

and so the zombies are moving faster.

(c) When plotted in the xy-plane, what is the concavity of the zombies' path at t = 2? (Justify your answer using calculus.) (3 pts)

Recall that

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

We have that

$$\frac{dy}{dx} = \frac{y_z'(t)}{x_z'(t)} = \frac{2t - 3t^2}{2t} = 1 - \frac{3}{2}t.$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(1-\frac{3}{2}t)}{2t} = \frac{-\frac{3}{2}}{2t} = -\frac{3}{4t}.$$

At t = 2 this is equal to -3/8 < 0, and so the zombies' path is concave down at t = 2.

(d) Do the zombies ever catch the students? Explain. (3 pts)

Setting $x_z(t) = x_s(t)$ yields $4 + t^2 = -t^2 + 6t + 4$, or $2t^2 - 6t = 0$. This factors to 2t(t-3) = 0 and so t = 0 or t = 3. We observe that $y_z(0) \neq y_s(0)$, but that $y_z(3) = y_s(3)$. Therefore, since the zombies and students are at the same place at the same time, we conclude this corresponds to the zombies catching the students, and it occurs at t = 3. 3. Consider the region in the first quadrant inside the curve $r = 3\sin(3\theta)$ and above the line y = 1. (Note: Even if you can't answer part (a), try to proceed to parts (b) and (c) and just express your answers in terms of θ_1 and θ_2 instead of their actual values.)



(a) Find θ_1 and θ_2 , the angles in the first quadrant at which the curve and line intersect. (Hint: You will need to use your calculator.) (3 pts)

Since $y = r \sin(\theta)$ on the curve we have that $y = 3\sin(3\theta)\sin(\theta)$. We find the θ values for the intersection by solving

$$3\sin(3\theta)\sin(\theta) = 1.$$

Solving numerically using a calculator yields $\theta_1 = 0.377178$, and $\theta_2 = 0.900776$.

(b) Use inequalities to describe the region using polar coordinates. (3 pts)

 θ will be between θ_1 to θ_2 , and r will range from what it is on the inner curve (the horizontal line y = 1) to what it is on the outer curve (given by $r = 3\sin(3\theta)$.) We recall that since in polar coordinates $y = r\sin(\theta)$, the horizontal line y = 1 has equation $r\sin(\theta) = 1$, or $r = \frac{1}{\sin(\theta)}$. Therefore, we have

$$0.377178 \le \theta \le 0.900776; \quad \frac{1}{\sin(\theta)} \le r \le 3\sin(3\theta).$$

(c) Write out, but do not evaluate, an expression involving integral(s) giving the area of the region. (3 pts)

The area is contained between θ_1 and θ_2 , inside the curve $r = 3\sin(3\theta)$, and outside the curve $r = \frac{1}{\sin(\theta)}$, and so the area is equal to

$$\frac{1}{2} \int_{0.377178}^{0.900776} \left((3\sin(3\theta))^2 - \left(\frac{1}{\sin(\theta)}\right)^2 \right) d\theta.$$

4. The error function, usually denoted by erf(x), is very important in statistics and probability. It is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

(a) Suppose x > 0. Is erf(x) increasing or decreasing? Is erf(x) concave up or concave down? Justify your answers using calculus. (4 pts)

By the second fundamental theorem, $\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2} > 0$, and so $\operatorname{erf}(x)$ is increasing. Then, $\operatorname{erf}''(x) = \frac{-4x}{\sqrt{\pi}}e^{-x^2} < 0$, and so $\operatorname{erf}(x)$ is concave down.

(b) Use MID(4) to estimate erf(0.4). (4 pts)

We have that

$$\operatorname{erf}(0.4) = \frac{2}{\sqrt{\pi}} \int_0^{0.4} e^{-t^2} dt.$$

For MID(4), we have that $\Delta t = \frac{0.4}{4} = 0.1$, and so we obtain

$$\mathrm{MID}(4) = \frac{2}{\sqrt{\pi}}(0.1) \left(e^{-(0.05)^2} + e^{-(0.15)^2} + e^{-(0.25)^2} + e^{-(0.35)^2} \right) \approx 0.4287.$$

(c) Is your answer to part b an overestimate or underestimate? Justify your answer using calculus. (Be careful.) (3 pts)

The integrand for the midpoint rule was the function $\frac{2}{\sqrt{\pi}}e^{-t^2}$, and so it is that function's concavity, not the concavity of $\operatorname{erf}(x)$, which is relevant (which is why there was the hint to be careful). We observe that

$$f(t) = \frac{2}{\sqrt{\pi}}e^{-t^2}, \qquad f'(t) = \frac{2}{\sqrt{\pi}}(-2t)e^{-t^2}, \quad f''(t) = \frac{2}{\sqrt{\pi}}(4t^2 - 2)e^{-t^2}$$

f''(t) < 0 if $|t| > \frac{1}{\sqrt{2}}$ and so if $0 \le t \le 0.4$ we see that f''(t) < 0. Thus f(t) is concave down on the interval of integration, and so the midpoint rule yields an overestimate.

5. Suppose that f(x), g(x), and h(x) are continuous for x > 0, and that

$$0 \le f(x) \le g(x) \le h(x)$$
 for $x \ge 1$.

Suppose that $\int_1^{\infty} g(x) dx$ converges, while $\int_1^{\infty} h(x) dx$ diverges.

State whether the following improper integrals converge or diverge. Explain your reasoning. (3-4-3 pts)

(a)
$$\int_{1000}^{\infty} h(x) dx$$

We can split the integral of h(x) as the following:

$$\int_{1}^{\infty} h(x)dx = \int_{1}^{1000} h(x)dx + \int_{1000}^{\infty} h(x)dx$$

Because h(x) is continuous, the first integral on the right side is not an improper integral, and so we see that if $\int_1^{\infty} h(x)dx$ diverges that $\int_{1000}^{\infty} h(x)dx$ must also diverge. (Note that it is crucial for h(x) to be continuous for this argument to work.)

(b)
$$\int_{1}^{\infty} f(3x+2)dx$$

We use the substitution $w = 3x + 2, \frac{1}{3}dw = dx$, to obtain

$$\int_{1}^{\infty} f(3x+2)dx = \frac{1}{3} \int_{w(1)}^{w(\infty)} f(w)dw = \frac{1}{3} \int_{5}^{\infty} f(w)dw.$$

Since $f(w) \leq g(w)$, and $\int_5^{\infty} g(x) dx$ converges (due to the fact that the interval $[5, \infty)$) is contained in the interval $[1, \infty)$), we see that $\int_5^{\infty} f(w) dw$ converges. Multiplying by 1/3 does not affect the fact that it converges, and so the original integral converges.

(c)
$$\int_{1}^{\infty} (h(x) - g(x)) dx$$
 (Hint: Thinking about it graphically may help.)

Suppose that this integral converges. Because we know $\int_1^\infty g(x) dx$ converges, together this would imply that

$$\int_{1}^{\infty} h(x)dx = \int_{1}^{\infty} \left((h(x) - g(x)) + g(x) \right) dx$$

converges to the value

$$\int_{1}^{\infty} (h(x) - g(x))dx + \int_{1}^{\infty} g(x)dx.$$

However, this is impossible since we know $\int_1^\infty h(x) dx$ diverges. Therefore, the original integral diverges.

(Note: this is how you have to do this, since the only time you are allowed to "distribute" an improper integral over the sum of two functions is when the improper integral of each function converges. It DOES NOT make sense to say

$$\int_1^\infty (h(x) - g(x))dx = \int_1^\infty h(x)dx - \int_1^\infty g(x)dx$$

when the first integral on the right side diverges, ie is undefined. You can easily get yourself into trouble by splitting up limits into pieces that do not all exist, and remember an improper integral is a limit.)

Alternatively, you could view this as the area between two curves such that the area under the upper curve is infinite while the area under the lower curve is finite. The area between them must be infinite, otherwise we'd have infinite area = finite area + finite area. This is the spirit of the argument above.

6. Suppose f(x) is a continuous, positive, twice-differentiable function with values given in the table below.

x	0	1	2	3	4
f(x)	12	10	6	3	2
f'(x)	-1	-3	-4	-1	1
f''(x)	-2	-1	2	1	3

(a) Compute
$$\int_{2}^{3} \frac{f'(x)\ln(f(x))}{f(x)} dx.$$
 (3 pts)

Setting
$$w = \ln(f(x))$$
, we have (by chain rule) $dw = \frac{f'(x)dx}{f(x)}$. Thus $\int_2^3 \frac{f'(x)\ln(f(x))}{f(x)}dx = \int_{\ln(f(2))}^{\ln(f(3))} w dw = \int_{\ln(6)}^{\ln(3)} w dw = \frac{w^2}{2} |_{\ln(6)}^{\ln(3)} = \frac{\ln(3)^2}{2} - \frac{\ln(6)^2}{2}$

(b) Compute
$$\int_{1}^{2} z f''(2z) dz$$
 (3 pts)

Setting
$$u = z$$
 (thus $du = dz$) and $dv = f''(2z)dz$ (thus, $v = \frac{f'(2z)}{2}$, by "mini w-
sub"), we get $\int zf''(2z)dz = \frac{zf'(2z)}{2} - \int \frac{f'(2z)}{2}dz = \frac{zf'(2z)}{2} - \frac{1}{4}f(2z) + C$. Thus
 $\int_{1}^{2} zf''(2z)dz = \left(\frac{zf'(2z)}{2} - \frac{1}{4}f(2z)\right)\Big|_{1}^{2} = \left[\frac{2f'(4)}{2} - \frac{1}{4}f(4)\right] - \left[\frac{f'(2)}{2} - \frac{1}{4}f(2)\right] = [1 - \frac{1}{2}] - [-2 - \frac{6}{4}] = [\frac{1}{2}] - [-\frac{7}{2}] = 4.$

(c) Let
$$g(x) = \int_{x}^{x^2} f(t)dt$$
. Compute $g'(2)$. (3 pts)

Suppose F(t) is an antiderivative of f(t) (ie F'(t) = f(t)). Then by the first fundamental theorem of calulus, $g(x) = F(x^2) - F(x)$. By the chain rule, $g'(x) = 2xF'(x^2) - F'(x) = 2xf(x^2) - f(x)$. Thus g'(2) = 4f(4) - f(2) = (4)(2) - 6 = 2.

(a) Determine whether each of the following improper integrals converges or diverges. You do not necessarily need to compute the values of the integrals, but you must give a rigorous justification of your answer. You must show your work to receive full credit. (4 pts each)

(a)
$$\int_{2}^{\infty} \frac{x^{2} + 1}{x^{3} - x^{2} - 1} dx$$

We note that for $x \ge 2$,
$$\frac{1}{x} = \frac{x^{2}}{x^{3}} < \frac{x^{2} + 1}{x^{3} - x^{2} - 1}.$$

(We see this either by cross multiplying OR by nothing that the function we wish to integrate has both larger numerator and smaller denominator than $\frac{x^2}{x^3}$, and therefore the fraction is bigger).

Thus
$$\int_{2}^{\infty} \frac{x^{2}+1}{x^{3}-x^{2}-1} dx > \int_{2}^{\infty} \frac{1}{x} dx$$
. We know $\int_{2}^{\infty} \frac{1}{x} dx$ diverges by *p*-test, therefore $\int_{2}^{\infty} \frac{x^{2}+1}{x^{3}-x^{2}-1} dx$ also diverges by comparison.
(b) $\int_{2}^{\infty} \frac{1}{x \ln^{5}(x)} dx$
First, we set $w = \ln(x)$ (thus $dw = \frac{dx}{x}$). After performing the *w*-substitution, we have $\int_{2}^{\infty} \frac{1}{x \ln^{5}(x)} dx = \int_{\ln(2)}^{\infty} \frac{1}{w^{5}} dx$. By the *p*-test ($p = 5 > 1$), we conclude this integral converges. Note that the upper limit after *w*-sub is still ∞ because $\lim_{b\to\infty} \ln(b) = \infty$.

(b) For which values of p does $\int_{1}^{\infty} \frac{\sqrt{1+x^{p}}}{x^{p}} dx$ converge? Justify your answer. (4 pts) We note \underline{p}

$$\frac{\sqrt{1+x^p}}{x^p} > \frac{\sqrt{x^p}}{x^p} = \frac{x^{\frac{1}{2}}}{x^p} = \frac{1}{x^{\frac{p}{2}}}.$$

For $p \le 2$, we have by *p*-test $\int_1^\infty \frac{1}{x^{\frac{p}{2}}}$ diverges, so by comparison, $\int_1^\infty \frac{\sqrt{1+x^p}}{x^p} dx$ diverges for $p \le 2$.

We note for $x \ge 1$ and $p \ge 0$ (which is all we need to consider since we settled the case of $p \leq 2$ above), we have

$$\frac{\sqrt{1+x^p}}{x^p} \le \frac{\sqrt{x^p+x^p}}{x^p} = \frac{\sqrt{2x^p}}{x^p} = \frac{\sqrt{2}x^{\frac{p}{2}}}{x^p} = \frac{\sqrt{2}}{x^{\frac{p}{2}}}.$$

For $p > 2$, we have by *p*-test $\int_1^\infty \frac{\sqrt{2}}{x^{\frac{p}{2}}}$ converges, so by comparison $\int_1^\infty \frac{\sqrt{1+x^p}}{x^p} dx$ converges for $p > 2$.

7.

For p >

8. The pigs at the local farm are fed out of a trough that is in the shape of a trapezoidal prism. The height of the trough is 10 inches, the bottom has width 4 inches and the top has width 19 inches. The trough is 30 inches long. When it is time to feed the pigs, the farmer adds enough $slop^1$ to the trough to fill it to a height of 8 inches. Slop has a density of $.04\frac{lbs}{in^3}$.



(a) Compute the work required to pump the slop out the top of the trough. Include units.(6 pts)

Consier first the work to pump out a slice that lies h inches from the bottom of the trough and is Δh inches thick. The work to pump this slice out is its weight $(= \delta(\text{Volume}))$ times the distance it must travel (= 10 - h, because we must pump it to the top of the trough). The volume of the slice is $(30)(\Delta h)(w)$, where w is the width of the trough at height h. By looking at a trapezoidal face, we note² that w = 1.5h+4, thus the work done to pump the slice out is $(.04)(30)(\Delta h)(1.5h+4)(10-h)(inch)(lbs)$. Because we must pump slices of slop out that lie anywhere from the bottom (h = 0) to the top of the slop (h = 8), the total work required to pump out the slop is $\int_0^8 (.04)(30)(1.5h+4)(10-h)dh = 499.2(inch)(lbs)$

(b) Compute the force the slop exerts on one of the trapezoidal faces of the trough. Include units. (6 pts)

First we compute the force exerted on a slice that lies h inches from the bottom and is Δh inches thick. The force on a slice is the area of the slice $(A = w(\Delta h))$ times the pressure at that depth. The depth at a height h from the bottom is (8 - h), and the pressure at that depth is $\delta(8 - h) = .04(8 - h)\frac{lbs}{in^2}$. Thus the force on that slice will be $w(\Delta h).04(8 - h) = (1.5h + 4)(\Delta h)(.04)(8 - h)$. Because there is force exerted on the wall where ever there is slop (h = 0 to h = 8), the total force the slop exerts on the face of trough will be $\int_0^8 (1.5h + 4)(.04)(8 - h)dh = 10.24$ lbs.

¹Slop is a soup-like liquid fed to pigs as food.

²When h = 0, we have w = 4 and when h = 10, w = 19. Finding the line w = mh + b that fits this data gives w = 1.5h + 4. We know w should be a linear function of h because trapezoids have linear sides.

9. A 7 inch tall beer glass can be modeled by the curve $x = 1 + .05y - 0.2 \sin(y)$ rotated about the *y*-axis (*x* and *y* measured in inches).

(a) Set up, BUT DO NOT EVALUATE, an integral that gives the volume the glass can hold. (3 pts)

The glass is a solid of revolution, so its volume is the integral (over the appropriate bounds) of πr^2 , where r is the distance from the axis of revolution. Here this distance is x = [1 + .05y - .2sin(y)], thus $V = \int_0^7 \pi [1 + .05y - .2sin(y)]^2 dy$

(b) Lucy the ladybug is sitting on top of the rim of the beer glass when she decides to walk down. Suppose she walks straight down along the outside of the glass to the bottom. Write an integral expression for how far she must walk to reach the bottom. You do not need to evaluate this integral. (3 pts)



The formula for arc length of a curve is $\int_a^b \sqrt{1 + f'(y)^2} dy$. Here f(y) = 1 + .05y - .2sin(y), thus f'(y) = .05 - .2cos(y). Therefore, the arc length from the bottom to the top of the glass is $\int_0^7 \sqrt{1 + [.05 - .2\cos(y)]^2} dy$, which is the discance Lucy must walk.

10. Consider a metal sheet in the shape of the region bounded by the curves y = 2x + 1and $y = x^2 - 4x + 1$. Suppose the density at any given point is directly proportional to the distance from that point to the y-axis with constant of proportionality k.

(a) Compute the area of the plate. (3 pts)

First, we note that the curves intersect when $2x + 1 = x^2 - 4x + 1$, thus (by algebra) when $x^2 - 6x = 0$, thus when x = 0 and x = 6. Moreover, we note that the quadratic is concave up, therefore it must lie below the line for $0 \le x \le 6$. Thus, the height of the region at a distance x from the y axis is $6x - x^2$. Finally, the area of the plate is $\int_0^6 (2x + 1) - (x^2 - 4x + 1)dx = \int_0^6 (6x - x^2)dx = 36$.

(b) Compute the mass of the plate. (3 pts) The mass of a vertical slice that lies x away from the y-axis and is Δx wide is $Mass = (Area)(Density) = (width)(height)(density) = (\Delta x)(6x - x^2)(kx)$. Thus the mass of the plate is $\int_0^6 (kx)(6x - x^2)dx = 108k$.

(c) Let (\bar{x}, \bar{y}) be the coordinates of the center of mass of the plate. Compute \bar{x} . (3 pts)

The center of mass of the plate can be computed as $\frac{1}{MASS}[\Sigma(x)(\delta(x))(A(x))]$, where A(x) is the area of the the vertical slices of the plate described above. We already computed these, so $\bar{x} = \frac{1}{108k} \int_0^6 (x)(kx)(6x - x^2)dx = 3.6$.