Election Forensics: Vote Counts and Benford's Law *

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Abstract

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How can we be sure that the declared election winner actually got the most votes? Was the election stolen? This paper considers a statistical method based on the pattern of digits in vote counts (the second-digit Benford's Law, or 2BL) that may be useful for detecting fraud or other anomalies. The method seems to be useful for vote counts at the precinct level but not for counts at the level of individual voting machines, at least not when the way voters are assigned to machines induces a pattern I call "roughly equal division with leftovers" (REDWL). I demonstrate two mechanisms that can cause precinct vote counts in general to satisfy 2BL. I use simulations to illustrate that the 2BL test can be very sensitive when vote counts are subjected to various kinds of manipulation. I use data from the 2004 election in Florida and the 2006 election in Mexico to illustrate use of the 2BL tests.

Fraudulent elections and disputes about election outcomes are nothing new. Gumbel (2005) reviews the sorry history of deceit and electoral manipulation in America, going back to the dawn of the republic. Throughout the world, in old and new democracies alike, allegations of vote fraud frequently occur (Lehoucq 2003). One new element is voting technologies that make some familiar methods for physically verifying the accuracy of vote totals impossible to use. The advent of electronic voting machines means that often now there are no paper ballots to be recounted. To steal an election it is no longer necessary to toss boxes of ballots in the river, stuff the boxes with thousands of phony ballots, or hire vagrants to cast repeated illicit votes. All that may be needed nowadays is access to an input port and a few lines of computer code. To detect such manipulations is a difficult and urgent problem. In terms of legitimacy it is not clear whether the worse problem is that erroneous election outcomes may occur or that many may not believe that correct outcomes are valid.

In this paper I study a statistical method intended to help detect election fraud. Other methods, using regression-based techniques for outlier detection, have previously been proposed to help detect election anomalies (e.g. Wand, Shotts, Sekhon, Mebane, Herron, and Brady 2001; Mebane, Sekhon, and Wand 2001; Mebane and Sekhon 2004). The method described here is distinctive in that it does not require that we have covariates to which we may reasonably assume the votes are related across political jurisdictions. The method is based on tests of the distribution of the digits in reported vote counts, so all that is needed are the vote counts themselves. Being based on so little information, the method cannot in itself diagnose whether an anomaly it may flag is a consequence of fraud or of some other kind of irregularity. But, as I show, some patterns of fraud will cause the method to trigger. So the method is best understood as an indicator for places where investigations that use other kinds of information—for instance, audits of election administration records and manual ballot recounts—might best be targeted.

Part of the potential practical relevance of the digit-testing method is that situations in which little more than the vote counts are available may arise frequently in connection with actual election controversies. I study the application of the method to both precinct-level and voting machine-level vote tabulations. At the precinct level the method may be expected to be remarkably sensitive to many patterns of distortion in the vote totals, but due to a prevalent feature of the way voting machines (or voting booths) are often deployed, the digit-testing method is probably not useful for screening the totals recorded for individual machines or individual ballot boxes. Changing the way voters are assigned to machines might eliminate this limitation.

The digit-test method is based on the expectation that the second digits of vote counts should satisfy Benford's Law (Hill 1995). Benford's Law specifies that the ten possible second digits should not occur with equal frequency. A fundamental question is why we should expect Benford's Law to apply to vote count data. Even though some have proposed to use the second-digit Benford's Law distribution to test for fraudulent votes (e.g., Pericchi and Torres 2004), prominent election monitors have strongly disputed such proposals (Carter Center 2005). I suggest that a close focus on the act of casting each vote suggests a statistical model that very often produces counts with second digits that have the distribution specified by Benford's Law. As important, to match observed vote count data, the counts the model generates do not have first digits that satisfy Benford's Law. What we often have in vote count data is not precisely Benford's Law but a process that strongly resembles the Benford's Law distribution in the second digits it produces. For lack of a better designation I will use the acronym 2BL to refer to this second-digit Benford's Law-like distribution.

A behavioral focus on the individualized uncertainty in each person's voting decision that follows the lines of familiar behavioral models may be inappropriate when thinking about vote counts for the purpose of trying to decide whether the counts are fraudulent. Indeed, leaving aside questions of vote fraud, to the extent that the familiar kinds of behavioral models cannot in general produce vote counts with second digits that follow the 2BL distribution—and, in general, they cannot—the fact that vote counts do often satisfy 2BL is evidence that the familiar behavioral models do not describe the votes people actually cast.

Even if 2BL typically describes vote count data, it does not follow that deviations from 2BL indicate election fraud. I present the results of some simulation exercises that suggest a test based on the 2BL distribution can detect many kinds of fraud. The 2BL test is sensitive to some kinds of manipulation of vote counts but not to others. In some cases it is very sensitive.

I apply the 2BL test to data from electronic early voting and election day votes in three Florida counties (Broward, Miami-Dade and Pasco) in the 2004 general election,[1] and to data

[1]See Gronke, Bishin, Stevens, and Galanes-Rosenbaum (2005) for a discussion of early voting in Florida during the 2004 election.

from the 2006 Mexican national election. The Florida data include voting machine event log files that have labels identifying the precinct or ballot style and the voting machine for every ballot cast.[2] The Mexican data include vote counts resolved to the individual "casilla" (i.e., voting booth or ballot box).

Benford's Law and Vote Counts

Benford's Law specifies that in a collection of numbers the different digits should not occur with equal frequency. That is, each of the nine possible first significant digits $(1, 2, \ldots, 9)$ should not each occur one-ninth of the time, each of the ten possible second significant digits $(0, 1, \ldots, 9)$ should not each occur one-tenth of the time, and so forth. Instead, according to Benford's Law the first and second significant digits should occur with the frequencies shown in Table 1. Tests against Benford's Law have been promoted for use to detect fraud in forensic financial accounting (Durtschi, Hillison, and Pacini 2004). In the realm of vote count data the relevance of Benford's Law has been controversial. Pericchi and Torres (2004) use tests of the second digits of vote counts against the Benford's Law distribution to raise the prospect of fraud in the Venzuelan recall referendum of August 15, 2004. This charge is specifically denied in the Carter Center report (Carter Center 2005, 132–133), based on technical analysis reported in Brady (2005) and Taylor (2005).

*** Table 1 about here ***

Why should Benford's Law apply to vote count data? A fundamental result is that Benford's Law does not in general hold for data that are simply random (Raimi 1976; Hill 1995). This property is one basis for its proposed use in financial fraud detection. If someone uses numbers taken directly from a table of random numbers to fill out faked financial records, the digits will occur with equal frequency. The positive case for using Benford's Law with financial data relies on the supposedly complicated origins of financial data:

> "[D]ata sets follow Benford's Law when the elements result from random variables taken from divergent sources that have been multiplied, divided, or raised to integer powers. This helps explain why certain sets of accounting numbers often appear to

[2] It is not possible to match an individual vote record (an individual ballot image) with a particular voting transaction.

> closely follow a Benford distribution. Accounting numbers are often the result of a mathematical process. A simple example might be an account receivable which is a number of items sold (which comes from one distribution) multiplied by the price per item (coming from another distribution)." (Durtschi et al. 2004, 20–21)

The complexity rationale runs afoul of the way behavioral political scientists usually think about voting. Students of voting behavior have developed a repertoire of models built on the idea that each individual's vote choice is essentially a coin flip (i.e., a stochastic choice), with the election outcome being simply the result of adding all the coin flips together. For different people the probabilities of the various outcomes are different, and for some elections the coin many have more sides than two. But the overall vote counts are seen as merely the sum of all the different coin flip outcomes. Such a sum of random coin flips lacks the kind of complexity needed to produce the Benford's Law pattern in the vote counts' digits. Taking voter turnout decisions into account does not essentially change the basic coin flip idea. In this case, to produce the coin flip probabilities the probability that each person votes is multiplied by the conditional probability that the person makes a particular choice among the candidates or ballot initiative options.

One can see this standard behavioral perspective at work in the analysis used to support the conclusions reached about the Venezuelan referendum by the Carter Center. This is most explicit in the analysis reported by Taylor (2005). Taylor writes, "we use the multinomial model (4) of a 'fair election' and find that its significant digit distribution is virtually identical to the observed distribution, which is different than Benford's Law" (Taylor 2005, 22). Taylor also generates data using a Poisson model. As a general matter these two models are essentially the same—as Taylor (2005, 9) observes, the multinomial arises upon conditioning on the total of a set of Poissons. Neither has the complexity needed to produce digits that follow Benford's Law.

The kind of complexity that can produce counts with digits that follow Benford's Law refers to processes that are statistical mixtures (e.g., Janvresse and de la Rue (2004)), which means that random portions of the data come from different statistical distributions. There are some limits that apply to the extent of the mixing, however. If the number of distinct distributions is large, then the result is likely to be well approximated by some simple random process that does not satisfy Benford's Law. So if we are to believe that in general Benford's Law should be expected to describe the digits in vote counts, we need to have a behaviorally realistic process that involves mixing among a small number of distributions.

Another important issue concerns whether Benford's Law should be expected to apply to all the digits in reported vote counts. In particular, for precinct-level data there are good reasons to doubt that the first digits of vote counts will satisfy Benford's Law. Brady (2005) develops a version of this argument. The basic point is that often precincts are designed to include roughly the same number of voters. If a candidate has roughly the same level of support in all the precincts, which means the candidate's share of the votes is roughly the same in all the precincts, then the vote counts will have the same first digit in all of the precincts. Imagine a situation where all precincts contain about 1,000 voters each, and a candidate has the support of roughly fifty percent of the voters in every precinct. Then most of the precinct vote totals for the candidate will begin with the digits '4' or '5.' This result will hold no matter how mixed the processes may be that get the candidate to roughly fifty percent support in each precinct. For Benford's Law to be satisfied for the first digits of vote counts clearly depends on the occurrence of a fortuitous distribution of precinct sizes and in the alignment of precinct sizes with each candidate's support. It is difficult to see how there might be some connection to generally occurring political processes. So we may turn to the second significant digits of the vote counts, for which at least there is no similar knock down contrary argument.

Benford's Law Test Example

For an example that illustrates these ideas, consider an application to votes recorded in the 2004 general election in Miami-Dade County, Florida. I examine the votes cast for the Republican and Democratic candidates for president (George W. Bush and John F. Kerry) and for U.S. Senator (Mel Martinez and Betty Castor). I also examine the votes Yes or No for eight state consitutional amendments that appeared on the ballot in Florida in 2004. These amendments are described in Table 2. As can be seen from the statewide vote counts shown in Table 2, all eight amendments passed, and most passed by a wide margin. Only the vote regarding Amendment 4 was somewhat close.

*** Table 2 about here ***

Because we are examining the results for tests for several different votes, we should adjust the test level we apply to hypothesis tests to take into account the false discovery rate (FDR) (Benjamini and Hochberg 1995; Benjamini and Yekutieli 2005). I use the form of the FDR that assumes independence across tests. Benjamini and Hochberg (1995) define this FDR as follows. Let $t = 1, \ldots, T$, $T = 20$, index the votes for a candidate or for or against an Amendment, and let the significance probability of the test statistic for each vote be denoted S_t. Sort the values S_t from all T types of votes from smallest to largest. Let $S_{(t)}$ denote these ordered values, with $S_{(1)}$ being the smallest. For a chosen test level α (e.g., $\alpha = .05$), let d be the smallest value such that $S_{(d+1)} > (d+1)\alpha/T$. This number d is the number of tests rejected by the FDR criterion. If assumptions hypothesized to define the tests hold, then we should observe $d = 0$.

Using the precinct-level counts for votes cast on election day, Table 3 reports Pearson chi-squared statistics for two kinds of tests. First is whether the distributions of the first digits of the precinct vote counts for the major party candidates for president and for U.S. Senator and for the eight constitutional amendments on election day 2004 in Miami-Dade match the distribution specified by Benford's Law. Second is whether the first digits occur equally often. For Benford's Law tests of the first or second significant digits, let $q_{B_k i}$ denote the expected relative frequency with which the k-th significant digit is i. For $k = 1$, the $q_{B_1 i}$ values are the values shown in the first line of Table 1. Let d_{ki} be the number of times the k-th digit is i among the J precincts being considered, and set $d_1 = \sum_{i=1}^{9} d_{1i}$. The statistic for a first-digit Benford's Law (1BL) test is

$$X^2_{B_1} = \sum_{i=1}^{9} \frac{(d_{1i} - d_1 q_{B_1 i})^2}{d_1 q_{B_1 i}} .$$

For the test that first digits occur equally frequently, the test statistic is

$$X^2_{E_1} = \sum_{i=1}^{9} \frac{(d_{1i} - d_1/9)^2}{d_1/9} .$$

Assuming independence across precincts, these statistics may be compared to the χ^2-distribution with 8 degrees of freedom.[3] That distribution has a critical value of 15.5 for a .05-level test, and the critical value using $\alpha = .05$ for each test but taking the FDR with $T = 20$ into account is 23.8. Because all of the statistics reported in Table 3 greatly exceed that value, the hypothesis that the

[3]The consequences of dependence are unclear. In other contexts such dependence may tend to produce sample statistics that are either excessively large or excessively small relative to the nominal χ^2-distribution.

first significant digits follow a 1BL distribution may be handily rejected, as may be the hypothesis that the nine values (1–9) occur equally often.

*** Table 3 about here ***

In contrast, consider Table 4, the first two columns of which reports Pearson chi-squared statistics for tests of the distribution of the precinct vote counts' second significant digits. For $k = 2$, the q_{B_2i} values are the values shown in the second line of Table 1, and $d_2 = \sum_{i=0}^{9} d_{2i}$. The statistic for a second-digit Benford's Law (2BL) test is

$$X_{B_2}^2 = \sum_{i=0}^{9} \frac{(d_{2i} - d_2 q_{B_2i})^2}{d_2 q_{B_2i}} .$$

For the test that second digits occur equally frequently (2EL), the test statistic is

$$X_{E_2}^2 = \sum_{i=0}^{9} \frac{(d_{2i} - d_2/10)^2}{d_2/10} .$$

These statistics may be compared to the χ^2-distribution with 9 degrees of freedom (χ_9^2), which has a critical value of 16.9 for a .05-level test. Using $\alpha = .05$ for each test but taking into account the FDR with $T = 20$, the critical value is 25.5 (using $\alpha = .10$ the critical value is 23.6). The results give little reason to doubt that a 2BL distribution applies. Two of the twenty statistics are larger than 16.9, but no statistic is larger than 25.5. The largest $X_{B_2}^2$ value in the first column of Table 4 is 17.9. The results give some reason to doubt that 2EL describes the vote counts. The largest $X_{E_2}^2$ value in the second column of Table 4 is 25.3.

*** Table 4 about here ***

The remaining columns of Table 4 show that what works for precincts need not work for voting machines. The middle columns report the results of applying the tests to the vote counts on the election day voting machines. Noting that some voting machines recorded votes from more than one precinct, the last two columns show results from applying the tests to vote counts for each unique precinct-machine combination. Both forms of the analysis firmly reject the idea that 2BL describes the vote counts on election day voting machines in Miami-Dade.

Generating Counts that Satisfy the Second-digit Benford's Law

Is there a family of processes that are behaviorally plausible and that are capable of producing precinct-level vote counts that satisfy 2BL but not 1BL? Can we explain why such a process would produce precinct counts that satisfy 2BL but not machine counts that do so? In this section I consider the first question. In the next section I take up the question of machine counts.

There are at least two behaviorally plausible mechanisms that generate counts that satisfy 2BL but not 1BL. Both mechanisms feature mixtures of a small number of component distributions. Use of the 2BL test to detect election fraud or other vote count anomalies might be based on the idea that precinct-level vote counts observed in actual elections typically reflect the combined action of versions of these mechanisms.

The first mechanism has voters who make choices that are subject to small frequencies of mistakes. The frequency of making mistakes varies across precincts but is constant in each precinct. There are three types of voters: voters who intend to favor the referent alternative, voters who intend to oppose the alternative and voters who intend to choose at random among alternatives. All precincts have the same number of voters, but the proportion of voters of each type varies across precincts according to a function of a uniform distribution.

The second mechanism features precincts whose sizes vary according to a function of a uniform distribution across precincts. There are three types of voters. The propensity of the voters of each type to choose the referent alternative is arbitrary but constant across precincts. The proportion of voters of each type varies across precincts according to a function of normal distributions.

Here is an **R** (R Development Core Team 2003) function that implements an example of the first mechanism. The function generates counts for `nprecincts` simulated precincts, each containing `size` voters.

```
mechA <- function(size, nprecincts=500, mf=1/3, lgp=1, hgp=1, lb=4, ha=4) {
  lgb <- exp(lgp)/(exp(lgp)+exp(hgp)+1);
  hgb <- exp(hgp)/(exp(lgp)+exp(hgp)+1);
  mgb <-        1/(exp(lgp)+exp(hgp)+1);
  sapply(1:nprecincts, function(x){
    p3 <- c( rbeta(1,1/2,lb), mf, rbeta(1,ha,1/2) );
    q <- runif(1,0,1);
    pf <- c(q*lgb, mgb, (1-q)*hgb );
    sum(size * p3 * pf/sum(pf))
```

```
  })
}
```

For each simulated precinct, the vector `p3` contains three numbers. The first element of `p3`, drawn from the beta distribution $B(1/2, \texttt{lb})$, represents the proportion of voters who intend not to vote for the referent alternative but nevertheless do so. The third element of `p3`, drawn from the beta distribution $B(\texttt{ha}, 1/2)$, represents the proportion of voters who intend to vote for the referent alternative who in fact do so. The extent to which $\texttt{p3}[1] > 0$ and $\texttt{p3}[3] < 1$ reflects the extent to which voters of the respective type make consequential mistakes. The second element of `p3`, which is constant and set by the argument `mf`, indicates the proportion of voters who are choosing at random who select the referent alternative. The default value $\texttt{mf} = 1/3$ might reflect the situation where there are two alternatives—say to vote either Yes or No on a constitutional amendment—and each voter of the at-random type has an equal chance of either selecting one of those or not selecting either one. The vector `pf` determines the proportion of the voters in each precinct who are of each type. The expected proportions are set by the arguments `lgp` and `hgp` through a logistic function, but the proportion realized in each precinct depends on the uniformly distributed variable `q`, $\texttt{q} \sim U[0, 1]$. With the default values $\texttt{lgp} = \texttt{hgp} = 1$, the expected proportions of voters of each types are approximately $\texttt{pf}/\texttt{sum}(\texttt{pf}) = (0.366, 0.269, 0.366)$.

The following **R** function implements an example of the second mechanism. Again the function generates counts for `nprecincts` simulated precincts, but now the `size` parameter does not correspond to the number of voters in each precinct.

```
mechB <- function(size, nprecincts=500, mf=1/3, onen=1, twon=1, onev=1, twov=1) {
  sapply(1:nprecincts, function(x){
    p3 <- c(0,mf,1);
    onex <- rnorm(1, onen, onev);
    twox <- rnorm(1, twon, twov);
    pf <- c( exp(onex), 1, exp(twox) )/(exp(onex)+exp(twox)+1);
    q <- runif(1,0,1);
    c(q * size * sum(p3 * pf))
  })
}
```

The three numbers in the vector `p3` again represent the proportion of the voters of each type who vote for the referent alternative, but now these values are constant across precincts. In the current example the values correspond to no votes for the alternative from voters of the first type, all the votes for the alternative from voters of the third type, and `mf` of the middle type votes for the alternative. The vector `pf` again determines the proportion of the voters in each precinct who are of each type. Now these proportions vary across precincts through a logistic function with arguments determined by a pair of random normal variables. A uniformly distributed variable `q`, $\texttt{q} \sim U[0,1]$ determines the number of voters in each precinct, through the construction `q * size`, so the expected precinct size is $\texttt{size}/2$.. With the default values $\texttt{onen} = \texttt{twon} = 1$, the expected proportions of voters of each types are approximately $\texttt{pf} = (0.422, 0.155, 0.422)$.

The interpretations of the processes the two mechanisms specify are not completely orthogonal. In particular, the three types of voters featured in the second mechanism might be considered to be produced by assigning all of the first mechanism's first or third types who vote for the alternative to the second mechanism's third type, and assigning all of the first mechanism's first or third types who do not vote for the alternative to the second mechanism's first type. The essential difference between the mechanisms is that precinct sizes are constant in the first mechanism but they vary in the second mechanism. So in some respects the second mechanism might be considered a generalization of the first. Notice, however, that if the variation in `q` is eliminated in the second mechanism—e.g., by setting $\texttt{q} = 1/2$—then the resulting function no longer produces counts that satisfy 2BL.

The two mechanisms are supposed to represent the results of processes that happen at the instant each vote is cast. Of course, timing is not specified in either mechanism, so this supposition is a matter of interpretation. There may be a latency between the time the voter acts to cast a vote and the time the vote is effectively recorded. Voters who decide not to cast a vote for any of the alternatives presented for a particular office or ballot initiative may be deemed to have acted at the time of their final opportunity to cast a vote. The decision and choice processes that are usually the focus of behaviorally minded studies of voting are to be considered as determining the numbers of voters who are of each type in the various precincts. That is $\texttt{size} * \texttt{pf}/\texttt{sum}(\texttt{pf})$ in `mechA` and $\texttt{q} * \texttt{size} * \texttt{pf}$ in `mechB`. In this way the two mechanisms are compatible with a very wide range of models of and ideas about voter behavior. The mechanisms constrain such models very little, if at all, except that each mechanism requires in its own way substantial variation across precincts.

These two mechanisms produce counts that satisfy 2BL for a wide range of parameter values and precinct size specifications. Tables 5 and 6 show the results of a small Monte Carlo simulation exercise using function `machA`. The parameter denoted Size in the table refers to the `size` argument, which is the number of voters in each precinct. In these simulations I set `lb` = 500 and `ha` = 500, which represent very small voter error rates: the expected values for `p3`[1] and `p3`[3] are `p3`[1] = 0.000999 and `p3`[3] = 0.999. I set `mf` = 1/2. For the simulations of Table 5 I set `lgp` = 3, `hgp` = 2, and for Table 6 I set `lgp` = 2.5, `hgp` = 1. In each Monte Carlo replication there are 1000 simulated precincts.

*** Tables 5 and 6 about here ***

In most cases in Table 5, the simulated vote counts satisfy 2BL. The only exceptions are for Size values in the range from 1,400 to 1,800. In Table 6 the simulated vote counts satisfy 2BL for all the indicated Size values, from 500 up to 10,000. In both tables one can see that the second digits of the simulated counts very often deviate significantly from 2EL. While not reported in the tables, it is noteworthy that counts simulated using `mechA` never satisfy 1BL.

Simulations using function `mechB` also produce counts that satisfy 2BL for a wide range of parameter values and `size` specifications. Indeed, if the normal distribution variance terms `onev` and `twov` are sufficiently large, the function almost always produces 2BL counts. The indicated default values `onev` = `twov` = 1 are sufficiently large to produce this effect. Counts simulated using `mechB` never satisfy 1BL.

Second-digit Benford's Law and Voting Machine Vote Counts

The mechanisms that generate 2BL counts at the precinct level do not do so at the level of voting machines because the way voters are often assigned to machines makes the counts on the different machines used for a precinct very similar for most of the machines but slightly or very greatly different for a few of them. The mechanism that produces this effect might be thought of using the phrase "roughly equal division with leftovers" (REDWL).

To see the general idea I mean to capture by REDWL, consider the plots of voting machine usage times shown in Figure 1. The plots show the times at which votes were cast on each voting machine on election day in four Miami-Dade precincts. Each row of letters in each plot indicates the time at which a "vote cast" transaction occurs for a voting machine in the event log files, with a letter being plotted at each point when a vote was recorded. There is one row of letters for each voting machine used in each precinct. Times are shown using a 24-hour clock and resolved to the second. In precinct 109, most of the machines were used throughout the day, but the machine labeled “e” was not used after 10am, and the machine labeled “k” was used much less often in the afternoon than in the morning. A reasonable guess is that the machine was pulled from service at that time. In precinct 233, the machine labeled “c” was not used after 8am, and the machine labeled “f” was not used before 2pm. In precinct 322, the machine labeled “b” was used only between 11:30am and 2:30pm. In precinct 326, the machines labeled “g” and “m” were used only after 1pm, and the machine labeled “p” was used heavily only after 6pm.

*** Figure 1 about here ***

These and other plots of machine usage times I have examined suggest the machines might be separated into two categories, namely machines that are fully occupied pretty much throughout election day versus the rest. The set of machines that are not fully occupied throughout the day might be further subdivided into a set that is used heavily during some periods but only sporadically at other times, and a set that is not use at all throughout much of election day. In any case, REDWL reflects the idea that most of the votes in a precinct are recorded on a subset of the machines, with the votes being roughly equally distributed among those machines, while the remaining votes are scattered among the rest of the machines. A minimal indicator that REDWL is a concern is that the total number of votes cast on some machines is similar among the machines and noticeably higher (or lower) than the number cast on the rest of the machines.

A simple modification of one of the mechanisms previously seen to generate precinct-level 2BL counts illustrates how the REDWL mechanism tends to produce voting machine counts that do not satisfy 2BL, even when the process that is taking place in the precinct as a whole does produce 2BL counts. Consider the following augmented version of `mechA`:

```
mechAm <- function(size, nprecincts=500, mf=1/3, lgp=1, hgp=1, lb=4, ha=4) {
  lgb <- exp(lgp)/(exp(lgp)+exp(hgp)+1);
  hgb <- exp(hgp)/(exp(lgp)+exp(hgp)+1);
  mgb <-        1/(exp(lgp)+exp(hgp)+1);
  pb <- ceiling(size/250);
  sapply(1:nprecincts, function(x){
    p3 <- c( rbeta(1,1/2,lb), mf, rbeta(1,ha,1/2) );
    q <- runif(1,0,1);
    pf <- c(q*lgb, mgb, (1-q)*hgb );
    sumv <- sum(size * p3 * pf/sum(pf))
```

```
    # allocate votes to the pb machines
    mbeta <- rbeta(pb, 20,20*pb);
    mbmean <- 1/(pb+1);
    mtrunc <- ifelse(mbeta < mbmean, mbeta, mbmean);
    sumv * mtrunc/sum(mtrunc);
  })
}
```

The `sumv` values are precinct counts generated exactly as in `mechA`. The votes for the alternative in each precinct are divided among `pb` machines, where `pb` is determined as a function of the `size` of each precinct. There is roughly one machine for every 250 voters. In each precinct, `pb` random (Beta-distributed) values `mbeta` are generated, where the expected value of `mbeta` is $\texttt{mbmean} = 1/(1+\texttt{pb})$. Values of `mbeta` greater than `mbmean` are set equal to the mean value, while values of `mbeta` smaller than `mbmean` are left unchanged. Each precinct's votes for the alternative are assigned to voting machines in proportion to this vector of truncated values.

Running a small Monte Carlo simulation exercise using function `machAm` with the same parameters that were used to compute the simulations reported in Table 6 shows that the REDWL mechanism in most cases destroys the 2BL property of the simulated counts. Results of this simulation are reported in Table 7. In contrast to the results in Table 6, the results in Table 7 show departures from 2BL for most precinct sizes. The only exceptions are for Size values $\texttt{size} = 500$ and $\texttt{size} = 700$. Running the simulation with different numbers of voting machines per voter does not materially change the results. Similar results are also obtained if the largest machine counts are not constrained to be exactly the same. For instance, we get similar results if instead of defining

```
    mtrunc <- ifelse(mbeta < mbmean, mbeta, mbmean);
```

we use

```
    mtrunc <- ifelse(mbeta < mbmean, mbeta, runif(pb,.95*mbmean,mbmean));
```

In this case the top machine proportions do not all have the value `mbmean` but instead vary uniformly on the interval $[.95 * \texttt{mbmean}, \texttt{mbmean}]$.

*** Table 7 about here ***

I conclude from such demonstrations that while under normal circumstances we might expect vote counts from precincts to satisfy 2BL, we should not expect vote counts from voting machines to do so, if several voting machines are typically used in each precinct and if the total number of votes cast on each machine exhibits signs of the REDWL mechanism.

Can the Second-digit Benford's Law Detect Election Fraud?

Do relatively large $X^2_{B_2}$ values for precinct-level vote counts suggest the counts have been fraudulently manipulated? The simulations reported in Tables 5 and 6 suggest that an electorally intelligible and benign process can produce counts that often satisfy the 2BL. Suppose we take a process that we know usually produces such counts and perturb it in ways that mimic some ways vote fraud may occur. Does the 2BL test signal that there has been a distortion? If so, we might conclude that the relatively large $X^2_{B_2}$ values suggest that maybe there has been fraud. Because significant perturbations may occur in the absence of fraud, such a result can do no more than suggest the possibility of fraud. But if the 2BL test does not catch perturbations that we inject into otherwise pristine data, then of course the test is not useful for detecting vote fraud. In this case clean precinct-level results should not give us any comfort.

I present illustrative simulations regarding 12 kinds of manipulations of vote counts. The simulations use a rubric of votes being switched from one candidate to the another, although the vote-switching perspective is not necessary for the results to be meaningful. In cases where a candidate is simulated as gaining votes, the added votes might in some cases be considered not to come from another candidate but simply to be introduced independent of anything that is happening to other candidates' votes. Such a reinterpretation may be especially appropriate in relation to the simulations that use what I refer to as "repeaters," because in those cases the number of votes being added does not depend on the number of votes the other candidates are receiving. In cases where a candidate is simulated as losing votes, the added votes might be considered as simply disappearing. This might correspond to situations where there are intentionally or accidentally spoiled or lost ballots.

Table 8 gives an overview of the simulated manipulations. The simulations all feature two candidates in a simulated election where the expected outcome is a tie. Votes may be added to or subtracted from a candidate, with consequently the same number of votes being subtracted from or given to the other candidate. The vote shifts may occur in all precincts, only in those precincts where the first affected candidate receives more votes than expected or only in those precincts where the first affected candidate receives fewer votes than expected. The votes the first affected candidate receives (or loses) may be either a proportion of the votes the other candidate would otherwise have received or a proportion of the total number of voters in the precinct. These latter cases are the ones I describe as having "repeaters."

*** Table 8 about here ***

My conception of repeaters harks back to the classic manipulation Gumbel (2005) describes as having been perfected by several American city political machines in the late nineteenth and early twentieth centuries. Repeaters in the nineteenth century's Tammany Hall were the primary referents of the familiar phrase, "vote early and often." As Gumbel writes, "The repeaters carried changes of clothing, including several sets of coats and hats, so they could plausibly come forward a second or third or fourth time in the guise of an entirely new person.... Many of the repeaters sported full beards at the beginning of the day, only to end it clean-shaven" (Gumbel 2005, 74). Nowadays repeaters might simply be a few lines of computer code.

The simulations are based on the first mechanism that produces 2BL counts. The following **R** function gives the simulations' typical form.

```
mechA2p <- function(size, nprecincts=500, mf=1/3, lgp=1, hgp=1, lb=4, ha=4, fa=0) {
  mp <- meanAp(lgp=lgp, hgp=hgp, mf=mf, lb=lb, ha=ha);
  lgb <- exp(lgp)/(exp(lgp)+exp(hgp)+1);
  hgb <- exp(hgp)/(exp(lgp)+exp(hgp)+1);
  mgb <-        1/(exp(lgp)+exp(hgp)+1);
  sapply(1:nprecincts, function(x){
    p3mat <- matrix(c(rbeta(1,1/2,lb), mf, rbeta(1,ha,1/2),
                      rbeta(1,ha,1/2), mf, rbeta(1,1/2,lb)),
                    2, 3, byrow=TRUE );
    p3mat <- apply(p3mat,2,function(v){ ifelse(sum(v)>c(1,1), v/sum(v), v) });
    p3 <- c( rbeta(1,1/2,lb), mf, rbeta(1,ha,1/2) );
    q <- runif(1,0,1);
    pf <- c(q*lgb, mgb, (1-q)*hgb );
    y <- c(size*p3mat %*% pf/sum(pf));
    chg <- ifelse(y[1] > size*mp, fa*y[2], 0);
    y <- y + c(chg,-chg);
    ifelse(y < 0, 0, y);
  })
}
```

For each simulated precinct the function computes vote counts for two candidates, the votes for each having the attributes of counts produced by `mechA`. A function `meanAp` computes the proportion of the votes the first candidate is expected to receive, and the resulting expectation is assigned to mp. The line

```
chg <- ifelse(y[1] > size*mp, fa*y[2], 0);
```

defines a change value for each precinct that equals a proportion fa times the votes received by the second candidate if the first candidate's vote count is greater than the expected count size*mp, and otherwise the change value equals zero. In the case where the vote shifts occur in all precincts, the expectation is ignored and the change value is defined by

```
chg <- fa*y[2];
```

The value fa*y[2] corresponds to the proportional vote switch scenario. For the repeater scenario that value is replaced with fa*size. To define a simulated election that would be expected to be a tie (i.e., very close) in the absence of any manipulations, I set the parameter values size = 2500, lgp = 2, hgp = 2, mf = 1/3, lb = 4 and ha = 4.

The first set of results, reported in Table 9, shows what happens when votes are proportionally added to the first candidate in all precincts. I present results for, respectively, 500, 1,000 and 2,000 simulated precincts. The fraction of votes shifted ranges from zero (no manipulation) to a substantial fa = .15. In this scenario the 2BL test does not detect that votes have been added to the first candidate. The 2BL test starts to signal the losses being suffered by the second candidate only when they are fairly substantial and the number of precincts is large. With 2,000 precincts and a switched fraction of seven percent or smaller, the average of $X^2_{B_2}$ is smaller than the critical value for χ^2_9 for a test at level $\alpha = .05$, which is 16.9. But $X^2_{B_2}$ is expected to exceed that critical value if fa $\geq .08$.

*** Table 9 about here ***

Table 10 shows that having the proportional additions occur only in precincts where the first candidate's support is either particularly strong or particularly weak makes the vote switching much more susceptible to detection by the 2BL test. With 2,000 precincts, the 2BL test tends to be triggered by the first candidate's vote counts when as few as three percent of the votes are being switched. With 500 precincts the 2BL test tends to be triggered when six or seven percent of the votes are being switched. Tests of the second candidate's vote counts are substantially less sensitive when the votes are being taken from that candidate in precincts where the other candidate is strong, and somewhat less sensitive when the votes are being taken in precincts where the other candidate is weak.

*** Table 10 about here ***

Table 11 shows that the 2BL detectability of repeater additions in all precincts is similar to the detectability of proportional additions. But in Table 12 it is apparent that repeater additions that occur in the first candidate's areas of either relative strength or relative weakness are more readily detected than the corresponding proportional additions are. With 2,000 precincts, switching as little as two percent of the vote in the affected precincts tends to produce $X^2_{B_2}$ values for the first candidate's vote counts that are greater than the $\alpha = .05$ critical value for χ^2_9. Tests of the second candidate's vote counts are somewhat less sensitive.

*** Tables 11 and 12 about here ***

Table 13 shows that proportional substractions in all precincts are if anything slightly less detectable than the corresponding proportional additions are. Even with 2,000 precincts, such subtractions do not tend to produce large $X^2_{B_2}$ values for either candidate. As Table 14 shows, however, having the proportional subtractions occur only in precincts where the first candidate's support is either particularly strong or particularly weak makes the vote switching much more susceptible to detection. With 2,000 precincts, the 2BL test triggers with as little as a two percent subtraction from the counts in the first candidate's strongest precincts.

*** Tables 13 and 14 about here ***

Tables 15 and 16 show that the 2BL detectability of repeater substractions is comparable to that of repeater additions. Indeed, Table 16 shows that with 2,000 precincts a repeater subtraction amounting to one percent of the voters is detectable by a 2BL test of the first candidate's vote counts, if the first candidate's strongest precincts are the ones affected. Detectability is only slightly less when the manipulations are directed at the first candidate's weakest precincts.

*** Tables 15 and 16 about here ***

On the whole the simulations suggest that the 2BL test can be highly sensitive to additions or subtractions from candidates' precinct-level vote totals. Sensitivity in the considered scenario of an extremely close election is generally weaker when manipulations equally affect all precincts.

For such pervasive changes in vote counts, the 2BL test is sometimes not at all sensitive given the number of precincts considered here. But when manipulations are concentrated in either one candidate's strongest or weakest precincts, the 2BL test can produce significant results when even very small shares of the vote are switched. In some instances, with 2,000 precincts, changes are detectable when they are as small as one or two percent of either the votes the other candidate received or the number of voters in each precinct. The 2BL test can in this sense sometimes—not always—detect changes in vote counts that are just large enough to change the outcome of a very close election.

More Data from Florida 2004

Data from the 2004 election in other Florida counties repeat the patterns found for Miami-Dade. Table 17 provides an overview of the kind of data to be considered. The table shows the number of precincts in Miami-Dade, Broward and Pasco counties. In 2004, Miami-Dade, Broward and Pasco counties all used Election Systems & Software "iVotronic" electronic voting machines (see Electronic Frontier Foundation (2004)). On election day, some voting machines were used to record votes from more than one precinct. This occurred in cases where more than one precinct shared a polling place. Most voting occurred on election day, November 2, 2004, but votes were also cast during a 15-day early voting period (October 18 through November 1, 2004). Table 17 also shows the number of early voting sites used in each county (earlyvoting.org 2004; Miami-Dade County 2004; Browning 2004). In Broward and Pasco, voters from all precincts could vote at any early voting site. In Miami-Dade, voters from each precinct could vote only at selected early voting locations. At early voting sites in Miami-Dade, each voting machine was used for voters from multiple precincts. The voting data for the early voting period do not directly indicate the voter's precinct but instead indicate which of several ballot styles the voter used. Table 17 shows the number of styles used during early voting for each county. The event log files do not contain any indication of the physical location where each voting machine was used during the early voting period. I use Personal Electronic Ballot (PEB)[4] codes recorded in the event log files to group machines together, the idea being that machines for which the same PEB

[4]For a description of how PEBs are used in Election Systems & Software "iVotronic" voting machines, see (Electronic Frontier Foundation 2004).

was used must have been located at the same early voting site.

*** Table 17 about here ***

A concern with early voting is that not every voting machine was used every day during the early voting period. I use the event log files to identify the dates during the early voting period when each voting machine was used. I group machines together only if they were used on all the same days. The "site-days" entries in Table 17 show the number of unique combinations of the PEB-based location groupings with these date goupings in Broward and Pasco, and the "style-site-days" entry shows the number of unique combinations of the PEB-based location and ballot style groupings with the date goupings in Miami-Dade. These serve as the "precincts" for the early voting vote counts. The "site-day-machines" and "s-s-d-machines" (for "site-style-day-machines") entries show the number of unique combinations of the site-days or style-site-days groupings with voting machines.

Applying the 2BL test to other vote count data from the 2004 election in Florida mostly suggests that 2BL applies to the data, but a few results raise questions. Table 18 reports results based on data from early voting in Miami-Dade. Applying the FDR to the twenty tests for site-style-days, the results look fine if we use a single-test level of $\alpha = .05$, since no $X^2_{B_2}$ value is greater than 25.46, but the results are problematic if $\alpha = .10$: $X^2_{B_2}$ for the Amendment 7 Yes votes is 24.6, which is greater than 23.6. The election day precinct results for Broward, shown in Table 19, are similar. They are fine using the FDR with $\alpha = .05$ but problematic using $\alpha = .10$: two of the Amendment vote counts have $X^2_{B_2} > 23.6$. The Broward early voting results for counts at the level of ballot styles are fine if the FDR is used. The largest $X^2_{B_2}$ value among these early voting tests is $X^2_{B_2} = 21.4$, for the votes for Kerry. The election day results for Pasco, shown in Table 20, have one value of $X^2_{B_2}$ large enough to reject the hypothesis that 2BL applies even using the FDR among the twenty tests with $\alpha = .05$. This is the value $X^2_{B_2} = 29.5$, which occurs for the Amendment 7 Yes votes. Considered on their own and using the FDR for twenty tests, the early voting machine-precinct results for Pasco are fine.

*** Tables 18, 19 and 20 about here ***

The results for voting machines in Tables 18, 19 and 20 further illustrate that the 2BL property mostly does not apply to the vote counts on machines in these Florida counties. The case that comes closest to being an exception is the machine results for early voting in Broward.

Many of those $X^2_{B_2}$ values are unproblematically small, but three are larger than the χ^2_9 critical value for a single test at level $\alpha = .05$, and two are large even when we use the FDR. For the Amendment 8 Yes votes we have $X^2_{B_2} = 27.9$, which is larger than the critical value for the FDR for twenty tests with $\alpha = .05$, and for the Amendment 7 Yes votes we have $X^2_{B_2} = 44.0$, which is very large by any standard.

The value $X^2_{B_2} = 29.5$ that occurs for the election day precinct data from Pasco is large enough to count as a rejection of the 2BL hypothesis even using the FDR among all 60 of the election day tests, pooling across the three counties: the quantile of χ^2_9 corresponding to a tail probability of .05/60 is 28.35. Pooling over all 120 of the election day precinct and early voting site-style-day, style and machine-precinct tests, the value $X^2_{B_2} = 29.5$ is not problematic according to the FDR with $\alpha = .05$, since the quantile of χ^2_9 corresponding to a tail probability of .05/120 is 30.13. But using $\alpha = .10$ we again have a problem even when pooling over all 120 tests, because using the FDR we again arrive at the χ^2_9 quantile of 28.35.

Data from Mexico 2006

The 2006 election in Mexico has proven to be highly controversial. A very close outcome between the top two finishers in the presidential election led to calls from Democratic Revolution Party (PRD) candidate Andrés Manuel López Obrador for a manual recount of all 41 million ballots. The declared winner, National Action Party (PAN) candidate Felipe Calderón, officially has 15,000,284 votes while López Obrador has 14,756,350. As of the time of this writing Calderón has declared willingness to support a recount if that is ordered by the Federal Electoral Tribunal (TRIFE), but expresses doubt that the results would change or that a recount of all the ballots is in any case necessary (Associated Press 2006b,a). Other parties receiving votes in the election include the Institutional Revolutionary Party (PRI), the Social-Democratic and Rural Alternative Party (ASDC) and the New Alliance Party (NA). In the presidential voting these parties received, respectively, 9,301,441 (22.3 percent), 1,127,963 (2.7 percent) and 401,804 (1.0 percent) votes.

Here I present 2BL test results for votes at two levels of aggregation, "secciones," which correspond to precincts, and "casillas," which correspond to voting machines.[5] The Mexican

[5] I downloaded official vote count data from the website of the Instituto Federal Electoral (IFE), `http://www.ife.org.mx/`, on July 13, 2006.

election used paper ballots, so these "machines" in fact correspond to the boxes into which voters placed their ballots. Typically there are multiple casillas for each seccion. In each seccion voters are assigned to casillas based on each voter's name, not haphazardly as often occurs in elections in the United States. Inspecting the total numbers of ballots cast at each casilla shows patterns that suggest the REDWL mechanism should be a concern. Hence the 2BL test is likely to be appropriate for seccion-level vote counts but not for casilla-level counts.

Table 21 shows seccion-level 2BL test results. The first row of the table reports results considering all the secciones from across the whole country as one set. The $X^2_{B_2}$ statistics are larger than the χ^2_9 critical value for test level $\alpha = .05$ for four of the five parties. For ASDC the statistic is larger than the critical value for test level $\alpha = .1$. The remaining rows of the table show results separately for the secciones in each of Mexico's 32 states. In this case, taking into account the FDR for 32 separate tests gives an adjusted critical value of 26.7 for level $\alpha = .05$ and a value of 24.9 for level $\alpha = .1$. The FDR for all 160 separate tests gives critical values of 30.9 and 29.1 respectively for levels $\alpha = .05$ and $\alpha = .1$. For each party there are $X^2_{B_2}$ statistics larger than these FDR-adjusted critical values. For the two largest parties areas in Mexico city (the state Mexico) are particularly problematic, but a few other areas also show high values for $X^2_{B_2}$. Notably, for all but the smallest party (NA), most of the $X^2_{B_2}$ values are smaller than even the single-test critical values.

*** Table 21 about here ***

Table 22 shows that an analysis at the casilla level conveys a very different impression. Now most of the $X^2_{B_2}$ values are large. The difference between the 2BL tests for the casilla and seccion vote counts mirrors the difference observed in the Florida counties between the voting machine and precinct vote counts. In both cases the REDWL mechanism is the most likely explanation for why the lower level of aggregation produces systematically worse results. The worse casilla results therefore probably do not signal more widespread or more serious problems than the seccion-level results do.

*** Table 22 about here ***

The 2BL test results for secciones certainly suggest there are problems with the 2006 presidential vote counts in many Mexican states, although probably not in most of them. More refined analysis is needed to reach sharper conclusions, but the general impression is that more intensive investigation of the election results is in order. That might include doing a manual recount of many—perhaps all—of the individual ballots. A cost efficient method may be to begin by recounting a random sample of the ballots—all the ballots in a sample of secciones—where the probability that a seccion is selected for recounting is greater in places where the 2BL test results are worse. For such an exercise it may be reasonable to conduct 2BL tests for secciones collected into sets that correspond to the legislative districts they are part of, with sampling for purposes of initial recounting done at the level of districts. Perhaps a two-stage sampling plan could be used, with districts selected at the first stage (weighted by the 2BL test results) and secciones within each district selected at the second stage. If such an initial sampling did identify problems with the vote tabulations, then the case for a comprehensive manual recount would become extremely strong.

Discussion

Tests based on the second-digit Benford's Law show strong promise to become a standard tool for detecting fraudulent election results. The 2BL test cannot detect all kinds of fraud, and significant 2BL test results may occur even when vote counts are in no way fraudulent. But one should perhaps not expect too much from a test that has only the vote counts themselves to work with.

An important apparent limitation of the 2BL test is that it seems not to be suitable for checking voting machine level counts, at least not when the way voters are assigned to machines brings the REDWL mechanism into play. An idea worth investigation is whether strict random assignment of voters to machines can avoid the REDWL mechanism, so that 2BL tests would be informative at the level of individual voting machine counts. The few simulations I have conducted so far suggest that such an approach might be effective.

Data Note

David Dill supplied ballot and event log files recovered from electronic voting machines in Broward, Miami-Dade and Pasco counties. The files were originally obtained by Martha Mahoney. The ballot files indicate the choices made for each office by each voter and include labels identifying for each ballot the voting machine and the precinct (for election day ballots) or ballot style (for early voting ballots). The event log files show the time (resolved to the second) at which various transactions occurred on each machine, including the time at which each vote was recorded. It is not possible to match vote choices in the ballot files to voting events in the event log files.

Early voting polling site locations for many of the Miami-Dade machines was taken from a file supplied by Martha Mahoney (file "ev.xls," received by me on August 16, 2005) that was obtained using open records requests funded by the Verified Voting Foundation. Of the 670 machines that recorded votes during early voting in Miami-Dade, 88 are not included in that file. Two files supplied by Martha Mahoney also were used to determine which Miami-Dade machines were operating with audio capability enabled. These are the "ev.xls" file and a file "Election.xls" (received by me on August 16, 2005) for the machines used on election day.

The data comprise files for electronic early voting and electronic polling place votes but do not include information about paper absentee votes.

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Table 1: Frequency of Digits according to Benford's Law

digit	0	1	2	3	4	5	6	7	8	9
first	—	.301	.176	.124	.097	.079	.067	.058	.051	.046
second	.120	.114	.109	.104	.100	.097	.093	.090	.088	.085

Table 2: Florida Constitutional Amendments on the Ballot in 2004

		Yes	No
Am. 1	Parental Notification of a Minor's Termination of Pregnancy	4,639,635	2,534,910
Am. 2	Constitutional Amendments Proposed by Initiative	4,574,361	2,109,013
Am. 3	The Medical Liability Claimant's Compensation Amendment	4,583,164	2,622,143
Am. 4	Authorizes Miami-Dade and Broward County Voters to Approve Slot Machines in Parimutuel Facilities	3,631,261	3,512,181
Am. 5	Florida Minimum Wage Amendment	5,198,514	2,097,151
Am. 6	Repeal of High Speed Rail Amendment	4,519,423	2,573,280
Am. 7	Patients' Right to Know About Adverse Medical Incidents	5,849,125	1,358,183
Am. 8	Public Protection from Repeated Medical Malpractice	5,121,841	2,083,864

Note: Yes and No vote counts show statewide results.

Table 3: Miami-Dade Election Day First-digit Benford's Law Tests

item	Benf.	equal	item	Benf.	equal
Bush	29.3	292.5	Am. 4 Yes	144.8	367.0
Kerry	39.9	287.0	Am. 4 No	119.6	605.6
Martinez	35.6	273.8	Am. 5 Yes	115.4	122.2
Castor	22.0	304.7	Am. 5 No	27.6	623.4
Am. 1 Yes	86.2	290.5	Am. 6 Yes	98.8	395.0
Am. 1 No	80.5	636.2	Am. 6 No	84.0	532.9
Am. 2 Yes	95.6	362.5	Am. 7 Yes	130.3	112.7
Am. 2 No	60.0	722.7	Am. 7 No	49.9	582.8
Am. 3 Yes	60.5	401.3	Am. 8 Yes	123.0	210.6
Am. 3 No	51.5	496.5	Am. 8 No	102.6	831.1

Note: $n = 757$ precincts. Each statistic is the Pearson chi-squared statistic, with eight degrees of freedom.

Table 4: Miami-Dade Election Day Second-digit Benford's Law Tests

item	precincts ($n = 757$) Benf.	precincts ($n = 757$) equal	machines ($n = 5,326$) Benf.	machines ($n = 5,326$) equal	precinct-machines ($n = 7,064$) Benf.	precinct-machines ($n = 7,064$) equal
Bush	7.9	10.8	28.0	20.5	17.2	39.5
Kerry	9.5	14.4	61.8	10.0	44.0	13.1
Martinez	8.9	10.8	33.4	11.9	11.5	29.2
Castor	12.0	12.8	44.5	15.6	12.7	43.5
Am. 1 Yes	2.5	8.0	72.4	10.3	43.6	12.6
Am. 1 No	5.5	15.5	73.9	9.2	19.8	31.9
Am. 2 Yes	16.7	23.6	68.5	3.5	38.7	27.3
Am. 2 No	7.2	16.4	49.5	17.3	11.9	48.8
Am. 3 Yes	3.3	8.5	98.4	9.2	78.0	5.5
Am. 3 No	12.9	12.7	76.9	9.0	25.7	26.8
Am. 4 Yes	3.3	9.0	49.1	5.8	43.5	14.4
Am. 4 No	5.7	15.4	89.5	5.4	25.4	15.3
Am. 5 Yes	17.9	19.6	81.4	3.9	57.6	2.9
Am. 5 No	5.8	23.3	5.9	56.8	25.6	135.6
Am. 6 Yes	4.3	10.2	50.3	5.8	29.7	16.3
Am. 6 No	9.1	11.3	47.3	6.5	15.3	30.8
Am. 7 Yes	17.1	16.0	51.7	21.0	53.2	21.1
Am. 7 No	8.4	16.5	78.9	220.0	136.7	318.7
Am. 8 Yes	12.7	25.3	69.6	1.5	54.2	8.3
Am. 8 No	6.5	10.6	67.8	13.9	23.2	29.1

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom.

Table 5: 2BL Tests for Vote Counts Simulated Using the First Mechanism

Size	Benf.	equal	Size	Benf.	equal	Size	Benf.	equal
500	11.6	20.2	3,800	10.0	29.9	7,100	11.2	14.9
600	10.9	14.8	3,900	8.9	22.2	7,200	12.9	13.0
700	9.7	14.4	4,000	9.1	23.2	7,300	12.7	13.5
800	8.1	13.6	4,100	9.0	17.3	7,400	14.4	14.1
900	10.3	15.3	4,200	12.1	17.0	7,500	11.4	12.2
1,000	11.8	12.0	4,300	10.9	15.5	7,600	11.2	13.0
1,100	12.5	16.2	4,400	10.0	16.5	7,700	9.4	15.0
1,200	16.9	34.3	4,500	10.0	19.6	7,800	11.6	16.0
1,300	15.9	39.5	4,600	8.3	18.1	7,900	10.2	14.1
1,400	17.2	42.9	4,700	9.6	21.0	8,000	10.6	16.4
1,500	18.6	46.1	4,800	9.1	18.9	8,100	9.2	12.7
1,600	21.6	51.2	4,900	9.9	17.1	8,200	11.0	14.4
1,700	19.9	47.0	5,000	10.5	18.4	8,300	9.8	11.7
1,800	17.5	41.6	5,100	10.0	16.0	8,400	11.0	13.9
1,900	14.0	35.1	5,200	10.6	13.0	8,500	13.5	13.0
2,000	14.1	27.9	5,300	10.9	13.3	8,600	11.7	12.9
2,100	9.7	18.0	5,400	10.6	12.7	8,700	12.7	15.9
2,200	8.7	18.1	5,500	10.5	16.1	8,800	13.4	16.9
2,300	11.6	29.0	5,600	11.6	15.0	8,900	9.6	12.1
2,400	12.2	32.7	5,700	8.9	12.9	9,000	10.2	14.2
2,500	12.4	33.8	5,800	10.5	15.5	9,100	10.5	12.1
2,600	12.4	35.1	5,900	8.3	14.5	9,200	13.6	12.4
2,700	11.6	34.0	6,000	10.6	14.3	9,300	15.4	14.5
2,800	12.4	31.8	6,100	8.8	12.9	9,400	11.1	12.7
2,900	10.5	25.8	6,200	11.1	13.5	9,500	13.3	10.9
3,000	11.6	26.4	6,300	14.1	12.3	9,600	12.9	9.6
3,100	11.7	21.6	6,400	10.6	10.5	9,700	14.2	12.6
3,200	9.9	18.2	6,500	12.4	12.9	9,800	11.4	12.0
3,300	9.6	20.4	6,600	12.9	15.6	9,900	12.5	11.2
3,400	10.9	24.2	6,700	11.9	15.3	10,000	11.5	11.8
3,500	10.5	25.9	6,800	7.9	14.3			
3,600	9.0	25.0	6,900	10.1	16.8			
3,700	11.7	27.2	7,000	11.2	16.2			

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom, averaged over 25 Monte Carlo replications. Simulations use the `mechA` function with `size` equal to the values shown in the Size column, `nprecincts` = 1000, `lgp` = 3, `hgp` = 2, `mf` = 1/2, `lb` = 500, `ha` = 500.

Table 6: 2BL Tests for Vote Counts Simulated Using the First Mechanism

Size	Benf.	equal	Size	Benf.	equal	Size	Benf.	equal
500	10.3	22.5	3,800	11.3	18.7	7,100	8.3	15.7
600	9.5	18.1	3,900	9.2	17.7	7,200	9.1	17.1
700	10.0	15.7	4,000	12.2	19.6	7,300	8.9	19.6
800	9.0	19.6	4,100	10.5	20.0	7,400	9.3	18.0
900	10.0	13.2	4,200	10.4	19.5	7,500	7.8	18.1
1,000	9.7	15.7	4,300	9.1	18.4	7,600	7.9	18.1
1,100	10.4	13.4	4,400	10.2	16.1	7,700	9.1	22.0
1,200	12.0	15.9	4,500	12.3	17.5	7,800	10.9	21.1
1,300	12.3	27.2	4,600	9.9	14.4	7,900	8.7	17.6
1,400	13.4	35.2	4,700	11.2	20.0	8,000	9.0	17.7
1,500	13.8	35.5	4,800	9.6	20.9	8,100	11.4	17.4
1,600	13.9	38.6	4,900	8.6	21.7	8,200	9.1	16.9
1,700	15.4	42.8	5,000	8.8	22.6	8,300	10.4	14.8
1,800	14.0	38.2	5,100	9.1	25.2	8,400	9.1	16.2
1,900	13.6	37.0	5,200	9.7	23.1	8,500	9.1	14.8
2,000	12.4	34.4	5,300	10.5	27.0	8,600	9.5	17.5
2,100	10.7	27.2	5,400	9.2	24.1	8,700	9.6	13.8
2,200	9.3	21.9	5,500	10.1	20.9	8,800	9.9	14.9
2,300	8.1	18.6	5,600	10.9	17.2	8,900	9.3	14.9
2,400	10.5	24.0	5,700	9.8	15.8	9,000	9.2	14.1
2,500	11.2	27.7	5,800	9.2	16.6	9,100	10.2	14.7
2,600	9.4	30.0	5,900	9.6	15.2	9,200	11.0	13.9
2,700	12.3	31.7	6,000	9.2	15.4	9,300	10.0	15.2
2,800	12.4	29.3	6,100	8.3	16.9	9,400	10.2	13.9
2,900	12.9	29.6	6,200	9.2	17.3	9,500	8.1	14.3
3,000	13.3	26.8	6,300	8.9	16.0	9,600	10.8	20.3
3,100	11.6	23.3	6,400	9.9	14.4	9,700	9.7	15.5
3,200	13.0	23.4	6,500	10.8	16.6	9,800	10.1	16.4
3,300	12.3	16.7	6,600	11.1	17.3	9,900	9.5	16.0
3,400	11.4	15.2	6,700	8.8	15.3	10,000	10.2	18.2
3,500	12.4	11.7	6,800	8.8	17.3			
3,600	12.7	16.9	6,900	10.0	14.7			
3,700	9.9	18.7	7,000	8.7	15.6			

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom, averaged over 25 Monte Carlo replications. Simulations use the `mechA` function with `size` equal to the values shown in the Size column, `nprecincts` = 1000, `lgp` = 2.5, `hgp` = 1, `mf` = 1/2, `lb` = 500, `ha` = 500.

Table 7: 2BL Tests for Vote Counts Simulated Using the First Mechanism

Size	Benf.	equal	Size	Benf.	equal	Size	Benf.	equal
500	9.9	44.3	3,800	65.2	379.9	7,100	105.9	673.0
600	22.0	80.4	3,900	53.5	345.3	7,200	106.8	668.5
700	14.5	64.7	4,000	52.8	323.8	7,300	131.0	752.0
800	27.8	113.1	4,100	79.7	430.9	7,400	119.3	699.6
900	18.6	89.8	4,200	63.6	368.2	7,500	100.2	627.2
1,000	19.1	96.3	4,300	68.4	434.8	7,600	120.5	721.1
1,100	25.8	114.8	4,400	58.9	394.3	7,700	105.8	662.5
1,200	23.8	115.2	4,500	59.6	391.9	7,800	110.7	673.1
1,300	26.5	139.5	4,600	64.5	435.7	7,900	120.4	740.9
1,400	24.4	130.8	4,700	72.1	435.0	8,000	117.3	725.1
1,500	27.3	140.2	4,800	73.4	469.0	8,100	130.4	785.3
1,600	28.1	145.5	4,900	78.3	484.4	8,200	111.7	723.8
1,700	30.7	162.5	5,000	77.0	467.7	8,300	133.7	822.0
1,800	33.7	178.5	5,100	82.3	507.2	8,400	103.9	734.5
1,900	29.0	185.2	5,200	73.3	482.5	8,500	120.7	752.4
2,000	29.1	179.9	5,300	72.5	483.5	8,600	148.0	862.6
2,100	39.4	214.9	5,400	78.9	499.5	8,700	141.6	794.8
2,200	36.6	212.4	5,500	89.1	506.1	8,800	120.9	825.7
2,300	29.9	212.1	5,600	74.1	526.2	8,900	111.1	768.2
2,400	35.2	211.4	5,700	80.9	531.7	9,000	124.3	771.4
2,500	39.0	231.6	5,800	97.6	589.0	9,100	134.0	821.4
2,600	40.8	250.5	5,900	107.5	589.0	9,200	113.2	798.4
2,700	44.5	261.5	6,000	102.5	574.1	9,300	124.9	810.2
2,800	49.5	272.9	6,100	82.5	533.2	9,400	124.5	840.2
2,900	48.1	289.6	6,200	95.6	577.3	9,500	121.3	814.7
3,000	46.1	271.6	6,300	83.3	557.9	9,600	174.0	994.5
3,100	41.1	275.2	6,400	106.6	588.3	9,700	123.8	851.1
3,200	45.8	290.8	6,500	89.5	542.8	9,800	145.1	933.2
3,300	48.3	302.2	6,600	81.4	548.8	9,900	151.9	923.8
3,400	44.3	295.4	6,700	91.8	612.3	10,000	136.3	895.2
3,500	62.1	330.4	6,800	102.3	653.2			
3,600	64.4	357.3	6,900	100.7	646.7			
3,700	54.6	325.5	7,000	95.8	631.3			

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom, averaged over 25 Monte Carlo replications. Simulations use the `mechAm` function with `size` equal to the values shown in the Size column, `nprecincts` $= 1000$, `lgp` $= 2.5$, `hgp` $= 1$, `mf` $= 1/2$, `lb` $= 500$, `ha` $= 500$.

Table 8: Index of Vote Manipulation Simulations

	Add Votes		Subtract Votes	
Range of Application	Proportional	Repeaters	Proportional	Repeaters
All	Table 9	Table 11	Table 13	Table 15
Above Expected Value	Table 10	Table 12	Table 14	Table 16
Below Expected Value	Table 10	Table 12	Table 14	Table 16

Table 9: Simulated Proportional Vote Switching: Gains

	Receiver (cand. 1)			Donor (cand. 2)		
fraction	500	1000	2000	500	1000	2000
0	11.2	11.0	12.5	9.7	10.9	14.6
0.01	10.6	9.9	10.6	9.5	11.5	14.1
0.02	8.4	12.5	12.3	9.8	11.0	10.9
0.03	10.7	10.5	12.0	10.3	11.4	12.6
0.04	10.8	10.4	13.5	10.7	11.2	12.0
0.05	10.7	10.3	13.3	9.3	12.0	11.9
0.06	10.0	10.9	12.9	8.6	10.9	13.9
0.07	9.2	11.1	13.1	12.1	11.5	16.0
0.08	10.8	12.1	12.9	10.9	13.3	18.7
0.09	10.1	10.9	14.3	11.0	14.0	20.1
0.1	10.4	11.4	14.7	11.9	16.4	21.6
0.11	9.1	11.3	14.0	11.9	17.0	24.7
0.12	10.2	11.9	13.8	13.3	17.7	27.5
0.13	10.3	11.3	13.7	14.7	19.9	29.9
0.14	11.4	12.4	16.8	15.2	23.1	35.0
0.15	10.7	12.5	14.3	18.2	23.4	41.5

Note: Each statistic is the Pearson chi-squared statistic $X^2_{B_2}$, with nine degrees of freedom, averaged over 50 Monte Carlo replications.

Table 10: Simulated Proportional Vote Switching with Thresholds: Gains

receive votes when above expectation

fraction	Receiver (cand. 1) 500	1000	2000	Donor (cand. 2) 500	1000	2000
0	10.4	9.6	11.3	9.3	10.9	12.4
0.01	10.0	11.6	13.0	9.1	10.1	10.3
0.02	11.2	12.2	14.8	9.0	10.5	11.0
0.03	11.3	14.4	18.6	8.8	10.1	13.1
0.04	12.7	16.3	24.4	9.6	10.7	11.6
0.05	13.6	19.3	30.8	8.6	12.4	12.1
0.06	17.5	23.4	39.5	9.7	10.5	14.7
0.07	17.9	27.6	49.4	9.9	12.5	17.2
0.08	20.8	33.9	60.3	10.9	12.4	17.8
0.09	26.4	40.7	69.2	10.8	13.1	18.2
0.1	26.0	40.4	73.9	10.8	14.3	18.0
0.11	27.1	44.6	79.9	10.8	13.7	18.7
0.12	29.2	44.9	86.6	12.8	14.9	19.0
0.13	30.1	52.5	92.3	12.0	14.6	19.8
0.14	31.8	57.5	99.5	12.2	18.3	22.9
0.15	37.0	64.7	110.2	12.8	17.9	24.7

receive votes when below expectation

fraction	Receiver (cand. 1) 500	1000	2000	Donor (cand. 2) 500	1000	2000
0	9.6	11.2	9.6	9.8	9.9	12.4
0.01	9.9	11.2	13.2	10.2	10.8	13.9
0.02	9.9	11.1	16.3	11.2	11.1	15.8
0.03	10.4	15.6	21.4	11.2	12.5	17.3
0.04	13.1	17.8	25.7	11.0	14.0	17.8
0.05	15.6	20.7	32.7	11.8	14.1	18.8
0.06	16.0	27.4	39.9	12.2	15.7	22.3
0.07	18.9	32.0	51.6	12.6	17.5	24.9
0.08	20.9	39.5	67.1	12.6	20.3	28.6
0.09	25.7	42.7	74.9	14.8	19.8	33.5
0.1	26.7	45.9	77.2	16.1	23.1	39.6
0.11	27.4	45.1	86.3	18.9	26.0	42.8
0.12	29.9	47.6	89.4	18.3	29.7	51.3
0.13	32.5	52.6	94.5	21.4	35.5	62.8
0.14	32.8	54.2	110.0	24.7	39.2	72.8
0.15	36.0	60.0	114.2	27.8	43.1	79.7

Note: Each statistic is the Pearson chi-squared statistic $X^2_{B_2}$, with nine degrees of freedom, averaged over 50 Monte Carlo replications.

Table 11: Simulated "Repeater" Vote Switching: Gains

	Receiver (cand. 1)			Donor (cand. 2)		
fraction	500	1000	2000	500	1000	2000
0	10.6	10.6	11.0	10.5	11.3	11.3
0.01	9.5	10.3	13.0	10.0	11.1	12.4
0.02	11.5	9.2	12.8	10.5	10.3	13.2
0.03	9.9	11.5	14.3	10.6	11.4	11.9
0.04	10.4	10.7	14.6	9.2	11.6	14.1
0.05	11.3	10.2	13.5	10.8	9.8	14.1
0.06	11.0	12.6	15.0	9.5	10.8	16.0
0.07	10.3	12.0	15.1	9.4	11.4	14.5
0.08	10.5	12.7	17.6	10.6	11.8	17.7
0.09	9.6	13.2	17.6	11.3	13.4	20.5
0.1	11.0	13.3	17.9	11.4	13.3	17.1
0.11	11.1	13.1	19.2	11.6	15.7	22.5
0.12	10.8	15.7	17.8	13.1	17.5	22.0
0.13	10.5	13.5	19.9	13.8	17.1	26.9
0.14	11.1	14.0	19.1	14.4	16.8	29.0
0.15	12.6	14.7	19.7	15.5	19.2	32.1

Note: Each statistic is the Pearson chi-squared statistic $X^2_{B_2}$, with nine degrees of freedom, averaged over 50 Monte Carlo replications.

Table 12: Simulated "Repeater" Vote Switching with Thresholds: Gains

	receive votes when above expectation					
	Receiver (cand. 1)			Donor (cand. 2)		
fraction	500	1000	2000	500	1000	2000
0	9.6	8.7	12.4	11.1	11.9	13.0
0.01	11.2	13.3	15.0	9.3	10.3	11.4
0.02	12.7	17.7	27.1	8.8	12.2	13.2
0.03	15.5	27.2	44.1	10.5	10.7	14.2
0.04	25.6	41.8	68.9	10.9	13.1	16.9
0.05	24.8	38.1	67.2	11.2	13.6	17.1
0.06	23.6	42.2	74.2	12.0	15.1	19.3
0.07	28.2	48.4	89.9	12.9	15.6	22.1
0.08	33.5	58.1	112.8	13.5	17.3	26.5
0.09	32.7	56.5	107.7	12.9	18.0	29.3
0.1	36.0	59.8	110.0	12.7	18.1	25.6
0.11	37.7	64.6	119.2	13.2	17.5	26.1
0.12	39.4	75.6	135.1	14.2	17.0	27.4
0.13	38.0	71.9	128.5	12.9	18.0	30.0
0.14	40.0	72.3	128.4	13.2	17.4	24.8
0.15	40.9	75.9	130.1	12.4	14.7	24.1

	receive votes when below expectation					
	Receiver (cand. 1)			Donor (cand. 2)		
fraction	500	1000	2000	500	1000	2000
0	9.6	10.3	12.8	9.7	10.3	12.2
0.01	10.0	13.1	15.0	10.4	11.4	14.3
0.02	12.6	18.3	28.0	11.8	12.7	19.9
0.03	18.6	26.8	50.3	13.5	18.3	22.8
0.04	25.9	44.5	80.0	12.4	19.4	26.7
0.05	26.5	45.4	74.8	16.1	21.5	31.4
0.06	28.5	46.6	87.1	14.8	21.5	37.9
0.07	33.1	57.1	102.2	17.0	24.9	42.1
0.08	39.0	71.8	128.4	16.8	26.3	45.4
0.09	38.0	68.1	126.9	19.6	27.0	40.9
0.1	36.0	76.3	132.7	17.2	26.5	46.3
0.11	45.8	73.8	143.5	19.3	29.2	45.2
0.12	54.3	93.5	180.1	19.0	31.9	55.9
0.13	45.2	88.1	172.4	19.0	31.1	53.1
0.14	44.6	87.5	172.1	20.4	32.6	52.5
0.15	52.6	95.2	181.4	21.7	35.6	62.1

Note: Each statistic is the Pearson chi-squared statistic $X^2_{B_2}$, with nine degrees of freedom, averaged over 50 Monte Carlo replications.

Table 13: Simulated Proportional Vote Switching: Losses

	Donor (cand. 1)			Receiver (cand. 2)		
fraction	500	1000	2000	500	1000	2000
0	10.0	9.6	11.6	9.6	10.1	13.5
0.01	10.2	10.6	12.0	10.2	10.1	12.6
0.02	10.3	11.5	12.8	11.2	10.9	11.9
0.03	9.1	10.7	12.6	9.5	11.4	12.2
0.04	9.6	10.6	12.2	10.1	11.0	11.6
0.05	10.6	9.7	12.5	9.4	10.3	12.0
0.06	9.9	9.5	12.4	10.3	10.7	11.9
0.07	9.9	10.8	12.9	10.2	10.9	12.6
0.08	9.0	10.3	11.2	10.5	11.5	14.4
0.09	9.6	9.4	12.2	9.8	11.3	15.3
0.1	10.5	10.1	12.2	11.5	11.0	15.5
0.11	9.1	10.1	11.4	9.6	13.5	13.7
0.12	9.2	9.8	11.4	10.1	11.2	14.8
0.13	9.1	9.4	12.9	9.3	12.8	15.9
0.14	9.2	10.1	12.3	10.2	10.9	16.3
0.15	9.1	10.3	11.1	10.0	11.5	15.6

Note: Each statistic is the Pearson chi-squared statistic $X^2_{B_2}$, with nine degrees of freedom, averaged over 50 Monte Carlo replications.

Table 14: Simulated Proportional Vote Switching with Thresholds: Losses

lose votes when above expectation

fraction	Donor (cand. 1) 500	1000	2000	Receiver (cand. 2) 500	1000	2000
0	10.4	10.0	11.8	9.4	10.5	11.1
0.01	9.6	10.9	14.8	10.4	9.7	10.6
0.02	11.8	14.2	18.0	10.0	10.6	13.3
0.03	14.9	16.0	19.6	9.1	13.2	13.1
0.04	13.0	19.6	25.8	11.5	11.1	14.9
0.05	15.0	21.8	33.4	11.1	13.4	18.0
0.06	17.2	26.3	41.9	12.2	14.0	19.2
0.07	19.1	32.0	54.3	13.4	14.4	21.7
0.08	23.1	34.1	58.0	12.4	15.5	24.5
0.09	22.2	38.9	71.4	13.7	16.1	27.2
0.1	25.3	43.6	77.8	12.6	19.8	25.4
0.11	25.9	44.3	81.9	13.8	17.5	30.6
0.12	27.4	44.1	80.1	15.3	18.3	30.7
0.13	26.5	47.6	91.5	14.5	18.2	30.4
0.14	33.1	50.8	92.1	14.5	21.1	32.2
0.15	32.9	51.7	95.4	15.3	21.7	30.9

lose votes when below expectation

fraction	Donor (cand. 1) 500	1000	2000	Receiver (cand. 2) 500	1000	2000
0	10.4	11.2	11.6	9.8	10.9	10.5
0.01	9.3	10.4	11.3	9.8	10.0	11.4
0.02	11.8	9.4	11.9	9.7	9.6	10.8
0.03	10.7	11.6	15.1	9.4	10.3	12.4
0.04	11.0	13.7	17.6	9.6	9.8	12.9
0.05	12.4	15.3	22.8	9.6	9.9	11.5
0.06	13.3	18.6	28.4	8.9	9.8	12.8
0.07	16.8	22.7	39.4	10.8	11.3	13.7
0.08	18.2	27.9	45.5	10.9	12.8	15.1
0.09	19.4	32.9	51.2	10.8	12.5	14.0
0.1	20.2	31.9	52.7	10.9	12.3	16.9
0.11	18.6	30.6	55.8	10.4	12.5	18.2
0.12	19.4	30.5	53.9	10.6	12.3	16.4
0.13	21.5	32.8	61.1	11.8	11.7	17.0
0.14	21.2	37.3	59.7	10.5	13.6	16.8
0.15	22.8	36.9	65.3	11.1	13.5	19.4

Note: Each statistic is the Pearson chi-squared statistic $X^2_{B_2}$, with nine degrees of freedom, averaged over 50 Monte Carlo replications.

Table 15: Simulated "Repeater" Vote Switching: Losses

	Donor (cand. 1)			Receiver (cand. 2)		
fraction	500	1000	2000	500	1000	2000
0	10.0	9.6	11.6	9.6	10.1	13.5
0.01	10.1	10.4	13.1	10.0	10.5	14.2
0.02	10.4	10.8	12.2	9.4	10.1	12.5
0.03	10.8	12.5	12.8	9.8	12.4	12.4
0.04	10.2	10.2	11.9	10.1	11.6	12.2
0.05	9.4	10.6	12.8	9.6	11.2	13.4
0.06	10.6	11.5	15.2	10.5	11.6	14.4
0.07	11.0	11.4	15.2	10.7	11.4	14.6
0.08	10.5	14.0	17.8	11.3	12.3	16.0
0.09	10.3	11.3	18.4	10.2	12.2	17.0
0.1	12.2	13.8	17.5	11.3	12.5	17.8
0.11	13.8	15.1	20.7	11.3	15.5	18.9
0.12	12.1	17.2	25.3	10.6	14.4	21.5
0.13	12.7	17.5	26.9	12.3	14.5	21.9
0.14	15.3	19.1	29.4	11.3	14.3	19.0
0.15	14.6	18.6	32.1	12.4	14.5	21.4

Note: Each statistic is the Pearson chi-squared statistic $X^2_{B_2}$, with nine degrees of freedom, averaged over 50 Monte Carlo replications.

Table 16: Simulated "Repeater" Vote Switching with Thresholds: Losses

	lose votes when above expectation					
	Donor (cand. 1)			Receiver (cand. 2)		
fraction	500	1000	2000	500	1000	2000
0	10.4	10.0	11.8	9.4	10.5	11.1
0.01	10.8	12.9	19.6	10.0	11.2	12.6
0.02	15.6	21.5	32.2	11.0	13.0	16.4
0.03	22.1	29.5	47.1	11.6	16.9	21.1
0.04	24.6	42.5	71.9	15.2	20.8	29.1
0.05	26.0	41.7	75.1	14.6	20.5	34.3
0.06	26.4	49.7	87.4	17.7	23.0	36.1
0.07	33.7	55.6	108.4	17.9	26.7	45.9
0.08	37.3	64.4	127.2	18.1	30.1	48.8
0.09	35.9	58.7	120.6	21.5	32.6	57.4
0.1	35.9	61.4	115.7	20.7	37.6	60.8
0.11	34.7	63.7	117.3	24.8	41.6	70.9
0.12	41.8	71.6	143.0	27.4	41.0	75.0
0.13	39.5	65.7	131.3	25.6	41.7	78.7
0.14	41.9	70.9	134.8	25.9	45.0	85.6
0.15	42.6	67.3	130.3	31.5	49.2	86.1
	lose votes when below expectation					
	Donor (cand. 1)			Receiver (cand. 2)		
fraction	500	1000	2000	500	1000	2000
0	10.4	11.2	11.6	9.8	10.9	10.5
0.01	9.8	10.7	14.3	10.1	9.6	11.1
0.02	11.7	14.5	21.1	11.0	9.5	10.9
0.03	16.4	23.5	33.9	10.7	10.5	13.9
0.04	21.4	37.1	60.7	10.9	13.2	18.5
0.05	20.0	33.1	58.2	11.2	13.7	18.3
0.06	22.6	35.7	63.7	11.3	13.0	20.1
0.07	26.1	39.7	73.5	13.8	15.6	23.0
0.08	29.8	50.3	96.3	13.4	19.2	27.6
0.09	26.1	50.4	85.3	14.4	19.8	24.7
0.1	28.1	45.5	85.3	13.0	20.0	30.6
0.11	26.9	39.7	74.8	13.5	20.5	33.6
0.12	30.2	49.4	94.6	13.9	21.8	37.6
0.13	29.6	48.9	91.4	14.8	22.6	38.6
0.14	27.4	46.0	87.4	14.6	23.2	34.6
0.15	24.0	43.4	75.9	16.0	24.3	38.2

Note: Each statistic is the Pearson chi-squared statistic $X^2_{B_2}$, with nine degrees of freedom, averaged over 50 Monte Carlo replications.

Table 17: Precinct, Machine and Ballot Statistics

Election Day	Broward	Miami-Dade	Pasco
Precincts	775	757	152
Machines	5,306	5,323	1,338
Precinct-machines	10,614	14,128	2,676
Ballots	431,488	435,902	127,526

Early Voting	Broward	Miami-Dade	Pasco
Sites	20	14	3
Styles	150	100	16
Site-days	110	—	4
Style-site-days	—	4,429	—
Machines	190	726	36
Site-day-machines	380	—	72
S-s-d-machines	—	24,374	—
Ballots	176,743	242,344	29,584

Table 18: Miami-Dade Early Voting Second-digit Benford's Law Tests

	site-style-days ($n = 5,186$)		machines ($n = 727$)		site-style-day-machines ($n = 33,126$)	
item	Benf.	equal	Benf.	equal	Benf.	equal
Bush	10.1	44.9	23.5	20.9	130.3	391.4
Kerry	17.3	60.4	61.7	12.1	115.5	387.3
Martinez	14.8	48.6	32.6	17.3	107.6	357.9
Castor	9.1	42.1	43.3	18.6	93.0	336.2
Am. 1 Yes	14.1	59.9	69.6	9.8	119.7	415.4
Am. 1 No	8.7	44.1	64.8	9.8	86.3	295.7
Am. 2 Yes	17.7	65.4	58.3	2.6	83.4	334.7
Am. 2 No	20.2	71.1	41.9	16.9	92.0	292.8
Am. 3 Yes	8.2	41.4	90.8	7.6	122.7	394.8
Am. 3 No	15.3	56.7	66.1	7.8	104.8	342.1
Am. 4 Yes	7.7	40.6	47.1	11.0	87.3	338.0
Am. 4 No	14.4	60.7	83.6	5.3	108.9	351.4
Am. 5 Yes	21.9	78.3	69.2	4.6	58.4	307.5
Am. 5 No	11.0	44.8	5.7	71.6	84.4	237.8
Am. 6 Yes	12.9	56.9	55.3	11.0	105.2	368.5
Am. 6 No	9.0	37.8	44.4	9.6	126.6	374.1
Am. 7 Yes	24.6	85.0	47.8	14.9	134.2	468.3
Am. 7 No	12.0	33.9	77.4	236.4	64.5	192.7
Am. 8 Yes	13.9	61.7	68.9	2.4	96.3	377.7
Am. 8 No	6.7	28.9	63.5	15.5	79.2	261.2

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom.

Table 19: Broward Second-digit Benford's Law Tests

	Election Day				Early Voting			
	precincts ($n = 775$)		machines ($n = 5,307$)		styles ($n = 150$)		machines ($n = 190$)	
item	Benf.	equal	Benf.	equal	Benf.	equal	Benf.	equal
Bush	9.6	6.6	23.4	25.6	9.1	12.2	8.4	9.5
Kerry	21.2	12.4	79.7	6.5	21.4	24.8	10.5	17.6
Martinez	10.7	8.3	28.2	20.1	6.6	9.8	5.2	8.6
Castor	13.6	5.9	69.7	11.4	9.2	6.7	11.4	17.5
Am. 1 Yes	24.1	16.3	31.2	8.5	10.1	12.2	14.9	10.0
Am. 1 No	17.1	18.1	60.3	8.4	7.0	3.7	7.0	7.2
Am. 2 Yes	12.2	7.3	47.5	21.7	13.6	11.7	19.4	16.8
Am. 2 No	11.6	22.4	47.6	18.8	8.7	9.8	4.8	3.9
Am. 3 Yes	7.4	6.4	65.8	9.1	8.1	11.8	11.0	14.9
Am. 3 No	24.9	6.7	40.5	11.7	11.9	17.7	5.4	4.6
Am. 4 Yes	9.8	7.7	61.3	5.8	14.4	15.5	14.2	22.7
Am. 4 No	8.6	16.2	55.8	10.1	4.7	10.1	10.5	8.2
Am. 5 Yes	7.9	8.8	76.9	17.5	13.8	13.0	15.6	20.9
Am. 5 No	7.4	20.6	24.8	113.4	5.2	4.1	9.7	8.4
Am. 6 Yes	19.4	9.9	84.9	10.3	4.4	4.4	11.9	16.8
Am. 6 No	6.2	10.9	43.7	5.6	7.8	10.1	16.6	16.4
Am. 7 Yes	13.1	16.7	72.1	6.6	5.0	8.6	44.0	64.2
Am. 7 No	14.3	44.3	157.7	346.9	8.9	9.6	5.7	8.7
Am. 8 Yes	7.1	3.8	74.6	6.3	4.3	6.2	27.9	42.9
Am. 8 No	13.9	26.1	15.9	21.7	6.7	7.3	4.0	7.7

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom. In Broward, on election day each machine recorded votes for only one precinct. In the early voting data the number of votes on each style-machine combination was too small (mean = 16.7, median = 2) to support analysis for those combinations.

Table 20: Pasco Second-digit Benford's Law Tests

	Election Day				Early Voting machine-precincts	
	precincts ($n = 152$)		machines ($n = 1,338$)		($n = 372$)	
item	Benf.	equal	Benf.	equal	Benf.	equal
Bush	6.9	5.6	16.4	16.2	14.6	23.8
Kerry	4.0	3.5	22.9	21.7	19.0	25.2
Martinez	6.5	3.7	30.6	6.4	13.4	24.3
Castor	11.2	10.5	40.5	7.7	14.7	20.7
Am. 1 Yes	9.0	10.4	24.1	11.3	5.4	10.5
Am. 1 No	7.0	5.1	9.8	5.0	18.6	28.3
Am. 2 Yes	5.4	4.8	28.6	10.3	9.6	16.2
Am. 2 No	8.6	12.7	15.8	1.9	10.4	17.7
Am. 3 Yes	10.4	9.3	34.6	11.0	12.5	18.6
Am. 3 No	8.5	4.4	10.1	16.2	13.1	19.2
Am. 4 Yes	6.0	8.4	20.7	2.8	8.6	14.7
Am. 4 No	8.6	5.2	19.8	9.3	21.5	33.4
Am. 5 Yes	3.6	9.4	16.6	8.2	11.9	20.9
Am. 5 No	3.8	6.4	10.2	19.1	10.3	17.2
Am. 6 Yes	12.8	15.5	33.5	7.7	10.5	18.7
Am. 6 No	4.4	4.7	20.1	10.0	14.4	16.4
Am. 7 Yes	29.5	43.3	20.5	18.3	14.1	22.3
Am. 7 No	5.1	7.2	19.9	10.7	5.2	6.9
Am. 8 Yes	8.0	13.8	16.5	7.7	6.3	8.6
Am. 8 No	8.0	14.6	29.9	6.6	11.1	18.1

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom. In Pasco, on election day each machine recorded votes for only one precinct. In Pasco there were only 16 early voting "precincts," too few to support analysis for those units.

Table 21: Mexico 2006 Presidential Vote Second-digit Benford's Law Tests: Secciones

		Seccion-level 2BL test statistics				
State	n	PAN	PRD	PRI	NA	ASDC
All	64,679	48.1	25.3	109.0	1254.8	16.6
Aguascalientes	593	6.6	14.2	6.0	18.9	8.3
Baja California	1,525	12.0	34.9	13.4	49.5	20.4
Baja California Sur	351	6.3	17.7	15.6	13.7	4.0
Campeche	491	8.3	19.2	7.7	11.8	8.6
Coahuila	1,938	8.3	22.5	4.7	23.2	15.8
Colima	2,918	9.9	11.7	25.1	131.6	43.6
Chiapas	1,537	16.1	15.3	5.6	44.6	12.8
Chihuahua	338	15.7	10.6	10.1	11.4	6.9
Distrito Federal	5,560	10.9	26.9	42.6	178.9	61.6
Durango	1,382	5.1	11.1	9.8	51.1	16.3
Guanajuato	3,043	13.7	7.2	25.4	117.9	6.6
Guerrero	2,767	14.4	8.8	28.0	65.8	11.9
Hidalgo	1,712	5.8	5.4	19.3	50.8	4.9
Jalisco	3,363	18.1	24.0	16.3	81.6	12.3
Mexico	6,193	51.4	49.8	15.0	238.4	18.0
Michoacan	2,687	14.0	5.5	16.8	49.7	25.0
Morelos	912	14.1	17.3	8.8	17.4	11.5
Nayarit	879	5.6	7.2	8.8	23.7	10.6
Nuevo Leon	2,148	8.6	11.8	9.3	7.9	7.3
Oaxaca	2,456	16.3	9.4	14.4	38.9	19.8
Puebla	2,566	18.5	20.9	16.7	104.3	10.7
Queretaro	761	16.8	12.6	6.3	25.5	10.1
Quintana Roo	445	11.6	6.7	5.9	8.8	7.5
San Luis Potosi	1,797	9.7	9.6	24.9	43.2	8.1
Sinaloa	3,771	39.1	25.9	17.1	36.4	87.9
Sonora	1,361	23.1	3.9	20.3	45.5	8.5
Tabasco	1,139	7.3	16.3	13.6	6.4	29.0
Tamaulipas	1,743	10.7	9.3	13.8	56.8	12.7
Tlaxcala	611	8.4	13.5	11.5	23.5	12.6
Veracruz	4,736	14.0	9.5	19.1	75.2	28.7
Yucatan	1,083	20.9	31.0	6.6	46.5	7.5
Zacatecas	1,873	10.2	12.9	14.5	35.3	38.5

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom. The values in the n column show the total number of secciones in each state (or, for "All," across the whole country), not the number of vote counts that have two digits. That number varies among the parties.

Table 22: Mexico 2006 Presidential Vote Second-digit Benford's Law Tests: Casillas

State	n	PAN	PRD	PRI	NA	ASDC
		Casilla-level 2BL test statistics				
All	130,768	660.7	104.7	217.7	1712.2	2751.1
Aguascalientes	1,229	14.8	13.6	17.3	46.4	29.1
Baja California	3,550	174.5	44.3	24.9	92.5	223.5
Baja California Sur	664	41.9	42.3	3.0	6.4	23.3
Campeche	929	50.7	50.0	46.2	6.4	54.9
Coahuila	4,784	15.2	19.5	24.8	23.4	118.1
Colima	4,742	49.7	33.7	57.1	150.4	374.6
Chiapas	3,094	68.9	25.9	30.7	16.5	167.4
Chihuahua	762	9.5	37.6	85.2	14.7	53.3
Distrito Federal	12,235	105.9	71.4	142.7	17.3	134.8
Durango	2,236	8.5	27.1	17.5	58.1	141.4
Guanajuato	6,136	10.3	19.5	9.2	86.0	267.3
Guerrero	4,478	17.7	49.4	14.6	38.9	155.8
Hidalgo	3,055	41.8	62.0	31.1	164.4	70.9
Jalisco	8,110	98.8	41.5	272.5	276.9	160.6
Mexico	15,553	144.1	70.2	37.7	460.8	113.5
Michoacan	5,427	23.0	239.6	20.9	13.1	249.1
Morelos	2,067	36.7	40.0	26.1	56.2	89.7
Nayarit	1,395	7.1	38.8	64.2	41.4	47.7
Nuevo Leon	5,066	28.8	32.1	182.7	177.2	286.1
Oaxaca	4,479	21.3	6.7	46.6	16.0	123.4
Puebla	6,037	56.5	327.1	22.3	74.5	112.2
Queretaro	1,807	25.6	71.9	17.7	37.6	66.4
Quintana Roo	1,154	27.9	31.1	12.8	11.5	64.4
San Luis Potosi	3,094	10.2	15.0	8.7	50.5	42.1
Sinaloa	4,300	84.1	35.7	11.9	10.9	136.7
Sonora	3,011	50.2	27.8	8.5	21.4	222.9
Tabasco	2,408	80.4	25.9	69.9	8.9	30.2
Tamaulipas	3,971	106.6	62.2	57.5	68.6	323.6
Tlaxcala	1,255	71.6	31.4	26.8	18.2	79.5
Veracruz	9,185	169.9	100.4	27.2	31.1	302.5
Yucatan	2,157	24.4	11.4	71.7	20.8	45.6
Zacatecas	2,398	29.0	43.8	13.6	55.8	62.3

Note: Each statistic is the Pearson chi-squared statistic, with nine degrees of freedom. The values in the n column show the total number of casillas in each state (or, for "All," across the whole country), not the number of vote counts that have two digits. That number varies among the parties.

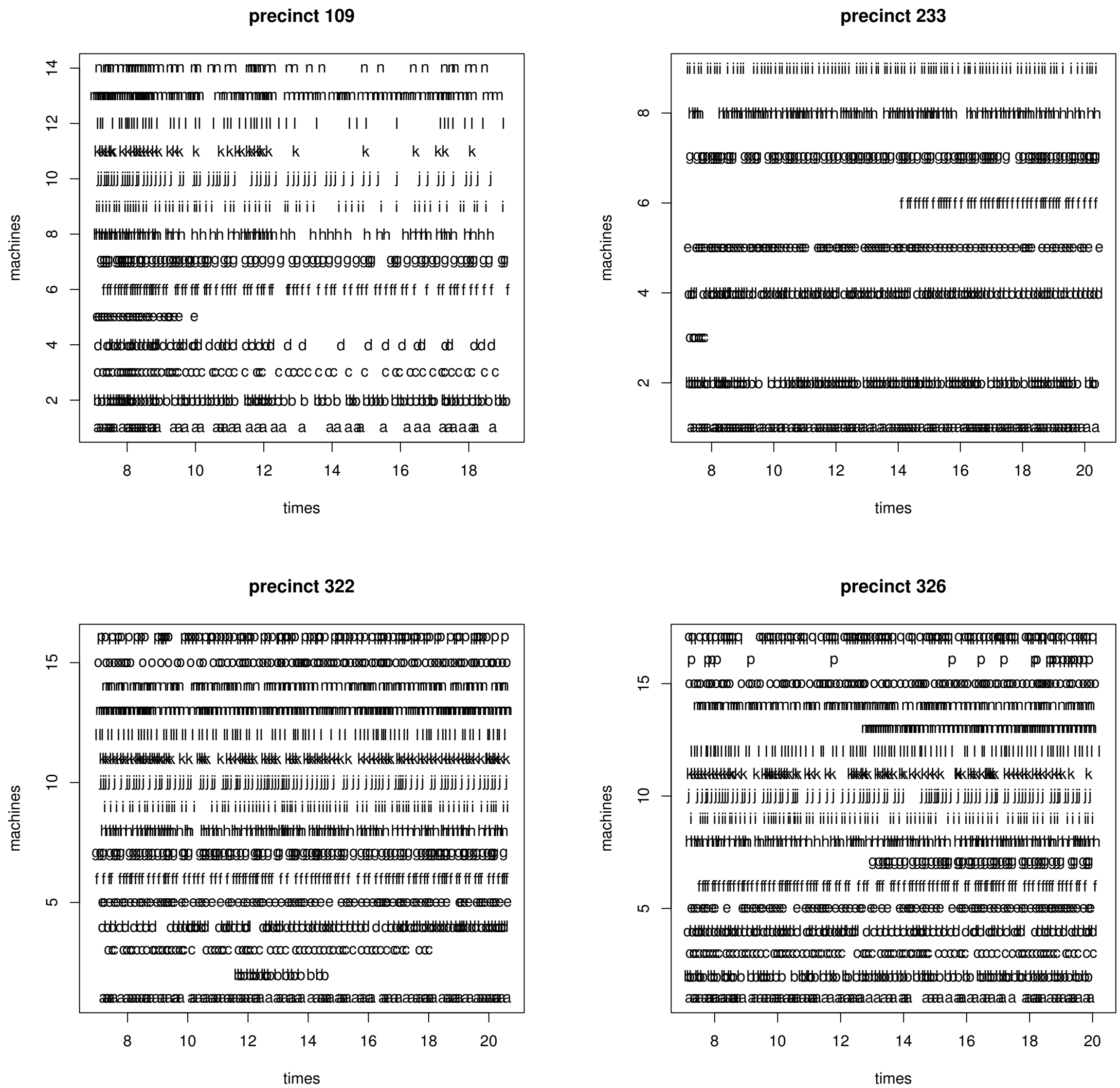

Figure 1: Times (Resolved to the Second and Shown on a 24-Hour Clock) When Votes Were Cast on Machines in Selected Precincts on Election Day, Miami-Dade County