Election Forensics: Frauds Tests and Observation-level Frauds Probabilities*

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Abstract

A key challenge for election forensics—the field devoted to using statistical methods to try to determine whether the results of an election accurately reflect the intentions of the electors—is to be able to distinguish election results caused by election frauds from results produced by strategic behavior or other normal politics. Election forensics studies counts of votes, counts of eligible voters and other traces of an election—preferably at low levels of aggregation such as tallies for each polling station—to produce evidence regarding what happened in the election. Considering the theory that motivates an important positive empirical model of election frauds shows that the very same mechanism that causes frauds to trigger the model also may cause strategic voting or coalitions to trigger the model. Using a finite mixture likelihood version of the frauds model, we analyze data from several countries to show, first, that the finite mixture model produces useful and informative estimates and tests, and second to explore the ambiguities of whether estimated frauds are genuine frauds. The parameters of the model that are supposed to measure election frauds sometimes also respond to strategic voting. However frauds may produce distinctive parameter values and estimated fraud magnitudes. Model parameters can help diagnose what kinds of frauds occurred.

1 Introduction

A key challenge for election forensics—the field devoted to using statistical methods to try to determine whether the results of an election accurately reflect the intentions of the electors—is to be able to distinguish election results caused by election frauds from results produced by strategic behavior or other normal politics. Election forensics studies counts of votes, counts of eligible voters and other traces of an election—preferably at low levels of aggregation such as tallies for each polling station—to produce evidence regarding what happened in the election. By starting with the numerical results and other measures of the voting process election forensics does not address the entirety of an election or address the full range of frauds that are possible (Lehoucq and Jiménez 2002; Lehoucq 2003; Magaloni 2006; Schedler 2006; Levitsky and Way 2010; Minnite 2010; Birch 2011; Hyde and Marinov 2012; Svolik 2012; Wang 2012; Simpser 2013; Stokes, Dunning, Nazareno and Brusco 2013; Norris 2014). For example, if parties are excluded from the ballot, such an action may not produce distinctive patterns in the votes that are cast. But some violations of election integrity such as unfair access to campaign resources, wrongfully manipulated voter lists, vote buying, voter intimidation and other coercive actions may produce distinctive patterns in votes that statistics can detect. Ambiguities arise when such patterns might also be produced by strategic voting and other normal political activites such as forming coalitions.

Many methods for trying to detect election frauds have been proposed (e.g. Myagkov, Ordeshook and Shaikin 2009; Levin, Cohn, Ordeshook and Alvarez 2009; Shikano and Mack 2009; Mebane 2010; Breunig and Goerres 2011; Pericchi and Torres 2011; Cantu and Saiegh 2011; Deckert, Myagkov and Ordeshook 2011; Beber and Scacco 2012; Hicken and Mebane 2015; Montgomery, Olivella, Potter and Crisp 2015). Methods based on the second significant digits of vote counts have been shown to respond both to normal political activities (strategic behavior, district imbalances, special mobilizations, coalitions) and to frauds (Mebane 2013*a*, 2014*a*,*b*). Methods that examine the last digit of vote counts can be fooled if malefactors have sufficient control over the numbers (Mebane 2013*b*).

Some of the methods focus on the modality of election data. Some methods in this vein emphasize that unproblematic elections feature unimodal distributions of turnout and regular flows of votes—the latter are most compatible with assumed unimodal distributions for parties' shares of the votes (Myagkov, Ordeshook and Shaikin 2008, 2009; Levin et al. 2009). Other contributions connect "spiky" (hence multimodal) distributions of turnout and vote proportions to ideas about agents committing frauds in ways that they intend to be detected (Kalinin and Mebane 2011; Mebane 2013*b*; Rundlett and Svolik 2015).

The sharpest contribution featuring multimodality is a model proposed by Klimek, Yegorov, Hanel and Thurner (2012) that stipulates a particular functional form according to which frauds occur. We describe the Klimek et al. (2012) conception in greater detail below, but a brief summary is that their model specifies two mechanisms by which votes are added to a winning party: inventing votes from genuine nonvoters; and stealing votes from the nonwinning set of parties. These mechanisms operate either in an "extreme" manner, so election data have turnout near 100 percent with nearly all votes going to the winner, or in an "incremental" manner, where a substantial number but not almost all votes are reallocated to the winner. The model includes parameters that express whether vote manufacturing or vote stealing is the predominant form of fraud that occurs. If frauds as described by their model occur, then turnout and vote proportion distributions are bimodal or trimodal. Visual displays of the joint distribution of turnout and winner's vote proportions can show "fingerprints of fraud" (Klimek et al. 2012). Their model is a positive empirical model of frauds: they describe a precise functional form for the frauds; and they present an algorithm for simulating the frauds that occur in particular elections.

The Klimek et al. (2012) method has many limitations, but the most important of those from a practical point of view is that their simulation method does not work to recover the form of their model that may generate the data in a particular election: their simulation method is not a valid method for estimation (Mebane, Egami, Klaver and Wall 2014). We remedy this problem by developing a finite mixture likelihood model that implements their concept. Using an EM algorithm (McLachlan and Krishnan 2008) we produce sound estimates of the model's parameters as well as tests for whether fraud (as defined by their model) is present in an election. Our estimation approach also allows point estimates to be computed for the magnitude of the frauds in terms of the number of fraudulent votes, for the probability that fraud occurs at individual electoral units (e.g., individual polling stations), and for the number of fraudulent votes at each electoral unit.

Review of the theory that motivates the Klimek et al. (2012) conception suggests, however, that multimodal distributions may be as readily produced by strategic voting and coalitions as by election frauds that stem from maleficent activity. That theory, as it has been developed so far, does not imply that the distributions produced by strategic voting and by frauds are the same: currently the theory is not specifically quantitative. But the theoretical ambiguity about the origins of multimodality may carry over to make the parameters of the Klimek et al. (2012) conception ambiguous. An election may appear to have a lot of fraud when in fact it has only robust politics featuring a lot of strategic activity.

We use estimates of the model for elections in several countries to show, first, that the finite mixture model produces useful and informative estimates and tests, and second to explore the ambiguities of whether estimated frauds are genuine frauds. While the precise ways that normal politics and fraudulent politics map into the Klimek et al. (2012) conception remain to be determined, our findings suggest that often strategic behavior causes a baseline level of multimodality—perhaps only bimodality—to exist in election data. Perhaps fraud magnitudes greater than a baseline level are evidence of frauds.

2 Theory

The Klimek et al. (2012) model is motivated by theory developed in Borghesi and Bouchaud (2010), and that theory provides our understanding of what are the kinds of election frauds the model can measure. The same theory also shows how the conception of Klimek et al. (2012) may respond to strategic voting as well as to frauds.

Motivated by data from an assortment of elections in France, Borghesi and Bouchaud (2010, 399) suggest that the electoral intention of individual i, denoted φ_i, can be represented as evolving over time according to the following equation,

$$\varphi_i(t) = \epsilon_i(t) + \phi(\mathbf{R}, t) + \sum_j J_{ij} S_j(t-1)\,, \tag{1}$$

where $\epsilon_i(t)$ is an individual-specific term, $\phi(\mathbf{R}, t)$ is a "cultural field" that is "an average of the intentions of the fellow denizens in a recent past" and $\sum_j J_{ij} S_j(t-1)$ describes the influence of the immediate decisions $S_j(t-1)$ of others. Each decision $S_i(t)$ is determined by a threshold rule from the intention $\varphi_i(t)$. Considering long-range spatial correlations between communes and other features of their data, Borghesi and Bouchaud (2010) argue that $\phi(\mathbf{R}, t)$ evolves according to a noisy diffusion equation, but more important for understanding the conception of frauds that Klimek et al. (2012) develop is their argument for why a strong imitation term $\sum_j J_{ij} S_j(t-1)$ does not occur. A strong imitation term is one in which which the magnitudes of the J_{ij} elements are large. If such a term were to occur, "the corresponding distribution of turnout becomes very wide, or even multimodal, and negatively skewed in a way that is incompatible with the unimodeal, positively skewed and rather narrow distributions observed empirically" (Borghesi and Bouchaud 2010, 399; see also Borghesi 2009, 52–62).

The Klimek et al. (2012) conception defines particular forms of multimodality that correspond to the two kinds of election frauds they identify (see the next section for details). Given a background of the model of Borghesi and Bouchaud (2010), this means that frauds occur when the imitation term "is so strong that the solution of the coupled equations giving the $\{S_i\}$ becomes multi-valued" (Borghesi and Bouchaud 2010, 399). It is immediate to think of examples. If votes are simply faked or miscounted to help a party or candidate, then clearly those "votes" are all imitating the decision of the malefactor. Likewise if votes are bought, or if voters are otherwise coerced. These kinds of frauds in general mean that the intentions the model represents as producing voting actions—that would have been somewhat independent—instead become tightly connected. Such connections can be represented by a strong imitation term in equation (1). This is the conceptual foundation that the positive empirical model of election frauds introduced by Klimek et al. (2012) rests on.

A reason for models based on Klimek et al. (2012)'s conception to respond to strategic voting as well as to frauds is immediate: if the strategic voting can be characterized as some kind of Nash equilibrium outcome (e.g. Cox 1994; Alesina and Rosenthal 1996), then the voters who are participating in that equilibrium may also have tightly connected intentions. The core idea in any kind of Nash equilibrium is that individuals act in response not merely to their own attributes and preferences but also to their expectations regarding the actions of others. Given rational expectations, a "best-response" equilibrium may correspond to a strong imitation term in equation (1).[1] Such connections should be strongest when their strategic voting means that individuals act differently than they would if they were acting sincerely. To act sincerely means to act in a way that ignores the expected actions of others. Coalitions may also induce or inherently involve strategic behavior. Because a coalition involves compromise among parties, a coalition inherently involves the kind of mutual coordination of expectations that is at the heart of political strategies. In terms of equation (1), connections originating from strategic activity may imply that an imitation term exists that is sufficiently strong to produce multivalued solutions and consequently multimodal distributions. In some cases a model based on Klimek et al. (2012)'s conception will respond to this multimodality.

[1] Contrast this point with, "Les agents ne peuvent alors imiter les choix ± 1 des autres agents, puisqu'ils n'ont pas encore été réalisés, ou ne le connaissent pas" (Borghesi 2009, 61). With best-response equilibrium in mind with might write (1) instead as $\varphi_i(t) = \epsilon_i(t) + \phi(\mathbf{R}, t) + \sum_j J_{ij} S_j(t)$, if the thought is that equilibrium is maintained at every moment (which seems to be an excessively strong assumption; see e.g. Fey 1997), but because $(t-1)$ in (1) is meant to refer to a time infinitesimally close to t there is no material difference from (1) as far as the implications of the values J_{ij} having critically large magnitudes is concerned.

While the model of Borghesi and Bouchaud (2010) gives a clear theoretical motivation for the positive empirical frauds conception of Klimek et al. (2012), that model also shows clearly why the conception's indicators for "frauds" may be ambiguous: genuine frauds (maleficent acts) may create multimodal distributions and hence trigger indicators based on the conception, but through the very same mechanism strategic voting may as well. Whether frauds and strategic activity trigger the empirical models in distinctive ways is a question that empirical and theoretical research will need to address.

3 Model

In the Klimek et al. (2012) model the baseline assumption is that votes in an election with no fraud are produced through the interaction of processes whose effects can be summarized by two Normal distributions: there is one distribution for turnout proportions and another, independent distribution for the proportion of votes going to the "winner" (that is, the party with the most votes). Conditioning on the number of eligible voters, Klimek et al. (2012) assume that election fraud means that votes are added to the votes for the winner. Some votes are transferred to the winner from the opposition, and some are transferred from nonvoters. Two kinds of election fraud refer to how many of the opposition and nonvoters votes are shifted: with "incremental fraud" moderate proportions of the votes are shifted; with "extreme fraud" almost all of the votes are shifted. Klimek et al. (2012) have parameters that specify the probability that each unit experiences each type of election fraud: f_{i} is the probability of incremental fraud and f_{e} is the probability of extreme fraud. Other parameters fully describe bimodal and trimodal distributions that the model characterizes as being consequences of election frauds.

The "winner" in the Klimek et al. (2012) conception is not necessarily the party that wins the election is the sense of gaining seats from it. In fact the significance of designating a party as the "winning" party in their model is that that's the only party that can gain votes from frauds. We refer this designated party instead as the "leading" party. Sometimes the leading party is the party that has the most votes in the election, but any party may be designated the leading party. A limitation of the model is that only one party may be represented as a party that gains votes from frauds.

Core idea: Klimek et al. (2012) describe a simulation protocol that includes three kinds of votes: votes without fraud; votes with "incremental fraud"; and votes with "extreme fraud." With no fraud the distribution of votes, given the number of eligible voters, is a product Normal distribution. The fraud conditions correspond to differing proportions of votes going to the leading party that should have gone to other parties or should not have been counted as votes at all. With incremental fraud a small proportion x of what should have been nonvotes are counted for the leading party while a proportion x^α, $\alpha > 0$, of votes that should have gone to opposition instead go to the leading party. With extreme fraud a large proportion $1 - y$ of the nonvotes are counted for the leading party and a proportion $(1 - y)^\alpha$ of genuine opposition votes instead go to the leading party.

A formal description of the key steps in the simulation protocol that relate to model specification follows. Features of the simulation protocal that relate to parameter selection are not described.[2] We describe the parts of the protocol that are the point of departure for the finite mixture likelihood.

Observed data come from n electoral units (e.g., polling stations), and the number of eligible voters in each unit is N_i, $i = 1, \ldots, n$. Votes for parties are observed as the count of votes for the leading party, denoted W_i, and the sum of votes cast for all other parties (the "opposition"), denoted O_i. The number of observed nonvotes ("abstentions") is $A_i = N_i - W_i - O_i$. The observed number of valid votes is $V_i = N_i - A_i$.

Using $\mathcal{N}(\mu, \sigma)$ to denote a normally distributed simulated random variable with mean μ and standard deviation σ, the protocol involves two kinds of fraud and is applied to each

[2]See Klimek et al. (2012) for a description and Mebane et al. (2014) for a critique of the simulation method.

observation i. For each electoral unit $i = 1, \ldots, n$, for some $\alpha > 0$ do:

1. Sample turnout: $\tau_i \sim \mathcal{N}(\tau, \sigma_\tau)$, subject to $0 \leq \tau_i \leq 1$.

2. Sample the leading party's vote proportion: $\nu_i \sim \mathcal{N}(\nu, \sigma_\nu)$, subject to $0 \leq \nu_i \leq 1$.

3. (Incremental fraud) With probability f_{i} sample the proportion of nonvotes that are turned into votes: $x_i \sim |\mathcal{N}(0, \theta)|$, subject to $0 < x_i < 1$. Set the number of votes for the leading party as

$$W_i = N_i \left(\tau_i \nu_i + x_i (1 - \tau_i) + x_i^\alpha (1 - \nu_i) \tau_i\right) , \tag{2}$$

   the number of votes for the opposition as $O_i = N_i (1 - x_i^\alpha)(1 - \nu_i) \tau_i$ and the number of nonvoters as $A_i = N_i(1 - x_i)(1 - \tau_i)$. x_i is the proportion of genuine nonvotes that are counted as votes for the leading party and x_i^α is the proportion of votes that were genuinely cast for other parties but instead are counted as votes for the leading party. Whether the fraud involves more vote stealing or more vote manufacturing is measured by the exponent α. If $\alpha = 1$ then both processes are equally affecting votes. If $\alpha < 1$ then $x_i < x_i^\alpha$ and vote stealing is more important, and if $\alpha > 1$ then $x_i > x_i^\alpha$ and manufacturing votes from nonvoters is more important.

4. (Extreme fraud) With probability f_{e} sample the proportion of nonvotes that are not turned into votes: $y_i \sim |\mathcal{N}(0, \sigma_x)|$, $\sigma_x = 0.075$, subject to $0 < y_i < 1$. Set the number of votes for the leading party as

$$W_i = N_i \left(\tau_i \nu_i + (1 - y_i)(1 - \tau_i) + (1 - y_i)^\alpha (1 - \nu_i) \tau_i\right) , \tag{3}$$

   the number of votes for the opposition as $O_i = N_i (1 - (1 - y_i)^\alpha)(1 - \nu_i) \tau_i$ and the number of nonvoters as $A_i = N_i y_i (1 - \tau_i)$. $1 - y_i$ is the proportion of genuine nonvotes that are counted as votes for the leading party and $(1 - y_i)^\alpha$ is the proportion of votes that were genuinely cast for other parties but instead are counted as votes for the leading party. If $\alpha < 1$ then $1 - y_i < (1 - y_i)^\alpha$ and vote stealing is more important, and if $\alpha > 1$ then $1 - y_i > (1 - y_i)^\alpha$ and manufacturing votes from nonvoters is more important.

5. (No fraud) With probability $f_0 = 1 - f_\text{i} - f_\text{e}$, the number of votes for the leading party is

$$W_i = N_i \tau_i \nu_i \,, \tag{4}$$

the number of votes for the opposition is $O_i = N_i \tau_i (1 - \nu_i)$, and the number of nonvoters is $A_i = N_i (1 - \tau_i)$.

The intuition is that, depending on the value of α, incremental fraud involves shifting to the leading party some of the votes from the opposition and from nonvoters, while extreme fraud involves shifting to the leading party almost all of those votes. Smaller values of α mean that larger fractions of votes are shifted from opposition to the leading party. Because the mean of x_i increases with θ, a higher value of θ implies that incremental fraud garners a higher number of votes for the leading party.

A finite mixture likelihood: We treat the model of Klimek et al. (2012) from a likelihood point of view. Variables that are simulated in Klimek et al. (2012) are treated as unobserved variables. Now using $\mathcal{N}(\mu, \sigma)$ to denote a theoretical Normal distribution:

$$\tau_i \sim \mathcal{N}(\tau, \sigma_\tau), \quad \text{subject to } 0 \le \tau_i \le 1 \tag{5a}$$

$$\nu_i \sim \mathcal{N}(\nu, \sigma_\nu), \quad \text{subject to } 0 \le \nu_i \le 1 \tag{5b}$$

$$x_i \sim |\mathcal{N}(0, \theta)|, \quad \text{subject to } 0 < x_i < 1 \tag{5c}$$

$$y_i \sim |\mathcal{N}(0, \sigma_x)|, \quad \text{subject to } 0 < y_i < 1 \tag{5d}$$

The "no fraud," incremental fraud and extreme fraud cases define three distinct components that can fit together in a finite mixture model. Let $\mathbf{W}$, $\mathbf{A}$ and $\mathbf{N}$ be vectors containing, repectively, the n observations of W_i, A_i and N_i. The finite mixture likelihood is

$$\mathcal{F}(\mathbf{W}, \mathbf{A} \mid \mathbf{N}; \mathbf{\Psi}) = \sum_{j \in \{0,\mathrm{i},\mathrm{e}\}} f_j \prod_{i=1}^{n} g_{jW}(W_i \mid N_i; \mathbf{\Psi}) g_{jA}(A_i \mid N_i; \mathbf{\Psi}) \,, \tag{6}$$

where f_0, f_i and f_e are probabilities with $f_0 + f_\mathrm{i} + f_\mathrm{e} = 1$. To adapt the language of Klimek et al. (2012), f_0 is the probability of "no fraud." $g_{jW}(W_i \mid N_i; \mathbf{\Psi})$ and $g_{jA}(A_i \mid N_i; \mathbf{\Psi})$ are conditional densities and scalar parameters are in a vector $\mathbf{\Psi} = (\alpha, \nu, \tau, \sigma_\nu, \sigma_\tau, \theta)'$.[3]

Let $\phi(x, \mu, \sigma)$ denote the Normal density with mean μ and standard deviation σ evaluated at quantile x. Let $\upsilon(x, \theta) = 2\exp(-x^2/2\theta^2)/(\theta\sqrt{2\pi})$ denote the density matching $x \sim |\mathcal{N}(0, \theta)|$ (a folded Normal density); we use the error function, $\mathrm{erf}\left(1/\sqrt{2\theta}\right) = \int_0^1 \upsilon(x, \theta) dx = 2\int_{-\infty}^{\sqrt{1/\theta}} d\phi(x, 0, 1) - 1$, to rescale the density for the censoring implied by $0 < x < 1$.

Densities for A_i: With no fraud the density of A_i is

$$g_{0A}(A_i \mid N_i; \mathbf{\Psi}) = \frac{\phi(A_i, N_i(1-\tau), N_i\sigma_\tau)}{\int_0^1 \phi(\tau_i, \tau, \sigma_\tau)\, d\tau_i} \tag{7}$$

Because incremental fraud implies $A_i = N_i(1 - x_i)(1 - \tau_i)$, with incremental fraud the density of A_i is

$$g_{\mathrm{i}A}(A_i \mid N_i; \mathbf{\Psi}) = \frac{\displaystyle\int_0^1 \phi\left(\frac{A_i}{1-x_i}, N_i(1-\tau), N_i\sigma_\tau\right) \frac{\upsilon(x_i, \theta)}{1-x_i}\, dx_i}{\mathrm{erf}\left(1/\sqrt{2\theta}\right) \int_0^1 \phi(\tau_i, \tau, \sigma_\tau)} \,. \tag{8}$$

[3]To insure the separate identifiability of the two "fraud" densities, and following Klimek et al. (2012), we fix $\sigma_x = 0.075$.

Because extreme fraud implies $A_i = N_i y_i (1-\tau_i)$, with extreme fraud the density of A_i is

$$g_{\mathrm{e}A}(A_i \mid N_i; \mathbf{\Psi}) = \frac{\displaystyle\int_0^1 \phi\left(\frac{A_i}{y_i}, N_i(1-\tau), N_i\sigma_\tau\right)\frac{\upsilon(y_i,\sigma_x)}{y_i}\,dy_i}{\operatorname{erf}\left(1/\sqrt{2\sigma_x}\right)\int_0^1 \phi(\tau_i,\tau,\sigma_\tau)} \,. \tag{9}$$

Densities for W_i: The density of W_i with no fraud (from equation (4)) is

$$g_{0W}(W_i \mid N_i; \mathbf{\Psi}) = \frac{\phi\left(\dfrac{W_i}{(1-A_i/N_i)}, N_i\nu, N_i\sigma_\nu\right)\dfrac{\phi((1-A_i/N_i),\tau,\sigma_\tau)}{(1-A_i/N_i)}}{\int_0^1 \phi(\nu_i,\nu,\sigma_\nu)\,d\nu_i \int_0^1 \phi(\tau_i,\tau,\sigma_\tau)\,d\tau_i} \tag{10}$$

With incremental fraud the density of W_i (from equation (2)) is

$$g_{\mathrm{i}W}(W_i \mid N_i; \mathbf{\Psi}) =$$
$$\frac{\displaystyle\int_0^1\int_0^1 \phi\left(\frac{W_i}{\tau_i\left(1-x_i^\alpha\right)}, \mu_{\mathrm{i}}, N_i\sigma_\nu\right)\frac{(1-x_i)^{\frac{1}{\alpha}-1}\upsilon(x_i,\theta)\phi(\tau_i,\tau,\sigma_\tau)}{\alpha\tau_i\left(1-x_i^\alpha\right)}\,dx_i\,d\tau_i}{\operatorname{erf}\left(1/\sqrt{2\theta}\right)\left(\int_0^1 \phi(\nu_i,\nu,\sigma_\nu)\,d\nu_i\right)\left(\int_0^1 \phi(\tau_i,\tau,\sigma_\tau)\,d\tau_i\right)} \tag{11a}$$

$$\mu_{\mathrm{i}} = N_i\left(\nu + \frac{x_i(1-\tau_i)}{\tau_i(1-x_i^\alpha)} + \frac{x_i^\alpha}{1-x_i^\alpha}\right) \tag{11b}$$

With extreme fraud the density of W_i (from equation (3)) is

$$g_{\mathrm{e}W}(W_i \mid N_i; \mathbf{\Psi}) =$$
$$\frac{\displaystyle\int_0^1\int_0^1 \phi\left(\frac{W_i}{\tau_i\left(1-(1-y_i)^\alpha\right)}, \mu_{\mathrm{e}}, N_i\sigma_\nu\right)\frac{(1-y_i)^{\frac{1}{\alpha}-1}\upsilon(y_i,\sigma_x)\phi(\tau_i,\tau,\sigma_\tau)}{\alpha\tau_i\left(1-(1-y_i)^\alpha\right)}\,dy_i\,d\tau_i}{\operatorname{erf}\left(1/\sqrt{2\sigma_x}\right)\left(\int_0^1 \phi(\nu_i,\nu,\sigma_\nu)\,d\nu_i\right)\left(\int_0^1 \phi(\tau_i,\tau,\sigma_\tau)\,d\tau_i\right)} \tag{12a}$$

$$\mu_{\mathrm{e}} = N_i\left(\nu + \frac{(1-y_i)(1-\tau_i)}{\tau_i(1-(1-y_i)^\alpha)} + \frac{(1-y_i)^\alpha}{1-(1-y_i)^\alpha}\right) \tag{12b}$$

In (11a), $f(x_i) = z = (1-x_i^\alpha)$ implies $f^{-1}(z) = (1-z)^{1/\alpha}$, so $|\partial f^{-1}(z)/\partial z| = (1-x_i)^{\frac{1}{\alpha}-1}/\alpha$ gives the Jacobian to use to express the density of $(1-x_i^\alpha)$ in terms of $\upsilon(x_i,\theta)$. In (12a), $(1-y_i)^{\frac{1}{\alpha}-1}/\alpha$ arises similarly.

Restrictions on parameters: To match an aspect of the Klimek et al. (2012) specification,[4] we specify upper bounds for parameters ν, τ, σ_ν and σ_τ: $\nu \leq \text{median}(W_i/V_i)$, $\tau \leq \text{median}(V_i/N_i)$, and using the sets $\mathfrak{W}_u = \{W_i/V_i : W_i/V_i \leq \text{median}(W_i/V_i)\}$ and $\mathfrak{V}_u = \{V_i/N_i : V_i/N_i \leq \text{median}(V_i/N_i)\}$,

$$\sigma_\nu \leq 2\left(\frac{1}{|\mathfrak{W}_u|}\sum_{W_i/V_i\in\mathfrak{W}_u}\left(W_i/V_i - \frac{\sum_{W_i/V_i\in\mathfrak{W}_u} W_i/V_i}{|\mathfrak{W}_u|}\right)^2\right)^{1/2},$$

$$\sigma_\tau \leq 2\left(\frac{1}{|\mathfrak{V}_u|}\sum_{V_i/N_i\in\mathfrak{V}_u}\left(V_i/N_i - \frac{\sum_{V_i/N_i\in\mathfrak{V}_u} V_i/N_i}{|\mathfrak{V}_u|}\right)^2\right)^{1/2},$$

where $|\mathfrak{W}_u|$ is the cardinality of $\mathfrak{W}_u$.

Estimation and Testing: To estimate parameter vector $\mathbf{\Psi}$ and probabilities f_0, f_i and f_e we use an EM algorithm (Dempster, Laird and Rubin 1977; Wu 1983; McLachlan and Peel 2000; McLachlan and Krishnan 2008) with random starting values, using GENOUD (Mebane and Sekhon 2011) to execute the maximization steps in the EM algorithm.[5] The algorithm includes a thresholding technique to handle instances in which frauds appear not to occur: if estimates $\hat{f}_\mathrm{i}$ or $\hat{f}_\mathrm{e}$ fall below 10^{-9} at any point in the algorithm, then the corresponding component is dropped from the likelihood and the referent probability is set to zero. When using GENOUD we suppress all use of BFGS. The functional form of the model that includes a folded-Normal variable x raised to a real exponent (x^α), when both the variance of x and the exponent need to be estimated, creates a situation in which Newton-Raphson and similar hill-climbing optimization algorithms fail (Hager and Bain 1970).

An important side effect of the EM algorithm is that for each observation we obtain estimates of the conditional probability that each observation belongs to each of the three

[4]Klimek et al. (2012) define τ and ν to equal the first local maxima of the empirical distributions respectively of V_i/N_i and W_i/V_i. They similarly compute σ_τ and σ_ν.

[5]All code to estimate the finite mixture model is written in **R** (R Development Core Team 2011).

mixture components (Dempster, Laird and Rubin 1977, 16). In other words, for each observation i—say for each polling station—we obtain estimates of the probability that that observation is a case of no fraud ($\hat{f}_{0i}$), incremental fraud ($\hat{f}_{\mathrm{i}i}$) or extreme fraud ($\hat{f}_{\mathrm{e}i}$). The EM algorithm method does not produce measures of uncertainty for these observation-specific probabilities. We use $\hat{f}_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$ for each polling station to measure the probability that frauds occur at each polling station.

We use likelihood ratio tests to assess whether using the three-component mixture model improves the fit to the data compared to a model that merely includes the "no fraud" component, in which case $g_{0A}(\cdot)$ and $g_{0W}(\cdot)$ fully describe the likelihood. The three-component model and the no-fraud model are nested, so that two times the difference in the models' loglikelihoods gives a statistic that can be compared to a chi square distribution to compute test probabilities.[6] We use the chi square distribution with four degrees of freedom: one for each of the parameters α and θ, which exist only if there are frauds, and one for each of f_{i} and f_{e}.[7] When an election includes elections in many separate districts (as occurs in many legislative elections) and therefore we compute separate estimates for each district, we use the false discovery rate (FDR) method to correct the test statistics for multiple testing (Benjamini and Hochberg 1995).

Fraud magnitudes: With either incremental or extreme fraud, some votes that should have gone to other parties instead go to the leading party, and turnout is inflated by some nonvotes being counted as votes for the leading party. For N_i observed electors at electoral unit i, unobserved proportion τ_i of electors who turn out to vote in the absence of frauds, and unobserved proportion ν_i of votes the leading party receives in the absence of frauds, with no frauds the leading party receives $N_i \tau_i \nu_i$ votes in electoral unit i (equation (4)). With incremental fraud the leading party receives votes as defined by (2) and with extreme

[6]To prevent problems due to multiple local optima for the likelihood function, we first estimate the no-fraud model—which has a globally concave likelihood—and then use that model's loglikelihood value as the minimum acceptable loglikelihood value for the model that includes frauds components.

[7]Strictly speaking θ exists only if the incremental fraud component exists, but we do not conduct separate tests for each of the two kinds of frauds.

fraud the leading party receives votes as defined by (3).

A point estimate for the number of votes produced by "frauds" can be determined by computing $M_{\mathrm{i}} = \sum_{i=1}^{n} N_i \hat{f}_{\mathrm{ii}} \xi_{\mathrm{i}}$ and $M_{\mathrm{e}} = \sum_{i=1}^{n} N_i \hat{f}_{\mathrm{ei}} \xi_{\mathrm{e}}$, using expected proportions $\xi_{\mathrm{i}} = E[x_i(1-\tau_i) + x_i^{\alpha}(1-\nu_i)\tau_i \mid \hat{\Psi}]$ and $\xi_{\mathrm{e}} = E[(1-y_i)(1-\tau_i) + (1-y_i)^{\alpha}(1-\nu_i)\tau_i \mid \hat{\Psi}]$. To express these fraudulent vote counts as proportions of the valid votes we use $p_{\mathrm{i}} = M_{\mathrm{i}} \left(\sum_{i=1}^{n} V_i\right)^{-1}$ and $p_{\mathrm{e}} = M_{\mathrm{e}} \left(\sum_{i=1}^{n} V_i\right)^{-1}$.

4 Data and Estimation Results

Estimates of the finite mixture model's parameters using data from elections in several countries are reported in Table 1 and Figures 1–4. We report $\hat{f}_{\mathrm{i}}$, $\hat{f}_{\mathrm{e}}$, $\hat{\alpha}$, $\hat{\theta}$, $\hat{\tau}$ and $\hat{\nu}$. For all but six elections, we have polling station or precinct observations. The exceptions are five elections in Mexico where we have ballot box (*casilla*) data, and the 2013 election in Kenya, where we have ward observations.[8]

In Table 1 we report estimates for elections in which votes throughout the whole country or state determine who gets seats (Brazil 2014, California 2008, Kenya 2013, Mexico 2006 and 2012, Russia 2004, 2007, 2008, 2011 and 2012, South Africa 2014, Uganda 2006 and 2016) and for two elections in which seats are determined by province (Albania 2013 and South Africa 2014). Figures 1–4 show the distributions of the estimates across districts in elections in which seats are determined in many districts (Bangladesh 2001, Germany *Erststimmen* in 2002, 2005 and 2009, Mexico Deputies in 2006, 2009 and 2012 (*Mayoría Relativa* votes only), and Turkey 2015 June and November). In every election except two the leading party is the party that received the most votes in the country, state or district. The exceptions are the elections in Turkey, where AKP (Justice and Development Party) is

[8]Data source information: for source details regarding Albania 2013, Bangladesh 2001, Kenya 2013, South Africa 2014 and Uganda 2006 see Mebane (2015); Brazil 2014 (Tribunal Superior Eleitoral 2014); California 2008 (University of California Berkeley 2009); Germany 2002 (Bundeswahlleiter 2011), 2005 (Bundeswahlleiter 2010*a*) and 2009 (Bundeswahlleiter 2010*b*); Mexico 2006 (Instituto Federal Electoral 2007); Mexico 2009 (Instituto Federal Electoral 2015); Mexico 2012 (Instituto Federal Electoral 2012); Russia 2004, 2007, 2008, 2011 and 2012 (Central Election Commission of the Russian Federation 2013); Turkey 2015 (SEÇSiS 2015); Uganda 2016 (Development Seed 2016).

always the leading party.[9] Table 1 also includes reports of the likelihood ratio test statistic for the hypothesis that there are no frauds (i.e., that $f_{\rm i} = f_{\rm e} = 0$), along with the number of *casilla*, polling station, precinct or ward observations for each election.

The electoral systems in the various elections are diverse. Some elections have plurality winner-take-all rules (Bangladesh 2001, Brazil 2014 round 2, California 2008, Germany 2002, 2005 and 2009 *Erststimmen*, Mexico 2006 and 2012 President and 2006, 2009 and 2012 Deputies *Mayoría Relativa*, Russia 2004, 2008 and 2012), supermajoritarian winner-take-all rules (Uganda 2006 and 2016) or plurality rules with a regional dispersion requirement (Kenya 2013). One election carries the two parties with the most votes forward into a runoff election (Brazil 2014 round 1). Other elections use proportional representation rules (Russia 2007 and 2011, South Africa 2014 National) or proportional representation rules in separate provinces or districts (Albania 2013, South Africa 2014 Provincial, Turkey 2015).

In most cases allowing for frauds improves model performance significantly. The likelihood ratio test statistics reported in the 'LR' column in Table 1 show that in all but two instances the hypothesis that there are no frauds can be rejected. Only for the two elections in Brazil is the hypothesis not rejected.[10] For the elections for which we estimate the model separately in several districts, FDR adjustments across the provinces in each election show that the hypothesis is rejected in all provinces at test level .05 for Albania and South Africa (Table 1). For the elections that have parameter results displayed in Figures 1–4, the districts for which the no-frauds hypothesis is not rejected given FDR adjustment across all the districts in each election are named in Tables 2 and 3. Frauds are statistically significant in a majority of the districts in every case. In Mexico 2006, frauds are significant in all the Deputy districts.

[9]An exception to the exception is that in the Tunceli region/district the finite mixture model could not be estimated with leading party AKP because the median vote proportion for AKP is too small. In Tunceli HDP (Peoples' Democratic Party), the party that received the most votes, is the leading party.

[10]For the estimates for Brazil reported in Table 1 only votes for a party are treated as valid votes, so a "Blank" ballot is not. If "Blank" is treated as a valid vote, then $\hat{f}_{\rm i} = \hat{f}_{\rm e} = 0$.

The magnitude of frauds in terms of the number of votes the model estimates were shifted to the leading party is a way to assess frauds. For example, such estimates could be used to assess whether frauds caused postelection seat allocations to change, although we will not conduct such exercises. Table 4 and Figures 5–8 report values computed from the model parameter estimates for $M_{\rm i}$, $M_{\rm e}$, $p_{\rm i}$ and $p_{\rm i}$. Also reported are the total proportions of fraudulent votes, $p_{\rm i} + p_{\rm e}$.

The estimated numbers of votes moved by frauds range from negligible (the Brazil 2014 elections) to large. In raw numbers the largest count is $M_{\rm i}$ for the South Africa 2014 National vote ($M_{\rm i} = 882,959$), followed by the Kenya 2013 presidential vote ($M_{\rm i} = 748,330$). In proportional terms the largest proportion of fraudulent votes in Table 4 occurs in Kenya ($p_{\rm i} + p_{\rm i} = .148$), although larger proportions occur in some districts in the Bangladesh 2001 election (Figure 5) and the Turkey 2015 elections (Figure 8). The larger proportions that occur for systems that use proportional allocation—in Albania, South Africa and Turkey—are probably large enough to have affected some seat allocations.

The scatterplot of $p_{\rm i} + p_{\rm e}$ versus $\hat{\nu}$ in Figure 9 shows that in Turkey $p_{\rm i} + p_{\rm e}$ tends to be larger as $\hat{\nu}$ is smaller. Given the D'Hondt system used in Turkey (Álvarez-Rivera 2015; Turkish Press 2010; Yüksek Seçim Kurulu 2015), such small proportions of fraudulent votes might affect a few seat allocations. As shown in Figure 10, the largest proportions of fraudulent votes occur in eastern Turkey and specifically in Kurdish districts where HDP is strong (see also Mebane 2016). Figure 10 shows town-level values of $p_{\rm i} + p_{\rm e}$. The Figure shows town-level values because while we lack information about the geographic location of each polling station we do have the location of each town (Hijmans 2015). Red polygons indicate towns that have high fraudulent vote proportions and blue polygons indicate towns that have low fraudulent vote proportions. The means of the town proportions are higher than the regional proportions displayed in Figure 8(b,d) suggest. The average value of $p_{\rm i} + p_{\rm e}$ computed by town in June is .0814 and in November is .0899.[11]

[11] The reason town proportion means are higher than regional proportions traces largely to four towns in June and five in November that have proportions higher than the largest proportion observed for a region.

For winner-take-all systems judging whether frauds affected who won the election depends on knowing how close to a tie the election was without the frauds. In some cases the parameter estimate $\hat{\nu}$ may be helpful in judging the closeness of the election, although in every case more information than that is needed. For example, in plurality systems it is crucial to know how divided is the opposition to the leading party. Certainly frauds as estimated by the model are not important in the Brazil 2014 elections, and given $\hat{\nu}$ they probably made no difference for the outcomes of the 2008 U.S. presidential election in California, for the 2004, 2008 and 2012 presidential elections in Russia, nor for the 2006 and 2016 presidential elections in Uganda.[12] In light of $\hat{\nu}$ and the geographic dispersion of the wards likely to have been affected by frauds in Kenya—see the map of ward values of $p_{\mathrm{i}} + p_{\mathrm{e}}$ in Figure 11[13] (for more details see Mebane 2015)—frauds might well have affected the outcome in Kenya 2013. The number of fraudulent votes estimated for the Mexico 2006 presidential election ($M_{\mathrm{i}} + M_{\mathrm{e}} = 568,039$) is almost twice as large as the difference between the counts of votes received by first- and second-place parties in the election (Instituto Federal Electoral 2007). The scatterplots in Figure 9 show that in Germany and Mexico the proportions of fraudulent votes are not only small but $p_{\mathrm{i}} + p_{\mathrm{e}}$ is unrelated to $\hat{\nu}$. In Bangladesh $p_{\mathrm{i}} + p_{\mathrm{e}}$ is often ten times as large as in Germany or Mexico, but the scatterplots in Figure 9 show $p_{\mathrm{i}} + p_{\mathrm{e}}$ is still largely unrelated to $\hat{\nu}$.

Strategies or Frauds? It is important to consider which if any of the "frauds" estimated by the finite mixture model represent genuine frauds in the sense of maleficent acts and which arise due to normal political activity such as strategic voting. In some cases we can draw on supplemental information to reach judgments, but in other cases not. The instances where we have additional information provide some rough rules-of-thumb we

For two of the towns in June a very large proportion of votes due to extreme fraud occur (Semdinli and Hasköy) and for the other two there are somewhat high proportions (Mus merkez and Sirvan). In November very large proportions of votes due to extreme fraud again occur in the same two towns (Semdinli and Hasköy). The other three towns in November have high proportions of votes due to incremental frauds (Sirvan, Tillo and Pervari).

[12] Regarding Uganda 2006, compare Supreme Court of Uganda (2006).

[13] White polygons in Figure 11 have missing data.

might use to decide what is happening in cases where we lack additional information.

We have the most additional information about Germany and Mexico. In Germany frauds model probabilities and parameters are strongly related both to explicit measures of strategic voting and to postelection complaints that originated with citizens (Mebane, Klaver and Miller 2016). While many of the complaints trace back to problems that occurred during the election, it is not clear whether those problems systematically have to do with maleficent acts. So in Germany the frauds model estimates most likely relate to strategic behavior and only minimally if at all to genuine frauds.

Strategic voting does not leave as many readily measured traces in Mexican data as in German data, nonetheless there are many signs (Mebane 2013*a*, 2014*a*). For example, let $\mathfrak{M}_{13}$ denote the proportional margin between the first-place and third-place candidates in each district, and let $\mathfrak{M}_{23}$ denote the proportional margin between the second-place and third-place candidates. These margins arguably relate to the amount of votes that are switched due to wasted-vote strategic behavior (Cox 1994). As the regression results in Table 5 show, except for $\hat{f}_{\mathrm{e}i}$ in 2009 and 2012, in the Deputies elections these margins relate significantly and quadratically to $\hat{f}_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$.[14] In Mexico estimated frauds probabilities relate in complex ways to petitions parties filed to try to nullify the results from particular *casillas* (Mebane and Wall 2015). While the nullification petitions relate in part to election-day problems, the petitions also clearly relate to parties' tactical incentives. The frauds probabilities' complex relations with nullification petition occurrences and their relationships with the margins of victory in districts suggest many of them are at least in part responding to strategic behavior.

There are many indications that voting in the U.S. presidential election in California in 2008 is affected by significant strategic activity. The particular signals of strategic movements in votes in California are present in many other elections in other U.S. states in 2008 and in other years (Mebane 2014*a*,*b*). While it is difficult to develop a measure of

[14]Extreme fraud hardly ever occurs in the Mexico 2009 or 2012 Deputies data. There are only 12 instances where $\hat{f}_{ei} > .0001$ in 2012 and 15 in 2009. In 2006 there are 94 instances.

strategic voting for California 2008 that can be associated with the frauds estimates, most likely the frauds estimates there relate to strategic behavior (and to precinct partisan imbalances) and not much at all to genuine frauds.

A common feature of the estimates from the finite mixture model in these three countries where it is likely that the model is being triggered mostly by strategic voting is that, for the most part, $\hat{f}_{\mathrm{e}}$ is zero or negligibly small. This is true for the German *Erststimmen*, for most of the Mexican Deputies elections and for the U.S. presidential election data from California. The distributions of turnout and leading party vote proportions produced by the model when $\hat{f}_{\mathrm{i}} > 0$ but $\hat{f}_{\mathrm{e}} = 0$ is bimodal.

Another common feature is that estimates of the proportion of votes being moved by the frauds are small. Always $p_{\mathrm{i}} + p_{\mathrm{e}} < .013$ in districts in Germany—and $p_{\mathrm{i}} + p_{\mathrm{e}} > .01$ only for one district (in 2005, see Figure 6). In California $p_{\mathrm{i}} + p_{\mathrm{e}} = .00020$. In Mexico usually $p_{\mathrm{i}} + p_{\mathrm{e}}$ is small: in 2009 and in 2012 $p_{\mathrm{i}} + p_{\mathrm{e}} < .015$ for all but a few districts; in 2006 $p_{\mathrm{i}} + p_{\mathrm{e}} < .015$ for more than three-quarters of the districts. Perhaps we can treat $p_{\mathrm{i}} + p_{\mathrm{e}} = .015$ as a rough threshold for "fraudulent" votes that actually occur because of strategic voting and normal politics. Perhaps $p_{\mathrm{i}} + p_{\mathrm{e}} - .015$ is a rough estimate of the proportion of votes moved by genuinely maleficent activites.

If we adopt such a rule-of-thumb an immediate question is what to think about the estimates for Russia. For all five Russian elections, $p_{\mathrm{i}} + p_{\mathrm{e}}$ is small. However in most of the elections $\hat{f}_{\mathrm{e}}$ is not negligible. Kalinin (2016) finds that when the polling station frauds probabilities $\hat{f}_{\mathrm{i}i}$ and $\hat{f}_{\mathrm{e}i}$ for 2011 and 2012 are compared to alternative sources of information, they seem to be effectively capturing electoral anomalies. Recent Russian elections are notoriously fraudulent (Myagkov, Ordeshook and Shaikin 2009; Mebane and Kalinin 2009*a*,*b*; Kalinin and Mebane 2011; Enikolopov, Korovkin, Petrova, Sonin and Zakharov 2013). Some of the reasons for the Russian elections to lack credibility trace to their failure to meet basic conditions to be fair and competitive (OSCE Office for Democratic Institutions and Human Rights 2008): for example, if genuine challengers are kept off the ballot, it doesn't much matter what happens on election day. Some types of fraud such as the vote fakery that produces spiky patterns in turnout and vote shares (Mebane and Kalinin 2009*a*,*b*; Kalinin and Mebane 2011; Mebane 2013*b*) do not produce multimodal patterns of kinds that trigger the finite mixture model. The "fingerprints of fraud" highlighted by Klimek et al. (2012) are detected ($\hat{f}_{\mathrm{i}} > 0$ and $\hat{f}_{\mathrm{e}} > 0$), but the maleficient acts that cause those patterns are not the only fraudulent activities that affect outcomes in those elections.[15] Because $\hat{\alpha} > 1$ in all the Russian elections, the frauds the model does measure involve more vote manufacturing than vote stealing.

Some elections have large fraud magnitudes but there may be a benign explanation for those magnitudes. Contrast Albania to South Africa. Several provinces in Albania have fraudulent vote proportions that are higher than that threshold (Diber, Tirane, Korce, Vlore) or more than three time greater (Durres, Elbasan, Fier, Gjirokaster). Also in South Africa some provinces have high fraudulent vote proportions (Eastern Cape, Kwazulu-Natal, Gauteng, Mpumalanga, Limpopo, Northern Cape, North West) or very high fraudulent vote proportions (Free State, Western Cape). A very high proportion of fraudulent votes is also estimated for the National part of the South African election. In Albania coalition dynamics may have have triggered the large frauds estimates—two large coalitions contested the election (Albanian Elections Observatory Brief 2013).[16] But in South Africa it is difficult to see coalition considerations having such effects. In Albania usually $\hat{f}_{\mathrm{e}} > 0$ but not in South Africa. In South Africa usually $\hat{\alpha} < 1$ so that the frauds the model measures involve more vote stealing than vote manufacturing, while in Albania mostly $\hat{\alpha} > 1$. If the coalitional interpretation of the statistics estimated for Albania is to be sustained, it needs to explain how the coalition dynamics produce a predominance of what looks like vote manufacturing.[17]

[15]The "spikes" in turnout that are spectacularly evident in 2008 diminish in 2011 and 2012 (Mebane and Kalinin 2009*a*,*b*; Mebane 2013*b*), but $\hat{f}_{\mathrm{i}}$ and $\hat{f}_{\mathrm{e}}$ are many times larger in 2011 and 2012 than they are in 2008. Perhaps the mode of committing fraud changed between the elections.

[16]Compare patterns in Mexico (Mebane 2014*a*).

[17]Compare the many values $\hat{\alpha} > 1$ estimated for Germany (Figure 2(b,e,h)) and Mexico (Figure 3(b,e,h)).

There is evidence of genuine frauds also for some districts in other elections. In Bangladesh $p_{\text{i}} + p_{\text{e}} > .015$ in more than ten percent of districts (see Figure 5), and in a few districts $p_{\text{i}} + p_{\text{e}} > .12$. Often $\hat{f}_{\text{e}} > 0$. In Bangladesh many problems were observed in the election (European Union 2001) and frauds were alleged (Centre for Research and Information 2002). Even though the elections in Bangladesh used plurality rules in single-member districts, coalitions were a major feature of the election (an alliance won the most seats). Whether actions to support a coalition can induce proportions of "fraudulent" votes as high as occur in Bangladesh is doubtful but should be investigated.

In Turkey $\hat{f}_{\text{e}} > 0$ in a few districts, $p_{\text{i}} + p_{\text{e}} > .015$ in a few districts, and the largest proportion is less than .03 (Figure 8). As suggested by Figure 10, the largest proportions occur in eastern Turkey. These may be evidence of small frauds in Turkey, but perhaps they are offshoots of the complexity of politics and violence in those regions of the country.

Other elections have strong signs of frauds. In Kenya $p_{\text{i}} + p_{\text{e}}$ is ten times greater than our rule-of-thumb threshold for genuine frauds, and $\hat{f}_{\text{e}} > 0$. In the Uganda elections a high proportion of fraudulent votes is estimated for 2006 but not for 2016. $\hat{f}_{\text{e}}$ is negligible in both Uganda elections.

5 Discussion

The idea to use multimodality as a key marker for election frauds is supported by a clear theory, and the specific conception introduced by Klimek et al. (2012) works well to capture the realization of some kinds of frauds. But the same theory that connects multimodality to frauds also shows that multimodality *per se* is an ambiguous marker, because the same mechanism that generates multimodal distributions from frauds also may generate multimodality from strategic voting and from other normal features of politics such as coalitions. A demonstration that there are multimodal turnout and vote choice distributions is not enough on its own to support a claim that there are election frauds.

Context needs to be considered, and additional evidence is always useful.

It may be that the magnitudes of "frauds" as are measurable using our likelihood variant of the Klimek et al. (2012) conception can be used to discriminate strategic voting effects from the effects of genuine frauds in the sense of maleficent acts. Analysis of the collection of elections examined in this paper suggests that strategic voting produces estimates for the proportion of "fradulent" votes that are small. Research is needed to determine whether the presence of coalitions can produce larger proportions of votes that appear fraudulent or produce apparently trimodal distributions of turnout and leader vote proportions. Elections in some countries feature model parameter and fraud magnitude estimates that suggest that frauds occurred, but some of those elections also feature important coalitions.

Estimates of the finite mixture model using data from the 2014 presidential election in Brazil shows that the model does not necessarily produce estimates that suggest frauds occur. For the estimates that use data from the first round of the election the null result is a bit surprising, as theory suggests voters have incentives to focus their votes strategically on the top two candidates in such systems (Cox 1997, 137). So the estimates that show no "frauds" may be a challenge to a conclusion that the model is always sensitive to strategic voting. But the fact that voters decide strategically does not imply that they act differently than they would if they decided sincerely. Also the fact that voting is mandatory in Brazil may affect the number of voters who treat their choice seriously and hence act strategically.

The effect of the election system on multimodal frauds estimates needs to be further investigated. The Kenya election appears to have a high proportion of fraudulent votes. No coalitional dynamics are immediately apparent in that election, but the election rules require that the winner receive votes that are dispersed across jurisdictions in the country. Perhaps that requirement motivates activity that triggers the model. Perhaps that activity reflects benign coalition-building arrangements, but a more plausible idea is that the rules motivate vote buying and patterns of voter coercion (Hassan 2016*a*,*b*).

To use the finite mixture model to distinguish frauds from strategic voting in this paper we have introduced only rules of thumb regarding (at most) bimodality and the magnitude of estimated “frauds.” The examples of Russian elections show that small magnitudes are no guarantee that elections are free of maleficent acts, and that to draw in contextual and auxiliary information can be crucial. And thinking of Albania and Bangladesh, more research is needed to understand the effects of coalitions.

It would also be good to generalize the model to be able to adapt to features of the election system. For example, in systems with proportional representation more than one party can be a “winner” by gaining seats in the election, but the model currently requires that only one party be designated as the party that might benefit from frauds. The Klimek et al. (2012) conception has other limitations such as not being able to represent consequences of voter suppression. Given such generalizations it may be possible to address sharper questions to help use statistical analysis to distinguish strategic voting and other aspects of normal politics from genuine frauds. While theory tied to equation (1) may not transfer directly to diverse election systems, it is a reasonable conjecture that multimodalities are distinctly interesting.

As a statistical matter, it will be good as well to shift the modelling framework from the likelihood implementation to a fully Bayesian implementation. Among the advantages a Bayesian implementation should confer is the ability to state well-motivated measures of the uncertainty in quantities such as the estimated magnitude of frauds. This is another development for the hopefully near future.

The parameters of a model inspired by Klimek et al. (2012) that purport to measure election frauds sometimes also respond to strategic voting. The Klimek et al. (2012) parameters describe particular bimodal and trimodal distributions that are viewed as anomalous. But such distributions might arise as a matter of course, because of voters' strategic behavior. Strategic behavior being essential in politics, multimodal distributions should not be viewed as being *per se* generically odd.

To be able to distinguish strategic behavior from election frauds is the key challenge for election forensics. If some voters change how they vote based on strategic considerations, then the distribution of votes differs from what it would have been had the voters not done that. Votes can also change due to fraudulent manipulations. Statistical methods for detecting frauds that focus on identifying anomalous patterns in votes need to be able to tell why anomalies arise.

Mebane (2013*a*, 2014*b*) argues especially that methods based on vote counts' second significant digits are highly sensitive to strategic behavior. The concern is that all election forensic methods are sensitive to strategic behavior. Perhaps putative fraud measures are inherently and always ambiguous. The task then is to tease out when they mean one thing and when another.

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Table 1: Finite Mixture Model Parameter Estimates

Election	$\hat{f}_{\mathrm{i}}$	$\hat{f}_{\mathrm{e}}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\tau}$	$\hat{\nu}$	LR	n
Albania 2013:								
Shkoder	.042	.0043	1.7	.63	.49	.37	524.53	474
Kukes	.0066	0	.77	.76	.58	.49	41.17	165
Lezhe	.059	.0084	8.3	.38	.49	.35	642.49	256
Diber	.12	.0039	2.0	.80	.58	.41	1257.46	258
Durres	.19	.011	2.5	.80	.48	.40	6089.516	444
Tirane	.082	0	2.3	.76	.56	.44	2272.94	1,421
Elbasan	.14	.0096	1.9	.79	.54	.44	3188.29	514
Fier	.11	.022	4.8	.79	.54	.44	2619.73	558
Berat	.034	0	2.0	.27	.51	.47	71.49	299
Korce	.18	.0023	2.0	.80	.55	.44	1628.26	447
Gjirokaster	.22	0	1.3	.79	.51	.41	734.72	237
Vlore	.060	.010	8.5	.80	.42	.46	785.93	431
Brazil 2014 President:								
round 1	1.3e-7	0	2.3	.71	.91	.44	0.04	3,032
round 2	0	0	—	—	.94	.54	0	3,032
California 2008 U.S. President	.00053	0	.1	.27	.78	.59	612.6	22,691
Kenya 2013 President	.35	.0015	2.3	.51	.25	.35	2147.62	1,231
Mexico 2006 President	.064	.00020	1.2	.37	.56	.34	53360	130,768
Mexico 2012 President	.043	2.7e-09	2.1	.39	.59	.39	50588	143,337
Russia 2004 President	.049	.000087	1.7	.44	.69	.72	20290	95,424
Russia 2007 Duma	.040	.00016	1.7	.53	.67	.66	18694	95,802
Russia 2008 President	.013	.0000017	1.7	.53	.76	.70	586	96,242
Russia 2011 Duma	.12	.0032	1.8	.36	.61	.48	69244	95,166
Russia 2012 President	.084	.0020	3.4	.35	.65	.65	55352	95,413
South Africa 2014 National	.25	5.4e-7	.54	.50	.67	.79	17,935.8	20,185
South Africa 2014 Provincial:								
Eastern Cape	.12	0	.98	.45	.63	.84	3261.12	4,287
Kwazulu-Natal	.18	0	.79	.53	.71	.67	3552.28	4,506
Free State	.32	0	.57	.48	.66	.80	1345.1	1,234
Gauteng	.11	0	.52	.54	.70	.71	1326.94	2,394
Mpumalanga	.18	0	.74	.56	.69	.85	1166.24	1,516
Limpopo	.18	7.7e-9	1.1	.46	.58	.83	2922.46	2,878
Northern Cape	.20	0	.71	.51	.67	.72	399.80	500
North West	.20	0	.73	.49	.62	.76	1487.16	1,550
Western Cape	.32	0	.37	.47	.68	.80	964.84	927
Uganda 2006 President	.13	0	.066	.46	.66	.63	11744	19,395
Uganda 2016 President	.052	1.8e-6	1.0	.36	.65	.61	7588.8	27,775

Note: LR is the likelihood ratio test statistic for the hypothesis that there are no frauds (i.e., that $f_{\mathrm{i}} = f_{\mathrm{e}} = 0$). n is the number of *casilla*, polling station, precinct or ward observations.

Table 2: Districts without Significant Election Frauds Components

Country	Year	Count	District
Bangladesh	2001	133	3, 5, 6, 8, 10, 12, 13, 14, 15, 20, 22, 24, 30, 33, 34, 35, 36, 37, 38, 39, 41, 44, 45, 46, 47, 49, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 64, 65, 67, 73, 74, 75, 76, 77, 79, 80, 81, 82, 83, 84, 85, 87, 88, 90, 91, 92, 96, 100, 102, 103, 104, 105, 106, 107, 108, 109, 111, 115, 125, 129, 132, 134, 135, 136, 138, 139, 142, 150, 151, 153, 156, 157, 158, 159, 160, 167, 170, 171, 172, 173, 174, 176, 177, 180, 181, 186, 189, 191, 192, 194, 195, 197, 198, 204, 206, 207, 209, 212, 215, 216, 231, 234, 240, 243, 245, 251, 256, 260, 263, 264, 265, 271, 274, 279, 280, 285, 290, 292, 293, 295, 300
Germany	2002	106	1, 5, 6, 8, 9, 20, 21, 23, 24, 25, 27, 28, 37, 39, 41, 42, 47, 50, 51, 53, 54, 55, 64, 72, 73, 74, 76, 77, 78, 79, 84, 85, 86, 88, 89, 92, 96, 99, 101, 102, 104, 106, 111, 112, 115, 116, 118, 119, 120, 121, 123, 126, 129, 131, 134, 135, 136, 139, 140, 141, 142, 144, 145, 153, 154, 157, 160, 162, 163, 164, 165, 166, 173, 174, 175, 178, 180, 181, 185, 186, 187, 193, 194, 206, 210, 212, 219, 220, 221, 227, 228, 229, 231, 232, 246, 248, 249, 253, 258, 260, 261, 263, 276, 279, 296, 299
Germany	2005	112	1, 5, 6, 7, 9, 14, 17, 20, 23, 28, 29, 30, 34, 35, 39, 41, 46, 47, 50, 51, 53, 59, 61, 63, 64, 68, 69, 72, 73, 76, 77, 78, 79, 81, 82, 83, 87, 88, 91, 92, 96, 98, 99, 104, 105, 106, 107, 110, 111, 112, 114, 115, 116, 118, 119, 120, 123, 126, 128, 129, 130, 131, 134, 136, 139, 141, 145, 154, 157, 158, 160, 162, 164, 166, 170, 172, 174, 175, 177, 178, 180, 186, 187, 208, 209, 213, 220, 221, 222, 224, 225, 230, 231, 243, 246, 248, 253, 258, 260, 261, 263, 267, 268, 272, 273, 276, 279, 283, 286, 294, 297, 298
Germany	2009	91	1, 4, 5, 8, 11, 20, 21, 23, 25, 27, 35, 37, 40, 42, 43, 44, 48, 56, 73, 76, 78, 80, 81, 84, 88, 89, 94, 96, 98, 99, 101, 102, 103, 104, 105, 107, 109, 110, 115, 116, 117, 120, 122, 123, 124, 128, 130, 133, 136, 139, 140, 141, 143, 146, 153, 154, 160, 163, 172, 178, 179, 185, 186, 193, 207, 208, 213, 218, 219, 221, 225, 227, 229, 230, 245, 252, 259, 260, 261, 262, 263, 265, 266, 267, 271, 277, 278, 284, 290, 294, 296

Note: “District” identifies election areas for which fraud components are not statistically significant according to a likelihood ratio test with false discovery rate correction across all districts in each election. Fraud components are significant for all other “districts.”

Table 3: Districts without Significant Election Frauds Components

Country	Year	Count	District
Mexico	2006	0	
Mexico	2009	65	Baja California 8, Baja California 5, Chiapas 9, Chihuahua 3, Chihuahua 4, Chihuahua 6, DF 1, DF 2, DF 3, DF 6, DF 7, DF 8, DF 9, DF 11, DF 12, DF 13, DF 14, DF 15, DF 16, DF 17, DF 18, DF 20, DF 22, DF 23, DF 24, DF 25, DF 27, Hidalgo 6, Jalisco 9, Jalisco 10, Jalisco 13, Jalisco 14, México 13, Jalisco 16, México 1, México 4, México 6, México 15, México 16, México 17, México 19, México 20, México 21, México 24, México 25, México 26, México 29, México 30, México 31, México 32, México 33, México 34, Nuevo Leon 2, Nuevo Leon 4, Nuevo Leon 10, Puebla 6, Puebla 9, Puebla 10, Puebla 11, Oaxaca 8, San Luis Potosi 6, Sonora 3, Sonora 5, Tamaulipas 2, Tamaulipas 4
Mexico	2012	15	Baja California 1, Baja California 4, Baja California Sur 2, Chiapas 12, Chihuahua 4, DF 19, Guanajuato 10, Guanajuato 14, Jalisco 7, México 20, Nuevo Leon 1, Oaxaca 8, Quintana Roo 3, Veracruz 10, Yucatan 5
Turkey	June 2015	5	Bayburt, Çorum, Erzurum, Gümüshane, Kütahya
Turkey	Nov. 2015	8	Bayburt, Bolu, Çankiri, Gümüshane, Karabük, Kirsehir, Kütahya, Kilis

Note: “District” identifies election areas for which fraud components are not statistically significant according to a likelihood ratio test with false discovery rate correction across all districts in each election. Fraud components are significant for all other districts.

Table 4: Estimated Fraudulent Vote Counts and Proportions

Election	M_{i}	M_{e}	p_{i}	p_{e}	$p_{\mathrm{i}} + p_{\mathrm{e}}$
Albania 2013:					
Shkoder	1,218	649	.00885	.00472	.0136
Kukes	305	0	.00662	0	.00662
Lezhe	388	191	.00486	.00240	.00726
Diber	1,362	1362	.0186	.0186	.0372
Durres	9,299	2557	.0600	.0165	.0765
Tirane	10,221	0	.0227	0	.0227
Elbasan	5,939	2644	.0356	.0158	.0514
Fier	5,866	5953	.0298	.00302	.0600
Berat	156	0	.00171	0	.00171
Korce	4,472	262	.0303	.00178	.0321
Gjirokaster	2,930	0	.0494	0	.0494
Vlore	1,445	1264	.0119	.0104	.0223
Brazil 2014 President:					
round 1	1	0	7e-9	0	7e-9
round 2	0	0	0	0	0
California 2008 U.S. President	2,528	0	.000199	0	.000199
Kenya 2013 President	748,330	21,282	.144	.00411	.148
Mexico 2006 President	553,592	14,447	.0135	.000353	.0139
Mexico 2012 President	534,620	0	.0109	0	.0109
Russia 2004 President	203,955	4,951	.00297	.0000721	.00304
Russia 2007 Duma	270,490	11,914	.00395	.000174	.00413
Russia 2008 President	84,933	113	.00116	.00000155	.00116
Russia 2011 Duma	680,082	260,254	.0105	.00403	.0146
Russia 2012 President	292,339	189,912	.00413	.00268	.00681
South Africa 2014 National	882,959	18	.0506	.00000103	.0506
South Africa 2014 Provincial:					
Eastern Cape	44,406	0	.0204	0	.0204
Kwazulu-Natal	126,519	0	.0330	0	.0330
Free State	57,282	0	.0565	0	.0565
Gauteng	119,612	0	.0273	0	.0273
Mpumalanga	31,863	0	.0238	0	.0238
Limpopo	42,484	0	.0291	0	.0291
Northern Cape	10,929	0	.0259	0	.0259
North West	47,422	0	.0436	0	.0436
Western Cape	122,836	0	.0795	0	.0795
Uganda 2006 President	165,678	0	.0240	0	.0240
Uganda 2016 President	50,319	11	.00487	.00000103	.00487

Table 5: *Casilla* Frauds Probabilities Regressed on Election Margins, Mexico Deputies

2006

$\hat{f}_{\text{i}i}$		
Regressor	Coef.	SE
One	1.195e-02	4.9e-04
$\mathfrak{M}_{13}$	1.173e-04	3.7e-06
$\mathfrak{M}_{13}^2$	−3.228e-09	1.4e-10
$\mathfrak{M}_{23}$	1.281e-05	6.0e-06
$\mathfrak{M}_{23}^2$	1.138e-09	2.3e-10

$F_{4,130443} = 1147$, $p < 2.2\text{e-}16$

$\hat{f}_{\text{e}i}$		
Regressor	Coef.	SE
One	−1.654e-03	8.0e-05
$\mathfrak{M}_{13}$	2.516e-05	6.0e-07
$\mathfrak{M}_{13}^2$	−7.315e-10	2.3e-11
$\mathfrak{M}_{23}$	−2.105e-05	9.9e-07
$\mathfrak{M}_{23}^2$	8.642e-10	3.7e-11

$F_{4,130443} = 631.7$, $p < 2.2\text{e-}16$

2009

$\hat{f}_{\text{i}i}$		
Regressor	Coef.	SE
One	2.661e-02	6.8e-04
$\mathfrak{M}_{13}$	3.958e-05	5.5e-06
$\mathfrak{M}_{13}^2$	4.419e-09	1.0e-09
$\mathfrak{M}_{23}$	6.169e-05	4.0e-06
$\mathfrak{M}_{23}^2$	−7.250e-09	1.2e-09

$F_{4,138340} = 74.04$, $p < 2.2\text{e-}16$

$\hat{f}_{\text{e}i}$		
Regressor	Coef.	SE
One	2.561e-05	3.6e-05
$\mathfrak{M}_{13}$	−2.094e-09	2.9e-07
$\mathfrak{M}_{13}^2$	−4.788e-12	5.3e-11
$\mathfrak{M}_{23}$	3.397e-07	2.1e-07
$\mathfrak{M}_{23}^2$	−4.862e-11	6.4e-11

$F_{4,138340} = .6471$, $p = .63$

2012

$\hat{f}_{\text{i}i}$		
Regressor	Coef.	SE
One	3.607e-03	3.6e-04
$\mathfrak{M}_{13}$	1.187e-04	2.8e-06
$\mathfrak{M}_{13}^2$	−2.114e-09	7.2e-11
$\mathfrak{M}_{23}$	1.797e-05	4.8e-06
$\mathfrak{M}_{23}^2$	−8.635e-10	2.0e-10

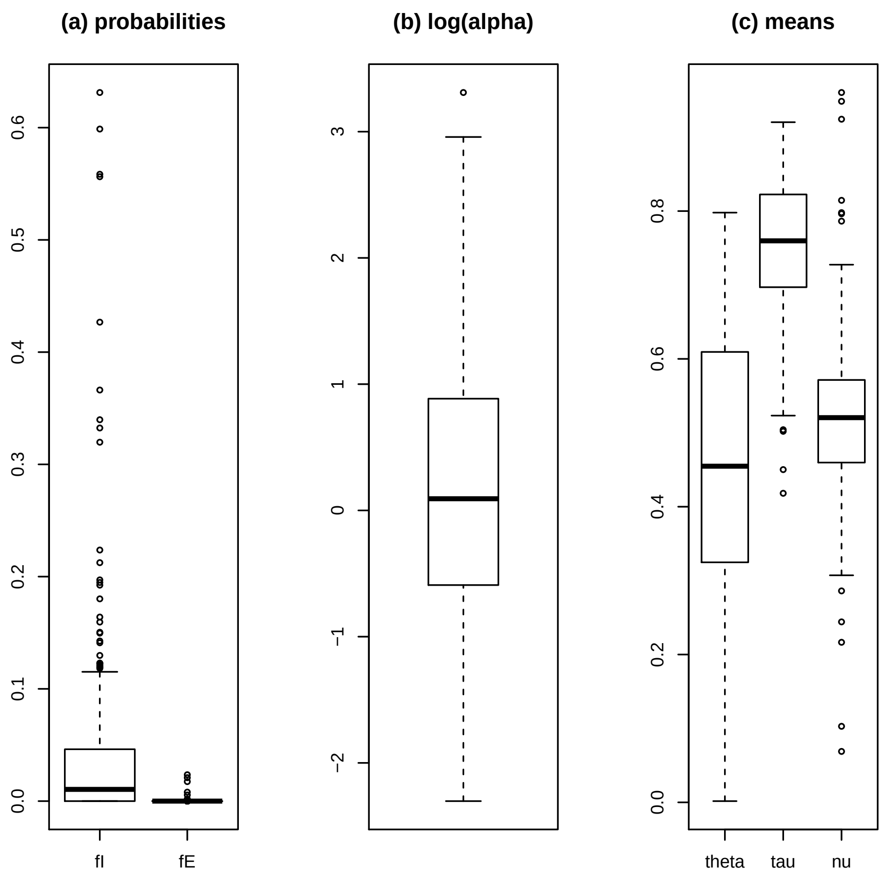
$F_{4,143010} = 2040$, $p < 2.2\text{e-}16$

$\hat{f}_{\text{e}i}$		
Regressor	Coef.	SE
One	8.078e-05	2.6e-05
$\mathfrak{M}_{13}$	−1.804e-08	2.0e-07
$\mathfrak{M}_{13}^2$	8.429e-13	5.2e-12
$\mathfrak{M}_{23}$	−1.560e-07	3.5e-07
$\mathfrak{M}_{23}^2$	5.030e-12	1.4e-11

$F_{4,143010} = .2477$, $p = .91$

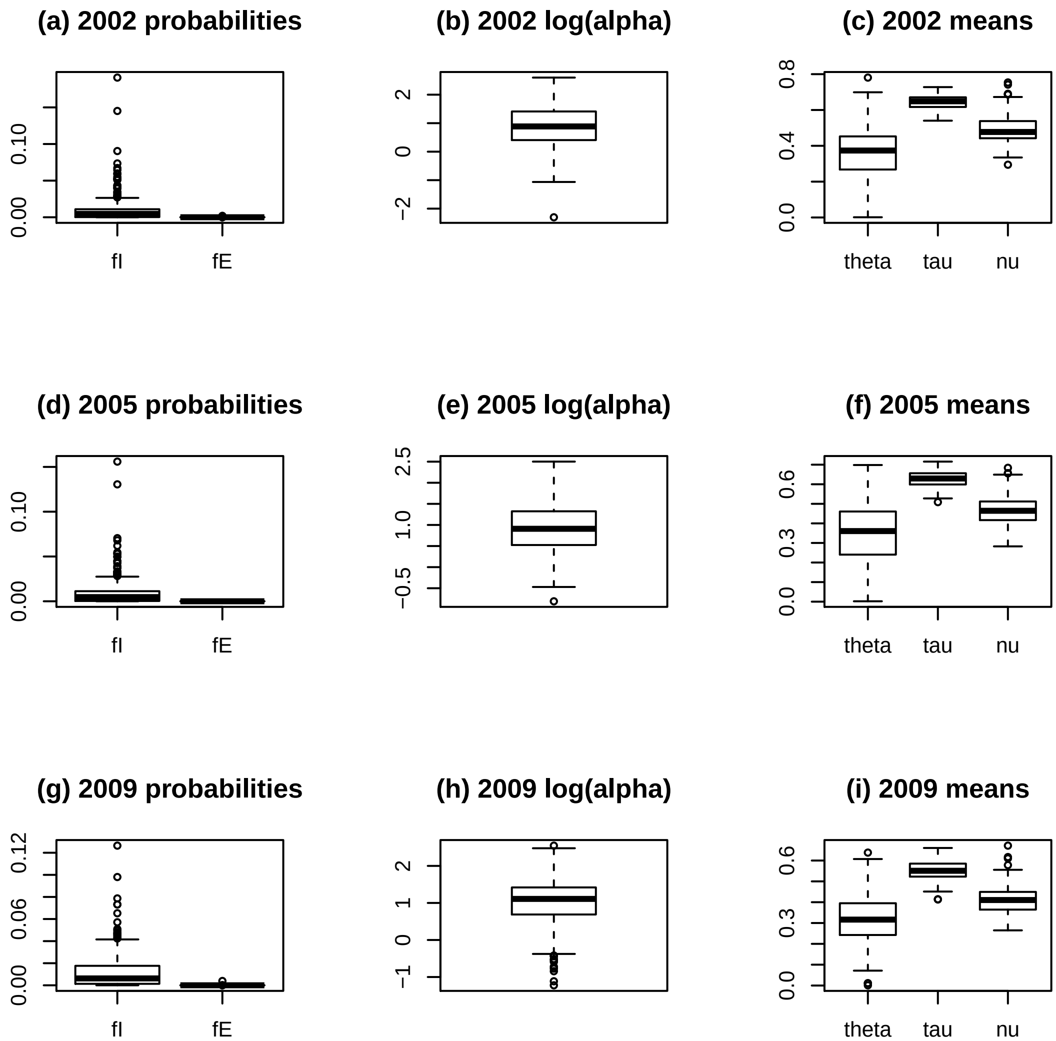
Note: linear regression of $\hat{f}_{\text{i}i}$ and $\hat{f}_{\text{e}i}$ on margins and squared margins for Mexico Deputies election *casillas*. 2006 $n = 130,448$ *casillas*; 2009 $n = 138,345$ *casillas*; 2012 $n = 143,015$ *casillas*.

Figure 1: Finite Mixture Model Parameter Estimates, Bangladesh 2001



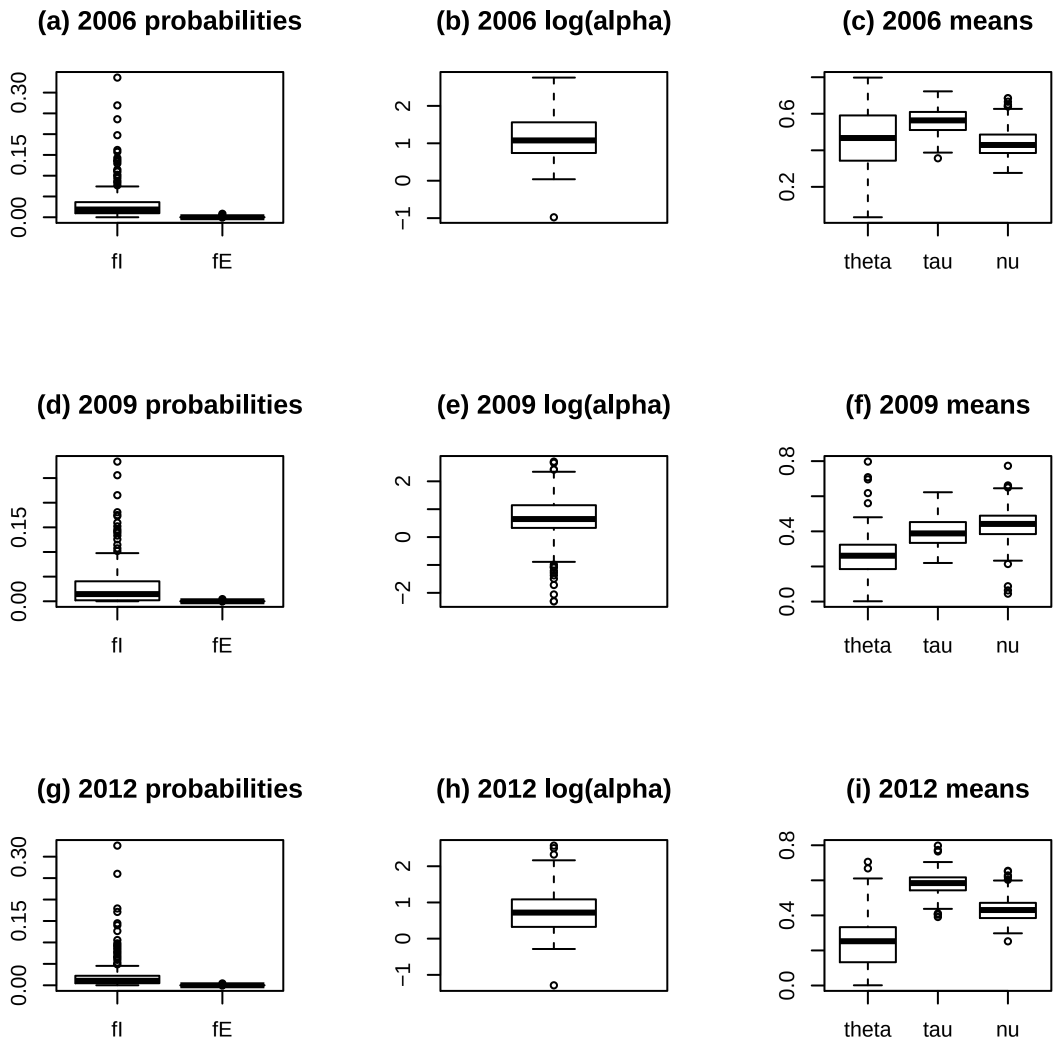
Note: distribution of district-specific estimates over 299 districts; (a) $\hat{f}_{\mathrm{i}}$, $\hat{f}_{\mathrm{e}}$; (b) $\log(\hat{\alpha})$; (c) $\hat{\theta}$, $\hat{\tau}$, $\hat{\nu}$.

Figure 2: Finite Mixture Model Parameter Estimates, Germany *Erststimmen*



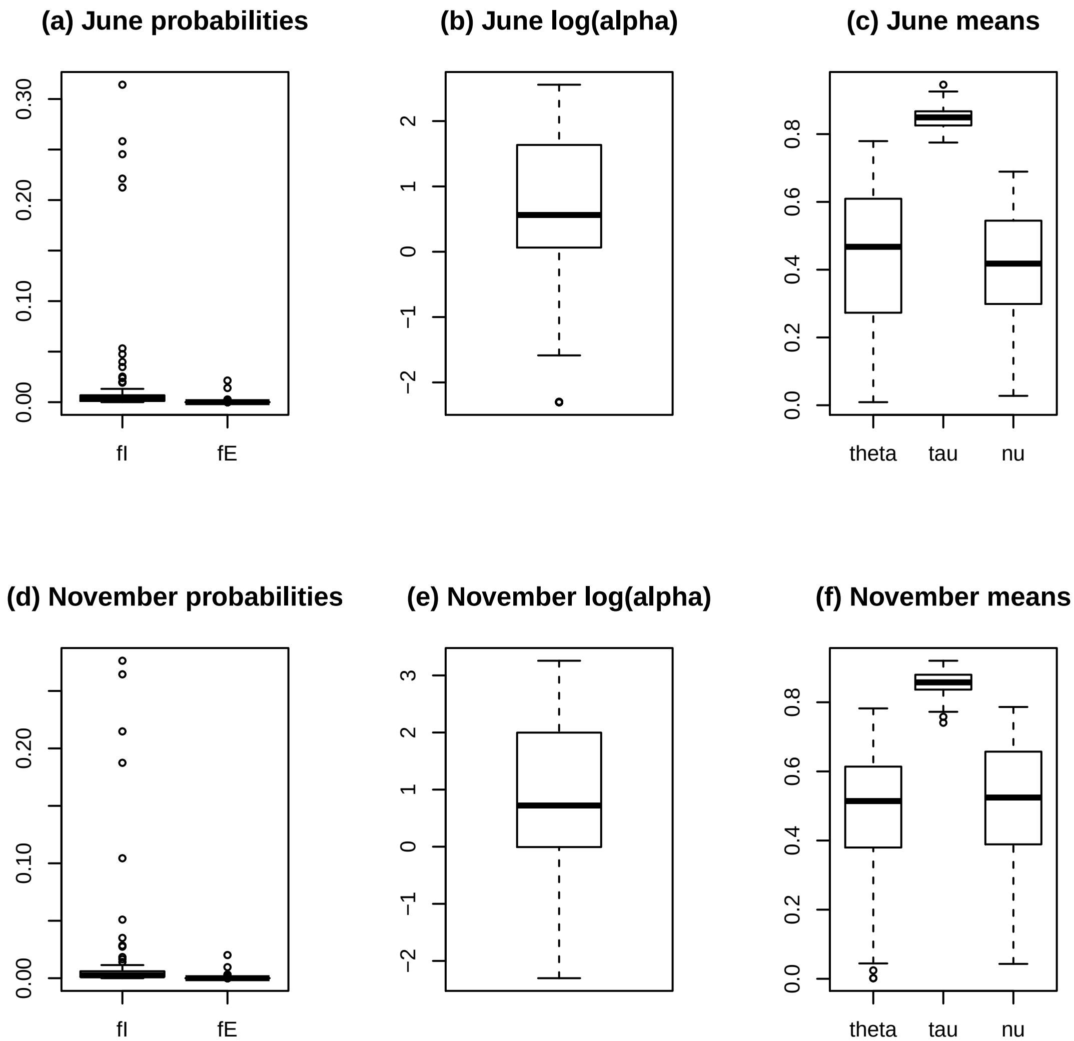
Note: distribution of district-specific estimates over 299 districts; (a) $\hat{f}_{\mathrm{i}}$, $\hat{f}_{\mathrm{e}}$; (b) $\log(\hat{\alpha})$; (c) $\hat{\theta}$, $\hat{\tau}$, $\hat{\nu}$.

Figure 3: Finite Mixture Model Parameter Estimates, Mexico Deputies



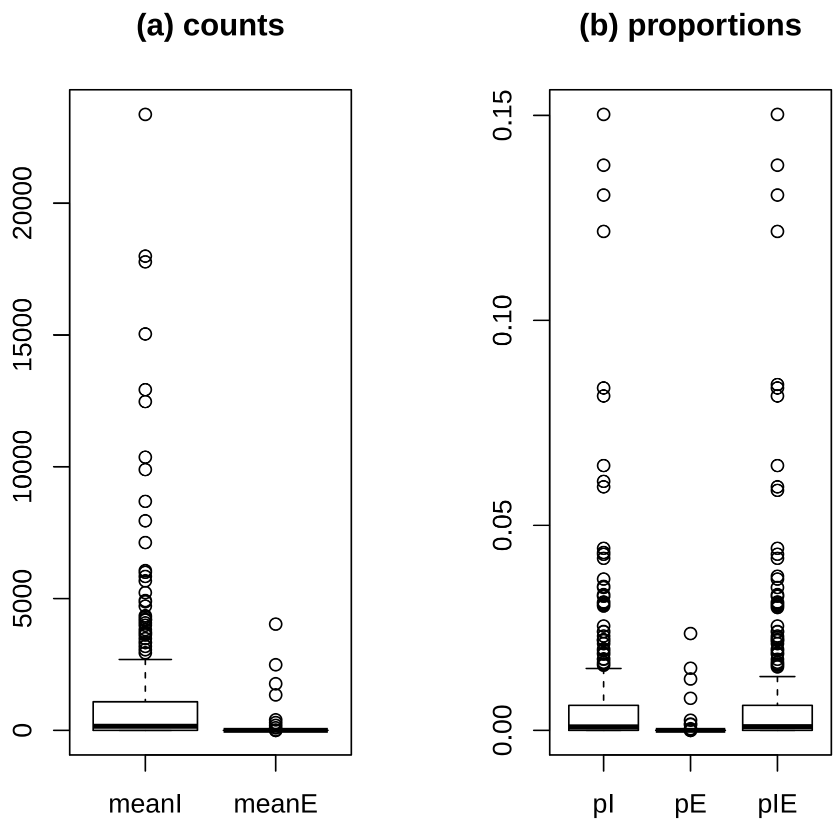
Note: distribution of district-specific estimates over 300 districts; (a) $\hat{f}_{\mathrm{i}}$, $\hat{f}_{\mathrm{e}}$; (b) $\log(\hat{\alpha})$; (c) $\hat{\theta}$, $\hat{\tau}$, $\hat{\nu}$.

Figure 4: Finite Mixture Model Parameter Estimates, Turkey 2015



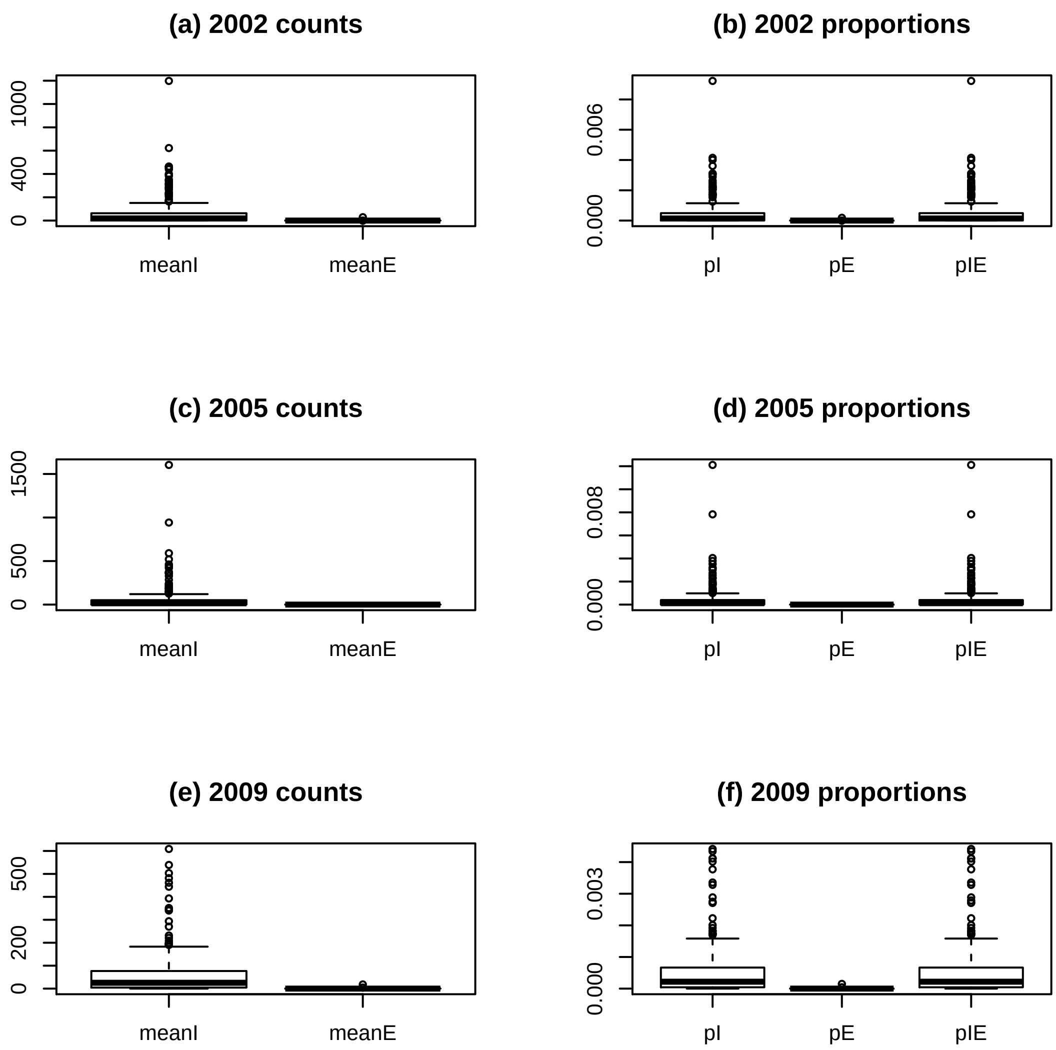
Note: distribution of district-specific estimates over 85 districts; (a) $\hat{f}_\mathrm{i}$, $\hat{f}_\mathrm{e}$; (b) $\log(\hat{\alpha})$; (c) $\hat{\theta}$, $\hat{\tau}$, $\hat{\nu}$.

Figure 5: Estimated Fraudulent Vote Counts and Proportions, Bangladesh 2001



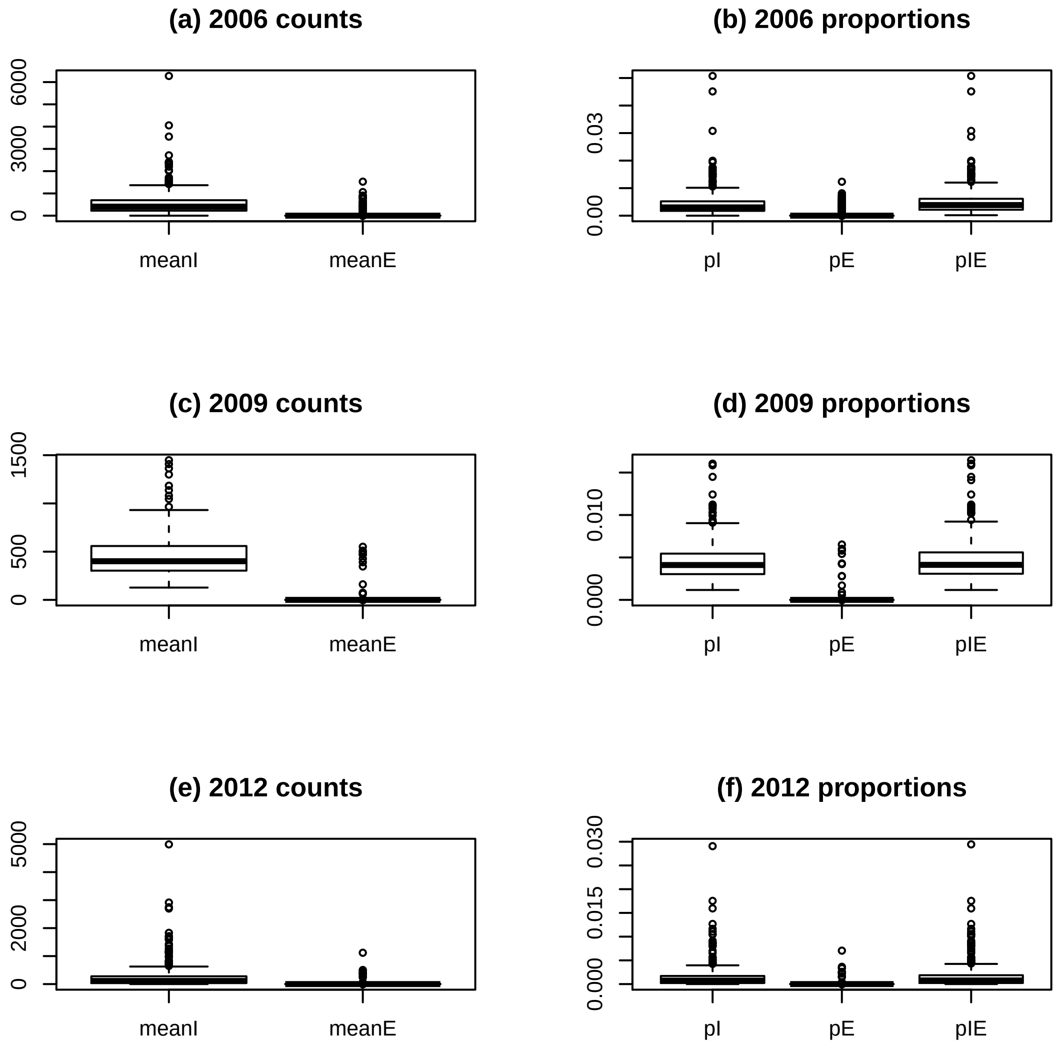
Note: distribution of district-specific estimates over 299 districts; (a) M_{i}, M_{e}; (b) p_{i}, p_{e}, $p_{\mathrm{i}} + p_{\mathrm{e}}$.

Figure 6: Estimated Fraudulent Vote Counts and Proportions, Germany *Erststimmen*



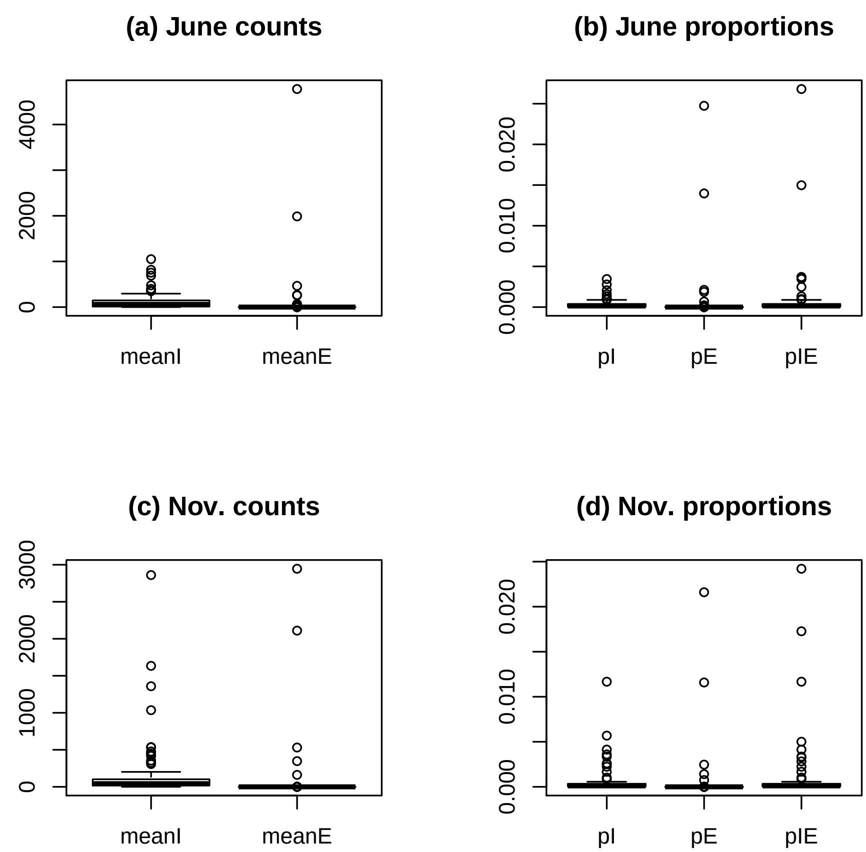
Note: distribution of district-specific estimates over 299 districts; (a) M_{i}, M_{e}; (b) p_{i}, p_{e}, $p_{\mathrm{i}} + p_{\mathrm{e}}$.

Figure 7: Estimated Fraudulent Vote Counts and Proportions, Mexico Deputies



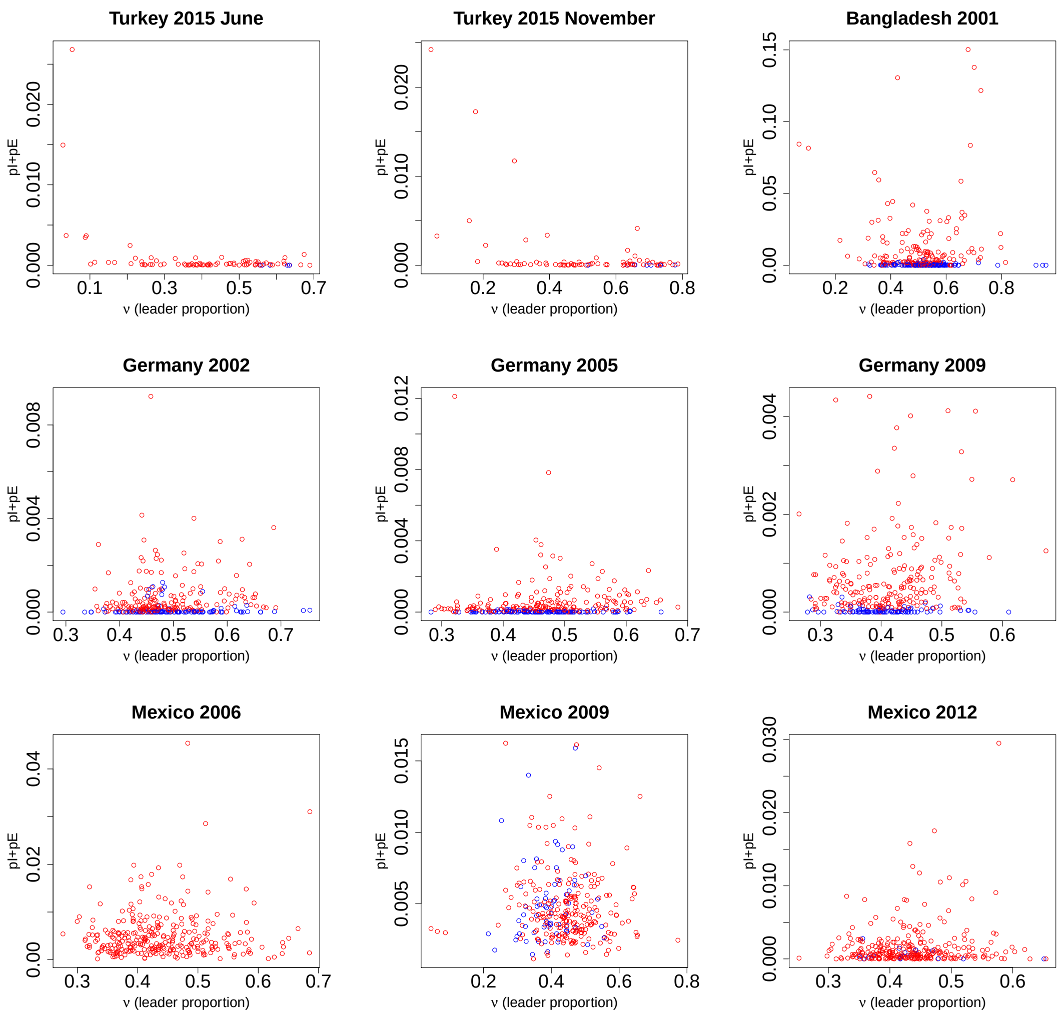
Note: distribution of district-specific estimates over 300 districts; (a) M_{i}, M_{e}; (b) p_{i}, p_{e}, $p_{\mathrm{i}} + p_{\mathrm{e}}$.

Figure 8: Estimated Fraudulent Vote Counts and Proportions, Turkey 2015



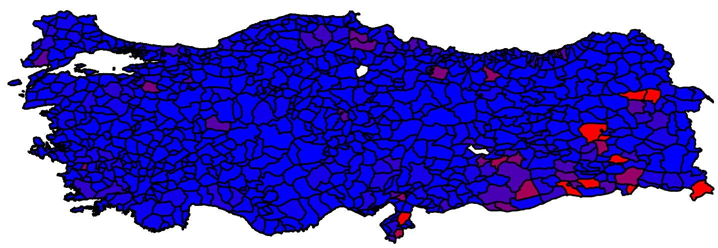
Note: distribution of district-specific estimates over 85 districts; (a) M_{i}, M_{e}; (b) p_{i}, p_{e}, $p_{\mathrm{i}} + p_{\mathrm{e}}$.

Figure 9: Mean Vote Proportions by Fraudulent Vote Proportions

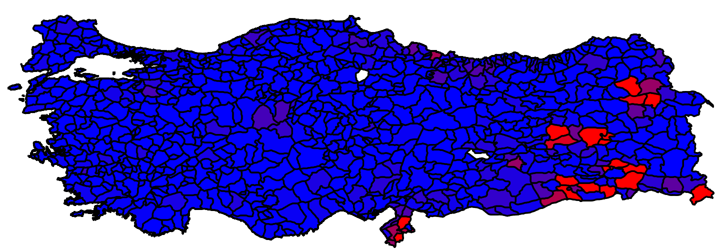


Note: scatterplots of $\hat{\nu}$ against district values of $p_{\mathrm{i}} + p_{\mathrm{e}}$. Red points are for districts in which the likelihood ratio test rejects the no-frauds hypothesis (given FDR adjustement for all districts in a given election), and blue points are for districts where the hypothesis is not rejected.

Figure 10: Fraudulent Vote Proportions by Town, Turkey 2015



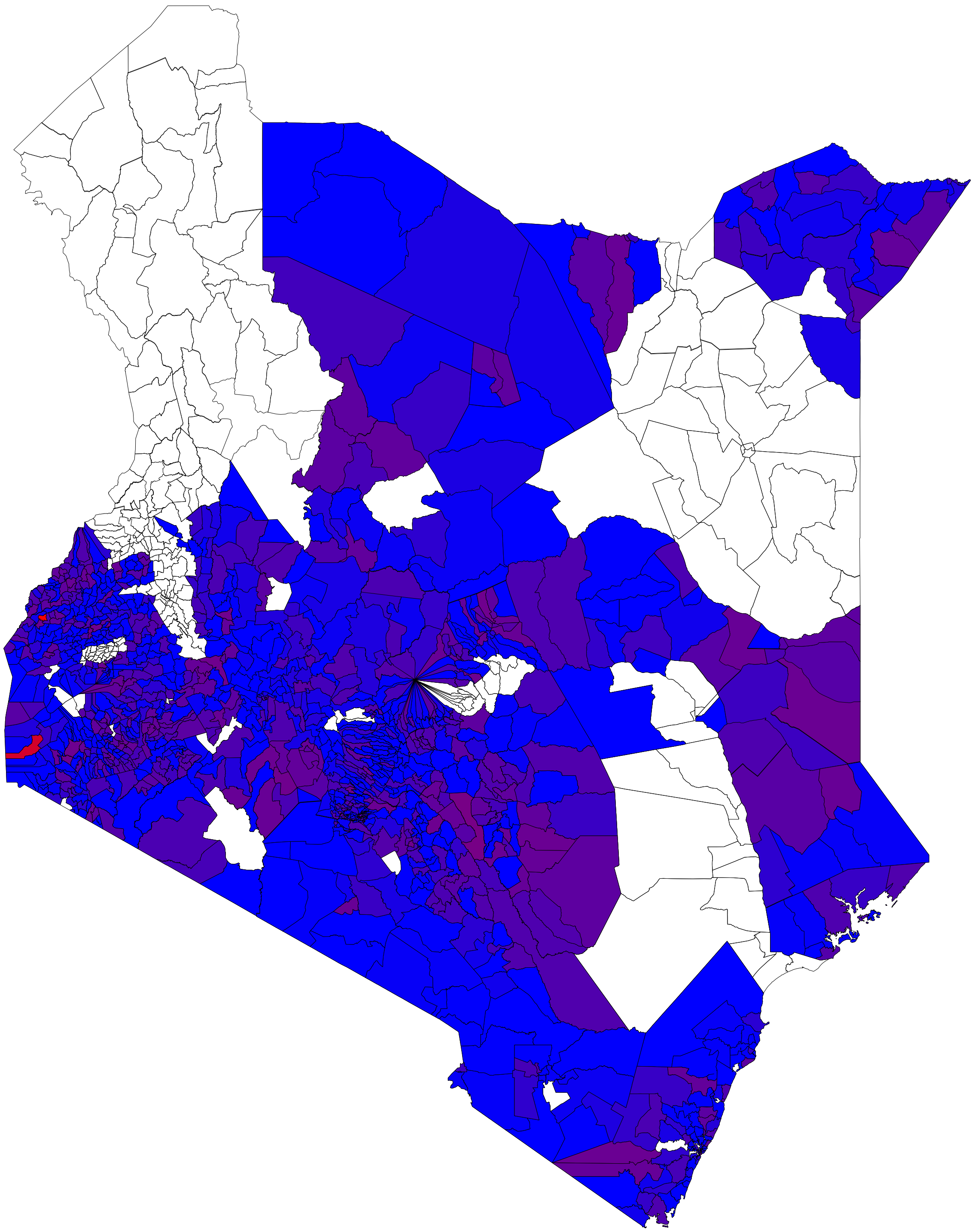
(a) June



(b) November

Note: Using AKP as the leading party in all districts (except Tunceli). Town fraudulent vote proportions computed from polling station fraudulent vote count estimates. Color red means $p_{\mathrm{i}} + p_{\mathrm{e}} = 1$, color blue means $p_{\mathrm{i}} + p_{\mathrm{e}} = 0$, and intermediate values of $p_{\mathrm{i}} + p_{\mathrm{e}}$ have colors that are weighted mixtures of red and blue.

Figure 11: Fraudulent Vote Proportions by Ward, Kenya 2013



Note: ward fraudulent vote proportions. Color red means $p_{\mathrm{i}} + p_{\mathrm{e}} = .64$ (the maximum value), color blue means $p_{\mathrm{i}} + p_{\mathrm{e}} = 0$, and intermediate values of $p_{\mathrm{i}} + p_{\mathrm{e}}$ have colors that are weighted mixtures of red and blue. White polygons have missing data.