

# Matrix Elements of the Translation Operator: Variations on the Jost-Hepp Theorem\*

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## Abstract

We consider matrix elements of the translation operator in any continuous, unitary representation  $U(a, A)$  of the covering group  $i\text{SL}(2, \mathbb{C})$  of the Poincaré group on a Hilbert space, between  $C^\infty$  and analytic vectors of the restriction  $U(0, A)$  of the representation to the homogeneous subgroup. By applying the Jost-Hepp technique, we extend the directions of rapid decrease (assuming any vacuum is excluded) to include not only spacelike ones, but all directions outside the center of momentum velocity cone generated by the four-momentum support. When the vectors are analytic vectors of  $U(0, A)$ , and have only physical momentum support contained in  $\bar{V}_+$  (again excluding any vacuum), we show that the decrease is exponential for large, spacelike translations. Then the  $p$ -space measure corresponding to the matrix element is analytic in a strip in the spatial components  $\mathbf{p}$  of the mass-momentum variables, analogous to the Jost-Hepp result that the measure is  $C^\infty$  in  $\mathbf{p}$ , for  $C^\infty$  vectors.

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## 1 Introduction

In the framework of Wightman field theory, the Jost-Hepp theorem [1, 2, 3] states that, if  $\psi$  and  $\phi$  are vectors in the domain  $D_0$  of polynomials in the smeared field operators acting on the vacuum  $\psi_0$ , and if the four-momentum spectrum associated via the SNAG theorem [4] with the unitary representation of space-time translations

$$T(a) = \exp i P \cdot a = \int dE(p) e^{ip \cdot a}$$

lies in the closed, future light-cone  $\overline{V}_+$ , and if the mass spectrum has a gap between the vacuum and the rest, then the matrix elements  $\langle \psi, T(a) \phi \rangle$  converge rapidly to the vacuum projection  $\langle \psi, \psi_0 \rangle \langle \psi_0, \phi \rangle$  for large, spacelike translations  $a^2 \rightarrow -\infty$ . That is, for any integer  $N$ , there is a finite bound  $C_N$  such that

$$|\langle \psi, T(a) \phi \rangle - \langle \psi, \psi_0 \rangle \langle \psi_0, \phi \rangle| \leq C_N (1 + |a^2|)^{-\frac{N}{2}}, \quad \text{for } a^2 < 0.$$

They made it clear in their beautiful discussion that aside from the spectrum condition, the key ingredient was the invariance of the domain  $D_0$  under the action of the infinitesimal generators  $P_\mu$  and  $M_{\mu\nu}$  of the unitary, continuous representation  $U(a, A)$ ,  $a \in \mathbb{R}^4$ ,  $A \in \text{SL}(2, \mathbb{C})$ , of the inhomogeneous  $\text{SL}(2, \mathbb{C})$  group,  $\text{iSL}(2, \mathbb{C})$ . The basic role of the field operators was just to generate a domain  $D_0$  with this property. Such a domain is often called a Gårding domain [5].

Associated with any continuous, unitary representation  $U(a, A)$  on a Hilbert space  $\mathcal{H}$ , there is a natural (essentially the largest) Gårding domain,<sup>1</sup> the  $C^\infty$  vectors of the representation [5, 6]. The  $C^\infty$  vectors are those vectors  $\phi$  for which the map  $(a, A) \rightarrow U(a, A)\phi$  is a  $C^\infty$  map of the group manifold into  $\mathcal{H}$ , in the strong topology of  $\mathcal{H}$ . We are saying nothing new when we say that the

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<sup>1</sup>It was Gårding's domain [5] that first made it possible to operate freely with infinitesimal generators. Nelson [6] emphasized that the natural, maximal extension of this property was to the  $C^\infty$  vectors.

Jost-Hepp theorem can be freed from field theory, that it applies in general to matrix elements of the translation operator between  $C^\infty$  vectors.

We are also saying nothing new when we say that the physical spectrum condition played no special role in their argument, except to pick out the spacelike directions of translation as being natural to study. They pointed that out themselves.

We do think it worth mentioning that Jost and Hepp's discussion can be extended to nonspacelike directions, even in the case of a physical spectrum, as long as we keep the translations outside the *center of momentum velocity cone*, to be described precisely in a moment, generated by the support of the measure  $d\langle E \rangle \equiv \langle \psi, dE(p) \phi \rangle$ , and that for directions inside the velocity cone, lower bounds can be put on the approach to the limit which are the same as the bounds occurring in Ruelle's elegant lemma<sup>2</sup> on smooth solutions of the Klein-Gordon equation (or in its analog for massless particles).

In fact, Ruelle's lemma becomes a special case of the Jost-Hepp theorem, when generalized in this way, if we restrict ourselves to the discrete mass spectrum. Or to turn it around, there is a generalization of Ruelle's lemma to  $C^\infty$  vectors of  $U(a, A)$ , or to  $C^\infty$  vectors of the sub-representation  $U(0, A)$  of  $SL(2, C)$ , having components in the continuous mass spectrum, which gives the Jost-Hepp theorem when we restrict to spacelike directions.

To help make the connection more precise, let us review Ruelle's results. He considers positive energy solutions of the Klein-Gordon equation,

$$\begin{aligned} \varphi(x) &= \int \frac{d^3\mathbf{p}}{2\omega} e^{-i\hat{p}\cdot x} f(\mathbf{p}), \\ \omega &= +\sqrt{m^2 + \mathbf{p}^2}, \quad m^2 > 0, \\ \hat{p} &\equiv (\omega, \mathbf{p}), \end{aligned}$$

where  $f \in \mathcal{D}(\mathbb{R}^3)$ , i.e.,  $f$  is  $C^\infty$  with compact support. (It is the smoothness, not the support property, that is essential.) Then  $\text{supp } f$  generates a timelike cone of possible classical orbits of free particles which pass through  $\mathbf{x} = 0$  when  $x_0 = 0$ :

$$\mathcal{C}(\text{supp } f) = \{x : x = \hat{p}\tau, \quad \mathbf{p} \in \text{supp } f, \quad -\infty < \tau < \infty\}.$$

$\mathcal{C}(\text{supp } f)$  is called the *velocity cone* generated by  $\text{supp } f$  because it is the set of points of the form  $\mathbf{x} = \mathbf{v}x_0$ ,  $\mathbf{v} \equiv \mathbf{p}/\omega$ , with  $\mathbf{p} \in \text{supp } f$ .

To get uniform estimates on the decrease of  $\varphi$  it is convenient to consider the enlarged cone  $\mathcal{C}(\eta)$  generated by a closed neighborhood  $\eta$  of the projection of the momenta  $\hat{p}$  in  $\text{supp } f$  (along rays through the origin in  $p$ -space) on the unit, four-dimensional sphere.

**Lemma** (Ruelle [9]). Parametrize  $x$  by  $x = u\lambda$ , where  $u$  is on the unit sphere,  $\|u\|^2 = u_0^2 + \mathbf{u}^2 = 1$ , and  $\lambda \geq 0$ . Let  $\eta$  be a fixed, closed neighborhood on the

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<sup>2</sup>A proof of this lemma was also given by H. Araki [7], for  $p$ -space wave functions in the Schwartz space  $\mathcal{S}(\mathbb{R}^3)$  [8].

unit sphere of the radial projection of  $\text{supp } f$ . Then there are finite bounds  $C$  and  $C_N$ , for every integer  $N$ , such that

- (i)  $|\varphi(u\lambda)| \leq C(1 + \lambda)^{-3/2}$ , uniformly in  $u$ ;
- (ii)  $|\varphi(u\lambda)| \leq C_N(1 + \lambda)^N$ , uniformly for  $u \notin \eta$ .

In particular, because  $\text{supp } f$  is compact,  $\eta$  can be chosen inside the light cone; and the estimate (ii) holds for all spacelike directions. If  $\text{supp } f$  is not compact, for example, if  $f$  is analytic on  $\mathbb{R}^3$ ,  $\eta$  can be chosen to extend arbitrarily little outside the light cone, and one still gets rapid decrease for spacelike directions, which need not be uniform near the light cone.

We get an immediate connection with the Jost-Hepp theorem by noting that:

(a)  $p$ -space wave functions  $f \in \mathcal{D}(\mathbb{R}^3)$  are  $C^\infty$  vectors of the irreducible representation  $[m, 0]$  of  $\text{iSL}(2, \mathbb{C})$ ;<sup>3</sup>

(b)  $\varphi(u\lambda)$  can be represented as the matrix element of the corresponding representation of the translation  $T(-u\lambda)$  between  $C^\infty$  vectors,

$$\begin{aligned} \varphi(u\lambda) &= \langle g, T(-u\lambda)f \rangle \\ &= \int \frac{d^3p}{2\omega} \bar{g}(\mathbf{p}) f(\mathbf{p}) e^{-i\hat{\mathbf{p}} \cdot u\lambda}, \end{aligned}$$

where  $g$  is any element of  $\mathcal{D}$  that is unity on  $\text{supp } f$ .

It is well known that for Ruelle's lemma to be valid it is not necessary to restrict  $f$  to  $\mathcal{D}$  [7]. In this discussion, the natural class for which Ruelle's lemma holds corresponds to the set of smooth measures  $(d^3p/2\omega)f(\mathbf{p}) = \langle \psi, dE(p) \phi \rangle$ , where  $\psi$  and  $\phi$  are  $C^\infty$  vectors of  $U(a, A)$ , or of the subrepresentation  $U(0, A)$ , having support for the moment only at a discrete point  $m \neq 0$  in the mass spectrum. It follows at once from the arguments of Jost and Hepp that, in the former case,  $f \in \mathcal{S}(\mathbb{R}^3)$ . Jost [10] has given an alternative proof of Ruelle's lemma for  $f \in \mathcal{S}$ , using classical methods. It follows from the arguments we give in this paper that in the latter case  $f \in D_{L^1}(d^3p/2\omega)$ , in a notation borrowed from L. Schwartz [8] (as are several others in this paper), the space of  $C^\infty$  functions which are  $L^1$ , along with all derivatives, relative to the measure  $d^3p/2\omega$ .<sup>4</sup> In this case the solution of the Klein-Gordon equation in  $x$ -space, which is still represented as a matrix element of the translation operator, need not be smooth; but it has the same laws of decrease as in Ruelle's lemma.

In this paper we discuss the behavior of matrix elements of the translation operator outside the velocity cone. We have also considered the behavior inside the velocity cone in an unpublished collaboration with Rudolph Seiler.

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<sup>3</sup>There is an analogous statement for type  $[m, S]$  for any spin  $S$ . Cf. Ruelle [9].

<sup>4</sup>It is natural to ask which functions in  $D_{L^1}(d^3p/2\omega)$  are generated by such measures. We do not know the precise answer, but certainly all  $C^\infty$  functions that are bounded at  $\infty$ , with all derivatives, by  $\omega^{-2}(\ln \omega)^{-1-\varepsilon}$ , for some  $\varepsilon > 0$ , are included.

Thus, let  $U(a, A)$  be a unitary, weakly continuous representation of  $i\text{SL}(2, \mathbb{C})$  on a Hilbert space  $\mathcal{H}$ . Let  $dE(p)$  be the spectral measure corresponding to the translations  $T(a) = U(a, I)$ . Until further notice, we put no restrictions on the spectrum of the four-momentum operator  $P$ . There need be no mass gap, and the momenta may be spacelike, lightlike, or timelike.

For any  $\varphi \in \mathcal{H}$ , we define  $\text{supp } \varphi$  to be the complement of the largest open set  $\Delta$  such that

$$E(\Delta)\varphi \equiv \int_{\Delta} dE(p) \varphi = 0.$$

Just as for solutions of the Klein-Gordon equation, we can associate a (center of momentum) velocity cone with  $\text{supp } \varphi$ :

$$\mathcal{C}(\text{supp } \varphi) = \{x : x = p\tau, \quad -\infty < \tau < \infty, \quad p \in \text{supp } \varphi\}.$$

And as before, we consider the velocity cone  $\mathcal{C}(\eta)$  generated by a closed neighborhood  $\eta$  on the unit sphere of the radial projection of  $\text{supp } \varphi$ . If  $0 \in \text{supp } \varphi$ , we follow the convention that the projection is the whole unit sphere, so  $\mathcal{C}(\eta)$  is the whole, four-dimensional space. By that device, the reader will shortly see that an effective, four-momentum gap condition enters; but there need be no mass gap. Since we are discussing matrix elements of energy-momentum conserving operators, it is convenient to assign a velocity cone to the support of the measure  $\langle \psi, dE(p) \phi \rangle$  rather than to one of the vectors  $\psi$  and  $\phi$ . Of course,  $\text{supp } d\langle E \rangle = \text{supp } \psi \cap \text{supp } \phi$ .

The Jost-Hepp theorem gets extended in the following way:

**Theorem A.** Let  $\psi$  and  $\phi$  be  $C^\infty$  vectors of  $U(0, A)$ . Let  $\mathcal{C}(\eta)$  be any fixed enlargement of the velocity cone  $\mathcal{C}(\text{supp } d\langle E \rangle)$ , of the sort just described.

- (i) Let  $\|u\| = 1$  and  $\lambda \geq 0$ . Then for each integer  $N$ ,

$$|\langle \psi, T(u\lambda) \phi \rangle| \leq C_N (1 + \lambda)^{-N},$$

for  $u \notin \mathcal{C}(\eta)$ . The finite bounds depend on  $\phi$ ,  $\psi$ , and  $\eta$ , but not on  $u$ .<sup>5</sup>

- (ii) Let the support be physical, excluding any vacuum;  $\text{supp } d\langle E \rangle \subset \overline{\mathbb{V}_+} - \{0\}$ . Let  $M^2 = p \cdot p$  and  $\omega_M = \sqrt{M^2 + \mathbf{p}^2}$ . Then the measure  $d\langle E \rangle$  is smooth in the coordinates  $\mathbf{p}$  in the variables  $(M^2, \mathbf{p})$ . That is, if

$$\langle \psi, dE(p) \phi \rangle = \mathcal{M}(p) d^4 p = \widehat{\mathcal{M}}(M^2, \mathbf{p}) dM^2 \frac{d^3 p}{2\omega_M}$$

and if  $h(M^2)$  is any  $dE(p)$ -measurable, essentially bounded function, then

$$\widehat{\mathcal{M}}_h(\mathbf{p}) \equiv \int \frac{dM^2}{2\omega_M} h(M^2) \widehat{\mathcal{M}}(M^2, \mathbf{p})$$

is  $C^\infty$  in  $\mathbf{p}$ .

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<sup>5</sup>In reference [11], we have used part (i) of Theorem A, stated for  $C^\infty$  vectors of  $U(a, A)$  in the physical spectrum, to prove a simple, macroscopic causality property of any Poincaré invariant  $S$  matrix.

Three remarks:

(a) An effective four-momentum gap condition enters in part (i), because if  $0 \in \text{supp } d\langle E \rangle$ , then the complement of  $\mathcal{C}(\eta)$  is empty. If  $0 \notin \text{supp } d\langle E \rangle$ , then there is a neighborhood of 0 outside  $\text{supp } d\langle E \rangle$ , since the support is closed, by definition. There need be no mass gap, but any mass zero vectors must correspond to “hard photons”.<sup>6</sup>

(b) The effect of choosing  $\psi$  and  $\phi$  to be  $C^\infty$  vectors of the full representation  $U(a, A)$  is to make the matrix element, in addition to its decrease properties, a  $C^\infty$  function of the translation four-vector  $a$ [1]. Smoothness is a trivial consequence of the definition of  $C^\infty$  vectors, and would also hold if only one of  $\psi$  and  $\phi$  were a  $C^\infty$  vector of the representation of the translation subgroup,  $T(a)$ . The effect on the measure  $d\langle E \rangle$  is to make it not only bounded, but rapidly decreasing in  $p$ .

(c) Note that the change of variables  $p \leftrightarrow (M^2, \mathbf{p})$  in part (ii) of Theorem A is regular, because  $p = 0$  is excluded from the support, so that  $p^0 = \omega_M \neq 0$ , due to the physical spectrum condition.

In the case of the physical spectrum, we could prove part (i) of Theorem A, at least if  $\psi$  and  $\phi$  were taken to be  $C^\infty$  vectors of the full representation  $U(a, A)$ , by applying Ruelle’s style of argument in the proof of part (ii) of his lemma to Jost and Hepp’s results on the regularity of the measure<sup>7</sup>  $\widehat{\mathcal{M}}(M^2, \mathbf{p})/2\omega_M$  in  $\mathbf{p}$ . An analogous argument should work for nonphysical spectra. We prefer to emphasize the functional calculus a bit more, and push the Jost-Hepp technique as far as we can. The proof of Theorem A is given in Sec. 3, starting from the basic properties of  $C^\infty$  vectors and functions of momentum listed in Sec. 2. We also mention in Sec. 3 that the regularity statements analogous to part (ii) of Theorem A can be made about  $d\langle E \rangle$  for nonphysical spectra, by choosing appropriate variables. This is a trivial generalization of a remark of Jost and Hepp [1].

In Sec. 4, we assume that  $\psi$  and  $\phi$  are analytic vectors [6] of  $U(0, A)$ , i.e., the maps  $A \rightarrow U(0, A)\phi$  and  $A \rightarrow U(0, A)\psi$  are analytic from the real analytic manifold  $\text{SL}(2, \mathbb{C})$  into  $\mathcal{H}$  in the strong topology of  $\mathcal{H}$ . We also put on the physical spectrum condition that the momentum support be in  $\overline{\mathbf{V}}_+$ . (As far as the argument is concerned, the support could have a piece in the backward cone as well.) Then rapid decrease of the matrix element of the translation operator outside the velocity cone gets sharpened to exponential decrease, which leads at once to the statement that the measure  $d\langle E \rangle$  is analytic in  $\mathbf{p}$  in the variables  $(M^2, \mathbf{p})$ .

Then it follows that the velocity cone fills the interior of the light cone, unless the measure is identically zero, or unless the spectrum has only a discrete, mass zero part. If the support has any part with  $p \cdot p > 0$ , only spacelike vectors

<sup>6</sup>We abuse the term *photon* by applying it to any massless particle, whatever its spin.

<sup>7</sup>The same word *measure* is used in this paper for  $d\langle E \rangle$ , for its density  $\mathcal{M}$  or  $\widehat{\mathcal{M}}/2\omega_M$ , and for the linear functional they define.

can be outside the cone  $\mathcal{C}(\eta)$ . Any discrete,  $pp = 0$  part could be separated out, and the additional timelike directions outside its velocity cone treated specially; but we do not do that, because a corollary to Theorem B below permits us to forget about it. It is convenient to use a Minkowski normalization for the directions, so we give them a different label  $v$ , with  $v \cdot v = -1$ . We assert the following:<sup>8</sup>

**Theorem B.** Let  $\psi$  and  $\phi$  be analytic vectors of  $U(0, A)$ , and let  $\text{supp } d\langle E \rangle \subset \overline{V}_+ - \{0\}$ . Parametrize  $T(a)$  by  $a = v\lambda$ ,  $v \cdot v = -1$ ,  $0 \leq \lambda < \infty$ . Let  $\mathcal{C}(\eta)$  be an enlargement of the light cone and its interior generated by a closed neighborhood of the intersection of  $\overline{V}_+$  with the unit, Euclidean four-sphere. Then

- (i) There are finite, positive numbers  $C$  and  $R$  such that

$$|\langle \psi, T(v\lambda) \phi \rangle| \leq C \exp -\lambda(R - \varepsilon),$$

uniformly for  $v \notin \mathcal{C}(\eta)$  and any  $\varepsilon > 0$ .

- (ii) The measure  $\widehat{\mathcal{M}}/2\omega_M$  is analytic in  $\mathbf{p}$  in the strip  $\|\text{Im } \mathbf{p}\| < R$ .

- (iii) If  $\psi$  and  $\phi$  are “sufficiently analytic”, the maximum  $R$  is controlled by

$$E_{\min} \equiv \inf_{\text{supp } d\langle E \rangle} p^0.$$

Remarks:

(a) Aside from a slight refinement in the definition of the derivative on functions of momentum, the technique of proof is just to estimate the Jost-Hepp derivative in more detail.

(b) In Wightman field theories with a mass gap, we know from the work of Araki, Hepp, and Ruelle [13] that the matrix elements of the spacelike translations between states generated from the vacuum by polynomials in the fields having compact support in  $x$ -space approach the matrix element of the vacuum projection operator exponentially, with the rate controlled by the smallest mass in the theory. Fourier transformation gives analyticity of the corresponding  $p$ -space measure in the variables  $\mathbf{p}$ , just as in part (ii) of Theorem B. Are such vectors analytic vectors of the representation  $U(0, A)$  appearing in the theory? We have not investigated what sort of converse to Theorem B holds.

(c) The corollary below says that analytic vectors with an energy gap must also have a mass gap. I.e., if there is no mass gap, there is no energy gap. We have already mentioned that  $C^\infty$  vectors may have an energy gap without a mass gap. Thus, “hard photon” vectors may be  $C^\infty$  vectors, but not analytic vectors.

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<sup>8</sup>An abbreviated version of the proof of this theorem was published in reference [12].

**Corollary.** If  $0 \notin \text{supp} \langle \psi, dE(p) \phi \rangle$ , and if there is an open set of  $p$  on the light cone ( $p \cdot p = 0$ ) and in the support, either  $\psi$  or  $\phi$  is not an analytic vector of  $U(0, A)$ .

*Proof.* If  $p = 0$ , and thus a neighborhood of  $p = 0$ , is excluded from the support, then  $\widehat{\mathcal{M}}_h(\mathbf{p}) = 0$  when  $h$  has support only near  $M^2 = 0$  and  $\mathbf{p}$  is sufficiently small. But if  $\psi$  and  $\phi$  are analytic vectors, Theorem B says that  $\widehat{\mathcal{M}}_h(\mathbf{p})$  is analytic, and hence vanishes everywhere. That contradicts the hypothesis that the support contains arbitrarily small masses.  $\square$

**Acknowledgments.** This work began during an unpublished collaboration with J. Bros, when we had occasion to study Jost and Hepp's elegant paper, and when we realized that K. Hepp [14] had already made some use of its result for  $C^\infty$  vectors of  $U(a, A)$  not generally in the field theory domain  $D_0$ .  $C^\infty$  vectors of  $U(a, A)$  have been studied from a more general viewpoint than I take here in independent work of John E. Roberts, to whom I am grateful for private communication of some of his results. Among other things, he has applied Jost and Hepp's regularity theorem corresponding to part (ii) of theorem A in this paper, to discuss the behavior of the matrix element for timelike translations. Rudolph Seiler and I have done essentially the same thing in an unpublished sequel to this paper.

Several authors have studied matrix elements of the translation operator in general representations of  $i\text{SL}(2, \mathbb{C})$  without investigating the approach to the limit. Borchers [15] shows that they decrease to zero, for any direction of translation and any vectors in  $\mathcal{H}$ , if any vacuum is excluded. H. Araki [16] asserts that the  $p$ -space measures corresponding to the one-dimensional translation subgroups  $T(u\lambda)$ , with  $u$  fixed and nonzero, have no singular part in their Lebesgue decomposition. D. Maison [17] has rediscovered the latter assertion and given a proof.

Finally, because most of the results of this and the following paper lie quite close to the surface of Jost and Hepp's, plus Ruelle's work, we think it well to emphasize that what we offer is basically a remark on what the range of validity of their theorems is. I regard the essential contributions of this paper to be the boundedness of the derivative on functions of the momentum along all directions outside the velocity cone, resulting from Eq. (3.3), and Lemma 2, which leads at once to the estimate (4.18) of the Jost-Hepp derivative between analytic vectors, and to Theorem B.

## 2 $C^\infty$ Vectors and Functions of Momentum

For the origins of the notion of  $C^\infty$  vectors of a strongly continuous representation of a Lie group by bounded operators on a Banach or Hilbert space, we refer to the works of Gårding [5] and Nelson [6], and to the references cited by Nelson. Here we just list some more or less well-known facts that we take as a starting point.

First, we discuss  $C^\infty$  vectors for a representation  $U(0, A)$  of  $SL(2, \mathbb{C})$  induced by a unitary continuous representation  $U(a, A)$  of  $iSL(2, \mathbb{C})$  on a Hilbert space  $\mathcal{H}$ . (We need a representation including translations because we want to consider functions of the momentum operators.) Then we mention how the restrictions on the rules of operation get relaxed when we consider the smaller set of  $C^\infty$  vectors of  $U(a, A)$ .

## 2.a $C^\infty$ Vectors of $U(0, A)$

Let the set of  $C^\infty$  vectors of  $U(0, A)$  in  $\mathcal{H}$ , as defined in the introduction, be denoted by  $\mathcal{K}_0$ . For any open set  $\mathcal{O} \in \mathbb{R}^4$ , we let  $\mathcal{K}_0(\mathcal{O})$  be the set of  $C^\infty$  vectors with momentum support contained in  $\mathcal{O}$ . Thus  $\mathcal{K}_0 = \mathcal{K}_0(\mathbb{R}^4)$ .

What we want are a few “algebraic” properties. The zeroth property below is never needed explicitly, but we mention it to indicate that there are enough  $C^\infty$  vectors in  $\mathcal{H}$  to make them interesting.

(0)  $\mathcal{K}_0$  is dense in  $\mathcal{H}$ . For example, the original Gårding domain is already a dense subdomain of  $C^\infty$  vectors [5], and so is the set of analytic vectors of  $U(a, A)$  [6].

(1)  $\mathcal{K}_0$  is invariant under  $U(0, A)$ :

$$U(0, A) \mathcal{K}_0 \subset \mathcal{K}_0. \quad (2.1)$$

(2)  $\mathcal{K}_0$  is invariant under the action of the six infinitesimal generators  $M_{\mu\nu}$  of  $U(0, A)$ . That is,  $\mathcal{K}_0$  is in the domain of all the self-adjoint operators  $M_{\mu\nu}$ , and

$$M_{\mu\nu} \mathcal{K}_0 \subset \mathcal{K}_0. \quad (2.2)$$

(3) Let  $D^n \equiv (\partial/\partial p^0)^{n_0} \dots (\partial/\partial p^3)^{n_3}$  be a differential monomial of order  $|n| = n_0 + n_1 + n_2 + n_3$ , and let  $\mathcal{B}_H$  be the set of  $C^\infty$  functions  $f$  on  $\mathbb{R}^4$  such that  $(1 + \|p\|)^{|n|} D^n f$  is uniformly bounded, for each  $n$ . If  $f \in \mathcal{B}_H$ , the corresponding function of the momentum operators

$$f(P) \equiv \int dE(p) f(p) \quad (2.3)$$

also has  $\mathcal{K}_0$  in its domain, and leaves  $\mathcal{K}_0$  invariant:

$$f(P) \mathcal{K}_0 \subset \mathcal{K}_0. \quad (2.4a)$$

Similarly, let  $\mathcal{B}_H(\mathcal{O})$  be the set of  $C^\infty$  functions defined on the open set  $\mathcal{O}$ , such that all derivatives satisfy the “homogeneous boundedness” condition above, uniformly on  $\mathcal{O}$ .<sup>9</sup> Then if  $f \in \mathcal{B}_H(\mathcal{O})$ ,

$$f(P) \mathcal{K}_0(\mathcal{O}) \subset \mathcal{K}_0(\mathcal{O}). \quad (2.4b)$$

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<sup>9</sup>The notation here is *not* exactly in the style of L. Schwartz. These functions need not have support contained in  $\mathcal{O}$ , and need not be defined outside  $\mathcal{O}$ .

(4) Lorentz transformations act as follows: let  $\Lambda$  be the homogeneous transformation corresponding to the element  $A \in \text{SL}(2, \mathbb{C})$ , and let  $\Lambda\mathcal{O}$  be the image of the set  $\mathcal{O}$  under  $\Lambda$ . Then

$$U(0, A) \mathcal{K}_0(\mathcal{O}) = \mathcal{K}_0(\Lambda^{-1}\mathcal{O}). \quad (2.5a)$$

If  $f \in \mathcal{B}_{\mathbb{H}}(\mathcal{O})$ , then  ${}^{\Lambda}f(P) \equiv f(\Lambda^{-1}p) \in \mathcal{B}_{\mathbb{H}}(\Lambda\mathcal{O})$ , and on  $\mathcal{K}_0(\mathcal{O})$  we have the operator identities

$$U(0, A) {}^{\Lambda^{-1}}f(P) U(0, A)^{-1} = U(0, A) f(\Lambda P) U(0, A)^{-1} = f(P), \quad (2.5b)$$

$$[M_{\mu\nu}, f(P)] = i \left[ P_{\mu} \frac{\partial f}{\partial p^{\nu}}(P) - P_{\nu} \frac{\partial f}{\partial p^{\mu}}(P) \right]. \quad (2.5c)$$

The last identity is well-defined because

$$M_{\mu\nu} \mathcal{K}_0(\mathcal{O}) \subset \mathcal{K}_0(\mathcal{O}). \quad (2.5d)$$

Properties (3) and (4) are not hard to prove from (1) and (2), plus the standard functional calculus for self-adjoint operators. From property (3) it is straightforward to prove property:

$$(\mathcal{O}') \mathcal{K}_0(\mathcal{O}) \text{ is dense in } \mathcal{H}(\mathcal{O}) \equiv E(\mathcal{O})\mathcal{H}.$$

## 2.b $C^{\infty}$ Vectors of $U(a, A)$

Let  $\mathcal{K}$  be the set of  $C^{\infty}$  vectors of  $U(a, A)$ , and let  $\mathcal{K}(\mathcal{O})$  be the subset of vectors in  $\mathcal{K}$  that have support contained in  $\mathcal{O}$ . Of course,  $\mathcal{K}(\mathcal{O})$  is contained in  $\mathcal{K}_0(\mathcal{O})$ . The set  $\mathcal{K}(\mathcal{O})$  differs from  $\mathcal{K}_0(\mathcal{O})$  primarily in being invariant under polynomials in  $P$ , and in being translation invariant:

$$P_{\mu} \mathcal{K}(\mathcal{O}) \subset \mathcal{K}(\mathcal{O}), \quad (2.6a)$$

$$T(a) \mathcal{K}(\mathcal{O}) \subset \mathcal{K}(\mathcal{O}). \quad (2.6b)$$

All the properties (0)–(4) above are valid for the domains  $\mathcal{K}(\mathcal{O})$ . In addition, the class of functions of momentum for which properties (3) and (4) are valid extends to  $\mathcal{O}_{\text{M}}(\mathcal{O})$ , the set of  $C^{\infty}$  functions defined on  $\mathcal{O}$  which are bounded at infinity, with all derivatives, by polynomials.

## 2.c Invariant Functions of Momentum

The smoothness restriction on the classes of functions of momentum considered in Secs. 2.a and 2.b can be removed to the following extent.

Let  $h(M^2)$  be any function that is measurable with respect to  $dE(p)$ . When the operator  $h(P \cdot P)$  acts on  $C^{\infty}$  vectors of  $U(a, A)$ , let  $h$  be essentially bounded by a polynomial, and when it acts on  $C^{\infty}$  vectors of  $U(0, A)$ , let it be essentially bounded by a constant.

(i) Then  $\mathcal{K}(\emptyset)$  resp.  $\mathcal{K}_0(\emptyset)$  is in the domain of  $h(P \cdot P)$  and

$$h \mathcal{K}(\emptyset) \subset \mathcal{K}(\emptyset) \quad \text{resp.} \quad h \mathcal{K}_0(\emptyset) \subset \mathcal{K}_0(\emptyset). \quad (2.7)$$

(ii) On  $\mathcal{K}(\emptyset)$  resp.  $\mathcal{K}_0(\emptyset)$ ,

$$[M_{\mu\nu}, h(P \cdot P)] = [P_\mu, h(P \cdot P)] = 0. \quad (2.8)$$

These statements remain true if  $h$  is replaced by any bounded operator  $S$  that commutes with  $U(a, A)$ . The proofs are straightforward applications of the calculus of self-adjoint operators.

### 3 Matrix Elements between $C^\infty$ Vectors of $U(0, A)$

To prove Theorem A, let's develop the method of Jost and Hepp a bit. Following them, we first consider matrix elements of more general functions of  $P$  than  $\exp i\lambda u \cdot P$ . Let  $\chi(p)$  be any uniformly bounded, continuous function on  $\mathbb{R}^4$ ; we write  $\chi \in \mathcal{B}(\mathbb{R}^4)$ . The linear space  $\mathcal{B}(\mathbb{R}^4)$  is equipped with the norm

$$\|\chi\| = \sup_{\mathbb{R}^4} |\chi|.$$

In general, we use the notation  $\|x\|$  for whatever norm belongs to the space of which  $x$  is a member. Thus for  $\chi \in \mathcal{B}(\mathbb{R}^4)$ , we define  $\|\chi\|$  as above. For  $\phi \in \mathcal{H}$ ,  $\|\phi\|^2 = \langle \phi, \phi \rangle$ . For  $p \in \mathbb{R}^4$ , the norm is Euclidean.

For any vectors  $\psi$  and  $\phi \in \mathcal{H}$ ,

$$\mathcal{M}(\chi) \equiv \langle \psi, \chi(P) \phi \rangle = \int d\langle E(p) \rangle \chi(p) \quad (3.1)$$

is a bounded measure, i.e., a bounded linear functional on  $\mathcal{B}(\mathbb{R}^4)$ :

$$|\mathcal{M}(\chi)| \leq \|\psi\| \cdot \|\phi\| \cdot \|\chi\|. \quad (3.2)$$

Now let  $\psi$  and  $\phi$  belong to  $\mathcal{K}_0$ . If we were to put on the physical spectrum condition  $\text{supp } \mathcal{M} \subset \overline{\mathbf{V}}_+$ , and if we were to restrict the direction of translation  $u$  to be spacelike, we could follow Jost and Hepp's argument very closely to show that  $\mathcal{M}/2\omega_M$  is smooth in  $\mathbf{p}$ , and that the matrix element of  $T(u\lambda)$  decreases rapidly in  $\lambda$ . The fact that we are talking here about  $C^\infty$  vectors of  $U(0, A)$  rather than  $U(a, A)$  would cause no trouble.

To discuss an unrestricted spectrum, and to include all directions  $u$  outside the velocity cone, we proceed as follows. Consider those  $\chi \in \mathcal{B}(\mathbb{R}^4)$  of the form  $\chi = \chi(u \cdot p) \equiv \chi_u(p)$ , where  $\|u\| = 1$ , and where  $\chi \in \mathcal{B}(\mathbb{R})$  as a function of its argument  $u \cdot p$ , i.e.,  $\chi$  is a  $C^\infty$  function on  $\mathbb{R}$  that is uniformly bounded, with all its derivatives. It follows that  $\chi_u \in \mathcal{B}(\mathbb{R}^4)$ . Of course, we have in mind that, for fixed  $\lambda$ ,  $\exp i\lambda u \cdot p$  is such a function; but let's leave that aside for now.

As a notation for the Euclidean dot product, we define  $\tilde{p}^\mu = p_\mu$ . Then  $p \cdot \tilde{p} = \|p\|^2$  and  $p \cdot \tilde{u} = p^0 u^0 + \mathbf{p} \cdot \mathbf{u}$ . Now let  $\mathcal{O}$  be a neighborhood of  $\text{supp } d\langle E \rangle$ ,

whose closure  $\overline{\mathcal{O}}$  excludes the origin and projects radially into the interior of  $\eta$  on the unit sphere. Such a neighborhood  $\mathcal{O}$  always exists. On  $\mathcal{K}_0(\mathcal{O})$ , the operator  $\chi(u \cdot P) \equiv \chi_u(P)$  satisfies the identity

$$\begin{aligned} \frac{\tilde{P}^\mu \tilde{u}^\nu}{G(u, P)} [M_{\mu\nu}, \chi(u \cdot P)] &= [M_{\mu\nu}, \chi(u \cdot P)] \frac{\tilde{P}^\mu \tilde{u}^\nu}{G(u, P)} \\ &= i\chi'(u \cdot P), \end{aligned} \quad (3.3)$$

where  $G(u, P) \equiv p \cdot \tilde{p} - (p \cdot \tilde{u})^2$  and  $\chi'(t) = d\chi/dt$ .

Why is this identity well defined? First, note that the function

$$Q^{\mu\nu}(u, p) \equiv p^\mu \tilde{u}^\nu / G(u, p)$$

is uniformly bounded in  $u$  and  $p$ , for  $u$  on the unit sphere outside  $\eta$  and  $p$  inside  $\mathcal{O}$ . The same is true of  $(1 + \|p\|)^{|n|} D^n Q^{\mu\nu}$ . Thus,  $Q^{\mu\nu}$  is in  $\mathcal{B}_H(\mathcal{O})$ , and the operators  $Q^{\mu\nu}(u, P)$  are uniformly bounded, for  $u$  outside  $\eta$ , on the Hilbert space  $\mathcal{H}(\mathcal{O})$ . So, in fact, are their  $n$ -fold commutators with  $M_{\mu\nu}$ 's.

The matrix element of the  $n$ -th derivative of  $\chi$  satisfies the identity

$$\begin{aligned} \mathcal{M}(\chi_u^{(n)}) &= \langle \psi, \chi^{(n)}(u \cdot P) \phi \rangle \\ &= (-i)^n \langle \psi, [M_{\mu_n \nu_n}, [\dots, \\ &\quad [M_{\mu_1 \nu_1}, \chi(u \cdot P)] Q^{\mu_1 \nu_1}] \dots] Q^{\mu_n \nu_n} \phi \rangle. \end{aligned} \quad (3.4)$$

By applying the commutation relations and properties of  $\mathcal{K}_0(\mathcal{O})$  described in Sec. 2, we can write the matrix element as a finite sum (in the manner of Jost and Hepp)

$$\mathcal{M}(\chi_u^{(n)}) = \sum_{j,k} \langle \psi_j, \chi(u \cdot P) \phi_k \rangle, \quad (3.5)$$

where  $\psi_j$  and  $\phi_k$  have the form  $\mathcal{D}(Q^{\mu\nu}) \mathcal{P}(M_{\lambda\rho}) \Phi$ , where  $\Phi$  is either  $\phi$  or  $\psi$ ,  $\mathcal{P}$  is a monomial in  $M$ 's, and  $\mathcal{D}$  is a monomial in  $Q$ 's and their commutators with  $M$ 's. It follows from our previous remark that  $\mathcal{D}$  is a bounded operator function of  $P$  on  $\mathcal{H}(\mathcal{O})$ , with bound uniform in  $u$  outside  $\eta$ . We conclude that

$$|\mathcal{M}(\chi_u^{(n)})| \leq C_n \|\chi_u\| < \infty, \quad (3.6)$$

where  $C_n$  is independent of  $u$ , outside  $\eta$ . Of course,  $\|\chi_u\|$  is independent of  $u$  for all  $u$ .

Part (i) of Theorem A follows by choosing  $\chi(u \cdot p) = \exp i\lambda u \cdot p$ , so that  $\lambda^n \chi(u \cdot p) = (-1)^n \chi^{(n)}(u \cdot p)$ .

To prove part (ii), note that the estimate in (3.6) is unchanged if we multiply  $\chi$  by any bounded,  $dE(p)$ -measurable function  $h(M^2)$ , because the operator  $h(P \cdot P)$  preserves domains and commutes with everything in sight:

$$|\mathcal{M}(h\chi_u^{(n)})| \leq C_n \|h\| \cdot \|\chi_u\|. \quad (3.7)$$

In case  $\mathcal{M}$  has support in the physical spectrum minus any vacuum, all spacelike, and in particular, all purely spatial directions are outside the velocity cone. If we let  $\chi$  be a function of  $\mathbf{p}$ , we can calculate the matrix elements of its partial derivatives along different spatial directions by formulas quite similar to those above. Letting  $\widehat{\mathcal{M}}_h(\chi) = \mathcal{M}(h\chi)$ , and

$$D^n \widehat{\mathcal{M}}_h(\chi) \equiv (-1)^n \widehat{\mathcal{M}}_h(D^n \chi), \quad (3.8)$$

where  $n = (n_1, n_2, n_3)$ , we get

$$|D^n \widehat{\mathcal{M}}_h(\chi)| \leq C_{n_1 n_2 n_3} \|h\| \cdot \|\chi\| < \infty, \quad (3.9)$$

which implies that  $\widehat{\mathcal{M}}_h(\mathbf{p})$  is  $C^\infty$ , by a theorem of L. Schwartz.<sup>10</sup>

We could arrive at the same conclusion by Fourier transformation, after noting that  $\langle \psi, h(P \cdot P) \exp(iu \cdot P \lambda) \phi \rangle$  decreases rapidly, uniformly when  $u \notin \eta$ , by applying the argument following (3.6) to the estimate (3.7).

Either of these remarks on the regularity of  $\widehat{\mathcal{M}}_h(\mathbf{p})$  under physical spectrum conditions is essentially contained in Jost and Hepp's paper, the only difference being that we have used another operator form of the derivative, and that we have refined their argument to permit zero masses and  $C^\infty$  vectors of  $U(0, A)$ , not just those of  $U(a, A)$ .

It is clear that analogous statements can be made for unphysical spectra. For example, the measure  $d\langle E \rangle$  is smooth in any three components of  $p$  when the remaining component is nonvanishing, if variables are chosen where the nonvanishing component is replaced by  $M^2$ .

## 4 Matrix Elements between Analytic Vectors of $U(0, A)$ in the Physical Spectrum

In Theorem B, the vectors  $\psi$  and  $\phi$  are analytic vectors of  $U(0, A)$ , and the intersection of their momentum supports lies in  $\overline{V}_+$  and excludes a neighborhood of  $p = 0$ . Since the spacelike direction  $v$ , with  $v \cdot v = -1$ , is always taken outside  $\mathcal{C}(\eta)$ , it is uniformly bounded away from the light cone; and its Euclidean norm varies in a compact set, depending only on  $\eta$ , which does not include zero. These restrictions on  $p$  and  $v$  make it somewhat easier to compute the operator form of the derivative with respect to  $v \cdot p$ , which is an advantage because, in order to prove Theorem B, we need more detailed estimates than those just given.

First, we choose a positive timelike four-vector  $n$ , satisfying  $n \cdot n = 1$  and  $n \cdot v = 0$ , which is to be a continuous function of  $v$ . Such an  $n$  has the property that  $n \cdot p$  is uniformly bounded away from zero, for  $p \in \text{supp } d\langle E \rangle$  and  $v \notin \mathcal{C}(\eta)$ . That is straightforward to verify, for example, if we choose  $n^\mu = \Lambda(v)^\mu_0$ , where  $\Lambda^{-1}(v)$  is the rotation-free, proper, orthochronous Lorentz transformation from  $v$  to a purely spatial vector.

<sup>10</sup>See [8], Théorème III, Vol. I, p. 54.

Now we define

$$D_v = \frac{n^\mu v^\nu}{n \cdot P} M_{\mu\nu} \equiv \frac{n^\mu v^\nu}{\Omega} M_{\mu\nu}. \quad (4.1)$$

Just as before, we choose a neighborhood  $\mathcal{O}$  which includes  $\text{supp } d\langle E \rangle$ , excludes a neighborhood of  $p = 0$ , and whose closure projects into the interior of  $\eta$ . Then on  $\mathcal{K}_0(\mathcal{O})$ , which of course contains the analytic vectors  $\psi$  and  $\phi$ , we have the identity

$$[D_v, \chi(v \cdot P)] = -i \chi'(v \cdot P). \quad (4.2)$$

Again, we choose  $\chi \in \mathcal{B}(\mathbb{R})$ . This is the same form of the derivative used by Jost and Hepp in the  $C^\infty$  case, except that we have included all spacelike and not just the purely spatial directions. They extended their estimates from spatial to spacelike directions by an invariance argument. No doubt we could do the same thing in the analytic case, but there may be a slight economy in treating all spacelike directions at once.

**Lemma 1.**  $D_v$  is *analytically dominated* by  $N \equiv n^\mu v^\nu M_{\mu\nu} = \Omega D_v$ . If  $\phi$  is an analytic vector for  $U(0, A)$  with  $\text{supp } \phi \subset \overline{V}_+ - \{0\}$ , the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|D_v^n \phi\| s^n \quad (4.3)$$

has a uniform radius of convergence for  $v$  outside  $\mathcal{C}(\eta)$ .

*Proof.* The terminology is that of E. Nelson [6]. According to Nelson (and using his notation) the series

$$(I): \quad \sum_n \frac{1}{n!} \|(\sum_{\mu < \nu} |M_{\mu\nu}|)^n \phi\| s^n$$

has a nonzero radius of convergence,  $R_I$ . The terms of the series

$$(II): \quad \sum_n \frac{1}{n!} \|N^n \phi\| s^n$$

are bounded by  $(C_{II})^n$  times the terms of the series (I) where

$$C_{II} = \max_{\mu, \nu} \sup_v |n^\mu v^\nu| < \infty.$$

Thus, the radius of convergence of (II) is at least  $R_{II} = R_I/C_{II}$ , uniformly in  $v$ . From now on, let us suppress the label  $v$  on  $D_v$ . To show that  $N$  analytically dominates  $D$  means to show that

$$\|D\phi\| \leq C \|N\phi\|$$

and

$$\|(\text{ad } D)^n N\phi\| \leq C_n \|N\phi\| ,$$

where  $\nu(s) \equiv \sum_{n=1}^{\infty} (C_n/n!)s^n$  has a positive radius of convergence. We have already mentioned that  $\Omega^{-1} \equiv (n \cdot p)^{-1}$  is a uniformly bounded function of  $p$  and  $v$  on  $\text{supp } \phi$  and outside  $\mathcal{C}(\eta)$ . Thus

$$\|D\phi\| \leq \|\Omega^{-1}\| \cdot \|N\phi\| , \quad (4.4)$$

leaving the multiple commutators to be evaluated.

To compute the iterated commutator  $(\text{ad } D)^n N$ , we look for a recursion. The first two commutators are

$$[D, N] = -i \frac{\beta}{\Omega} N, \quad \beta \equiv \frac{v \cdot p}{n \cdot p} , \quad (4.5)$$

$$[D, [D, N]] = \frac{\beta^2 - 1}{\Omega^2} N. \quad (4.6)$$

Note that  $-1 \leq \beta \leq 1$ . For  $n \geq 2$ , we make the *Ansatz*:

$$(\text{ad } D)^n N = -i^n (n-2)! \frac{h_{n-2}(\beta)}{\Omega^n} N ,$$

the form of which is easily verified by induction. By applying the rules

$$\left[ D, \frac{1}{\Omega^n} \right] = i \frac{n\beta}{\Omega^{n+1}} , \quad (4.7)$$

$$[D, f(\beta)] = i \frac{\beta^2 - 1}{\Omega} \frac{df}{d\beta} , \quad (4.8)$$

the latter of which is certainly justified when  $f$  and its derivatives are bounded, we find the recursion:

$$(\beta^2 - 1) \frac{d}{d\beta} h_{n-1} + n\beta h_{n-1} - n h_n = 0 .$$

This is one of the recursions satisfied by the Legendre polynomials,  $P_n(\beta)$ , but the starting function is

$$h_0 = \beta^2 - 1 = \frac{2}{3}(P_2 - P_0) .$$

The recursion is linear, so we conclude that

$$\begin{aligned} h_{n-2} &= \frac{2}{3}(P_n - P_{n-2}), \quad n \geq 2, \\ (\text{ad } D)^n N &= -\frac{2}{3} i^n (n-2)! \frac{P_n(\beta) - P_{n-2}(\beta)}{\Omega^n} N, \\ \|(\text{ad } D)^n N\phi\| &\leq \frac{4}{3} (n-2)! \left\| \frac{1}{\Omega^n} \right\| \cdot \|N\phi\| , \end{aligned} \quad (4.9)$$

where we have used the fact that  $|P_n(\beta)| \leq 1$  for  $-1 \leq \beta \leq 1$ . Finally, since

$$\left\| \frac{1}{\Omega^n} \right\| \leq \sup_{p,v} |(n \cdot p)^{-n}| \equiv \mu^{-n},$$

we have bounds  $C_n = \frac{4}{3} (n-2)! \mu^{-n}$ , and the series for  $\nu(s)$  certainly converges if  $|s/\mu| \leq 1$ .

According to Nelson, the radius of convergence of (4.3) is nonzero if  $\nu(s)$  has a nonzero radius of convergence, and is in fact determined by  $\nu(s)$ . Thus, we get a radius of convergence that is uniform in  $v$ .  $\square$

We use this lemma to get an estimate on

$$(-1)^n D^n \mathcal{M}(\chi) \equiv \mathcal{M}[(-1)^n \chi^{(n)}] = \mathcal{M}[(\text{ad } D)^n \chi]. \quad (4.10)$$

More explicitly, we have to estimate

$$\langle \psi, (\text{ad } D)^n \chi \phi \rangle = \sum_{r=0}^n \binom{n}{r} \langle D^{*n-r} \psi, \chi D^r \phi \rangle (-1)^r. \quad (4.11)$$

Lemma 1 tells us that  $\psi$  and  $\phi$  are analytic vectors for  $D$ , so we know how to estimate  $\|D^n \psi\|$  and  $\|D^n \phi\|$  in terms of the uniform (in  $v$ ) radii of convergence  $R_\psi$  and  $R_\phi$ , respectively. For example,

$$\|D^n \phi\| \leq C_\phi n! R_\phi^n. \quad (4.12)$$

To get an overall estimate, we must discuss the adjoint of  $D$ ,

$$D^* = N \Omega^{-1} = D + i \frac{\beta}{\Omega}, \quad (4.13)$$

and the binomial expansion for  $D^{*n}$ .

**Lemma 2.** With the usual support conditions,

$$\begin{aligned} \langle \psi, (\text{ad } D)^n \chi \phi \rangle &= \sum_{r=0}^n \sum_{\ell=0}^{n-r} \binom{n}{r \ell} \ell! (-i)^\ell (-1)^r \\ &\quad \times \langle D^{n-r-\ell} \psi, \frac{P_\ell(\beta)}{\Omega^\ell} \chi D^r \phi \rangle, \end{aligned} \quad (4.14)$$

where  $P_\ell$  is a Legendre polynomial, and  $\binom{n}{r \ell}$  is a multinomial coefficient.

*Proof.* From the *Ansatz*

$$(D + g_1)^n = \sum_{r=0}^n \binom{n}{r} g_r D^{n-r}, \quad (4.15)$$

with  $g_0 = 1$  and  $g_1 = i\beta/\Omega$ , it is easy to verify the recursion [6]

$$g_{n+1} = [D, g_n] + g_1 g_n, \quad (4.16)$$

which does not even depend on the fact that we chose  $g_1$  so that  $D^* = D + g_1$ . It is also easy to verify that the solution is

$$g_n = i^n n! \frac{P_n(\beta)}{\Omega^n}, \quad (4.17)$$

and that proves the lemma, by elementary calculation.  $\square$

Estimating  $\|P_\ell/\Omega^\ell\|$  in Lemma 2 uniformly in  $v$  by  $\mu^{-\ell}$ , and putting in the estimate (4.12) for the analytic vector  $\phi$  and the analogous one for  $\psi$ , we get

$$\begin{aligned} & |\langle \psi, (\text{ad } D)^n \chi \phi \rangle| \\ & \leq C_\phi C_\psi \|\chi\| \sum_{r=0}^n \sum_{\ell=0}^{n-r} \binom{n}{r \ell} \ell! (n-r-\ell)! r! \\ & \quad \times \left(\frac{1}{R_\psi}\right)^{n-r-\ell} \left(\frac{1}{R_\phi}\right)^r \left(\frac{1}{\mu}\right)^\ell, \\ & \leq C_\phi C_\psi \|\chi\| \left(\frac{n+2}{2}\right)! \frac{1}{R^n}, \quad R \equiv \min[R_\psi, R_\phi, \mu] > 0. \end{aligned} \quad (4.18)$$

The estimate (4.18) is the essential result of this discussion.

It leads immediately to the exponential decrease of the matrix element of  $T(\lambda v)$ . To see that, put

$$\chi(v \cdot P) = \exp i\lambda v \cdot P \exp \lambda(R - \varepsilon), \quad (4.19)$$

where  $\varepsilon$  is any number between 0 and  $R$ , and  $0 \leq \lambda < \infty$ ; and substitute the power series expansion for  $\exp \lambda(R - \varepsilon)$ :

$$\begin{aligned} & \left| e^{\lambda(R-\varepsilon)} \langle \psi, e^{i\lambda v P} \phi \rangle \right| \\ & \leq \sum_{n=0}^{\infty} \frac{(R-\varepsilon)^n}{n!} |\langle \psi, (\text{ad } D)^n e^{i\lambda v P} \phi \rangle|, \\ & \leq \frac{C_\psi C_\phi}{2} \sum_{n=0}^{\infty} \left(\frac{R-\varepsilon}{R}\right)^n \frac{(n+2)!}{n!} \equiv C(\varepsilon) < \infty. \end{aligned} \quad (4.20)$$

The bound  $C(\varepsilon)$  is uniform in  $v$ , as long as  $v$  is outside the cone  $\mathcal{C}(\eta)$ .

Just as in the  $C^\infty$  case, and for the same reason, all estimates are preserved if we multiply  $\chi$  by a  $dE(p)$ -measureable, essentially bounded function  $h(M^2)$ , up to a factor  $\|h\|$  in certain bounds. The estimate above becomes

$$|\langle \psi, h(P \cdot P) \exp i\lambda v \cdot P \phi \rangle| \leq \|h\| C(\varepsilon) e^{-\lambda(R-\varepsilon)}. \quad (4.21)$$

It is immediate from this estimate that  $\widehat{\mathcal{M}}_h(\mathbf{p})$  is analytic in the strip  $\|\text{Im } \mathbf{p}\| < R$ , by Fourier transformation.

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