The Dirac Algebra for Any Spin*

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I. Introduction

The proliferation of resonances in elementary particle physics has brought about a renewed interest in formalisms for describing processes involving particles of higher spin. Of course a number of formalisms, based on higher spin wave equations, have existed for many years, for example, those of Dirac [1], Fierz [2], Fierz and Pauli [3], Bargmann and Wigner [4], and Rarita and Schwinger [5]. All of these theories use first-order wave equations, and for spins higher than one-half require subsidiary conditions in order to restrict the number of distinct spins to one.

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For several reasons, there have not been any successful Lagrangian field theories for higher spin based on these equations. First of all, as emphasized for example by Fierz [2] and Fierz and Pauli [3], it is difficult to maintain consistency between the wave equations and the subsidiary conditions when interactions are included. Secondly, there is the nonrenormalizability of such interactions.[1]

In recent years, the situation has changed somewhat. One no longer insists on the Lagrangian formalism (which is not to say that it is no longer interesting); but instead one tries to discover as much as possible from general principles such as symmetry, causality, and analyticity. On this view, wave equations are restricted to a kinematic role, when they have any role at all, being applied only to the description of free particles. Their main value is that their solutions form Hilbert spaces on which representations of the Lorentz group are concretely realized.

With this view, none of the objections mentioned before is applicable. Moreover, there is no longer any reason to restrict the wave equations to be first order, especially if by doing so one can eliminate the subsidiary conditions. In fact, the emphasis is just the other way. One decides what representation of the group one wants to consider, and only then asks what is the most convenient wave equation corresponding to it.

In this lecture, we describe free, massive particles of spin j by means of a $2(2j{+}1)$-component formalism, based on the representation $(j,0)\oplus(0,j)$ of the homogeneous Lorentz group. Here (j',j), with j' and j half integers, refers to an irreducible, finite-dimensional representation of the connected component of the homogeneous Lorentz group, $\mathrm{L}_+^\uparrow$; and the direct sum $(j,0)\oplus(0,j)$ corresponds in the well-known way to an irreducible representation of the group L_+, which is $\mathrm{L}_+^\uparrow$ plus space reflection.

The Dirac equation corresponds to the representation $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$; and there is a very direct analog of the Dirac equation for the representation $(j,0)\oplus(0,j)$, as well as a direct analog of the Dirac algebra. In fact, the main advantage of the $2(2j{+}1)$-component formalism is that all analogies with the Dirac theory are very close; and one does not really have to learn anything new in order to use it.

The $2(2j{+}1)$-component formalism, with the generalized Dirac equation, is certainly not an essentially new development. The ideas are to a large extent anticipated in the work of Fierz [2]. In the form given here, the formalism has occurred to a number of people. The first explicit discussion known to me is contained in a paper by H. Joos [7], which also gives the connection with the theory of Fierz. Subsequently, S. Weinberg [8] has published a similar construction, arrived at in the course of developing Feynman rules for any spin. These lectures are a result of Weinberg's encouragement to revive and improve an earlier unpublished study on the generalized Dirac algebra.

What I propose to do is first to review the spinor calculus and the construction of generalized Pauli matrices for any spin, and then to list a few properties

[1] *Cf.* H. Umezawa [6].

of the wave equation and its solutions. Next, the main business of these lectures is to show how to compute and classify the analogs of the matrices in the Dirac algebra, I, γ_μ, $\sigma_{\mu\nu}$, $\gamma_5\gamma_\mu$, and γ_5, and to derive the relations among them and their traces. This turns out to be quite easy, and it all follows from the simple observation that the classification of matrices in the Dirac algebra corresponds to a Clebsch-Gordan analysis.

Finally, I intend to mention very briefly the representation of scattering amplitudes. In that respect, it is pertinent to remark that the $2(2j{+}1)$-component formalism offers no substantial advantage over the nonredundant, $(2j+1)$-component formalism, even when discrete symmetries are included, and that, on the other hand, it is also not particularly more difficult to manipulate. Thus, for such applications it seems to be primarily a matter of taste which formalism one should choose.

II. Spinor Calculus

Although I shall try to keep it as painless as possible, it is necessary to use a few detailed properties of spinors and Clebsch-Gordan coefficients in order to get the basic results across. The more important notions are reviewed here.

Recall that $\mathrm{L}_+^\uparrow$ is the set of real, orthochronous, unimodular, 4×4 matrices, Λ, satisfying

$$\Lambda^{\mathrm{T}}\, G\, \Lambda = G\,, \qquad G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1}$$

where "T" means transpose. I shall use the notation "$*$" for complex conjugate, and "$\dagger$" for Hermitean conjugate. The spinor claculus is based on the well-known two-to-one homorphism between $\mathrm{SL}(2,\mathrm{C})$, the group of 2×2 unimodular matrices, and $\mathrm{L}_+^\uparrow$.

This correspondence can be expressed explicitly with the help of the 2×2 Pauli matrices, written as a four-vector σ_μ with

$$\sigma_0 = I\,, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

We use a special symbol for σ_μ with the space components multiplied by -1, namely,

$$\widetilde{\sigma}_\mu \equiv \sigma^\mu\,. \tag{3}$$

Because the σ_μ are a complete set, we have a 1-1 correspondence between four-vectors v^μ and 2×2 matrices V expressed by

$$V = v^\mu \sigma_\mu \equiv v\cdot\sigma\,, \qquad \det V = v\cdot v\,, \tag{4}$$

where the summation convention is used. The reality of v implies that V is Hermitean. Any $A \in \mathrm{SL}(2, \mathrm{C})$ induces a Lorentz transformation Λ, according to

$$A\, v{\cdot}\sigma A^{\dagger} = (\Lambda v){\cdot}\sigma \,, \tag{5}$$

from which one finds

$$A\, \sigma_{\mu} A^{\dagger} = \Lambda^{\nu}{}_{\mu}\, \sigma_{\nu} \,, \tag{6a}$$

$$\Lambda_{\mu\nu} = \frac{1}{2} \operatorname{Tr} \left(\widetilde{\sigma}_{\mu} A\, \sigma_{\nu} A^{\dagger} \right) \,. \tag{6b}$$

For $\widetilde{\sigma}_{\mu}$ one finds that

$$A^{\dagger -1}\, \widetilde{\sigma}_{\mu}\, A^{-1} = \Lambda_{\mu}{}^{\nu}\, \widetilde{\sigma}_{\nu} \,. \tag{7}$$

According to (5), three-dimensional, proper rotations (the group $\mathrm{O}_{+}(3)$) are represented by unitary matrices $U \in \mathrm{SL}(2, \mathrm{C})$, and Lorentz transformations from the rest-frame vector $(m, 0, 0, 0)$ to k, where $k{\cdot}k = m^2 > 0$, by

$$A(k) = \sqrt{k{\cdot}\sigma/m}\, U \,, \tag{8}$$

where the Hermitean, positive-definite square root is intended, and where U is an arbitrary unitary element of $\mathrm{SL}(2, \mathrm{C})$.

In the spinor calculus,[2] one writes indices according to the following scheme. Let ζ be a two-component spinor. If it transforms according to A, *i.e.*,

$$\zeta' = A\zeta \,,$$

then we write ζ_{α}, with a lower undotted index,[3] taking the values $\pm\frac{1}{2}$. If it transforms according to A^{*} we write $\zeta_{\dot{\alpha}}$; and if it transforms according to the respective contragredient transformations, $A^{-1\mathrm{T}}$ and $A^{-1\dagger}$, we write ζ^{α} and $\zeta^{\dot{\alpha}}$. The summation convention applies to repeated upper and lower indices of the same type. Thus $\zeta_{\alpha}\eta^{\alpha}$ and $\zeta_{\dot{\alpha}}\eta^{\dot{\alpha}}$ are invariant forms.

Raising and lowering of indices is effected by the metric symbol

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon^{-1}, \tag{9}$$

which has the property, for any $2{\times}2$ matrix M,

$$\varepsilon M \varepsilon^{-1} = M^{-1\mathrm{T}} \det M \,. \tag{10}$$

Thus ε takes any $A \in \mathrm{SL}(2, \mathrm{C})$ into its contragredient form, $A^{-1\mathrm{T}}$. We write the spinor indices of ε as

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}}, \qquad \varepsilon^{-1}{}_{\alpha\beta} = \varepsilon^{-1}{}_{\dot{\alpha}\dot{\beta}} \,,$$

[2] *Cf.* Bade and Jehle [9] and Corson [10], and for the general spin notation, References [7] and [13].

[3] We use letters from the first part of the Greek alphabet for spinor indices, and letters from the latter part for tensor indices.

and we follow the rule of always contracting with the right index of ε for raising and with the right index of ε^{-1} for lowering of indices.

With these conventions, the Pauli matrices have indices $\sigma_{\mu\alpha\dot\beta}$ and $\widetilde{\sigma}_\mu{}^{\dot\alpha\beta}$, from (6a) and (7); and they satisfy orthogonality relations

$$\tfrac{1}{2}\,\sigma_{\mu\alpha\dot\beta}\,\widetilde{\sigma}^{\mu\dot\beta'\alpha'} = \delta_\alpha{}^{\alpha'}\delta_{\dot\beta}{}^{\dot\beta'}, \tag{11a}$$

$$\tfrac{1}{2}\,\sigma_{\mu\alpha\dot\beta}\,\widetilde{\sigma}^{\mu\dot\beta\alpha} = g_{\mu\nu}\,. \tag{11b}$$

The whole apparatus carries over for all finite-dimensional representations of $\mathrm{L}_+^\uparrow$ (*i.e.*, of $\mathrm{SL}(2,\mathrm{C})$) [9, 10]. We define a set of matrices $\mathrm{D}^j(A)$, where j is a half-integer, inductively by means of Clebsch-Gordan (CG) coefficients $\mathrm{C}(j_1j_2j_3;a_1a_2a_3)$ (in Rose's notation [11]):

$$\begin{aligned}\mathrm{D}^j(A)_{ab} = \sum_{a_1a_2b_1a_2} &\mathrm{C}(j-\tfrac{1}{2},\tfrac{1}{2},j;a_1a_2a)\\ &\times \mathrm{C}(j-\tfrac{1}{2},\tfrac{1}{2},j;b_1b_2b)\,\mathrm{D}^{j-\frac{1}{2}}(A)_{a_1b_1}\mathrm{D}^{\frac{1}{2}}(A)_{a_2b_2}\,,\end{aligned} \tag{12}$$

where $\mathrm{D}^{\frac{1}{2}}(A)\equiv A$. Equation (12) defines $\mathrm{D}^j(O)$ for any 2×2 matrix O, which may even have operator-valued matrix elements. For $A\in\mathrm{SL}(2,\mathrm{C})$, the $\mathrm{D}^j(A)$ form a representation of $\mathrm{L}_+^\uparrow$. They satisfy the identities: $\mathrm{D}^j(A^*)=\mathrm{D}^j(A)^*$, $\mathrm{D}^j(A^\mathrm{T})=\mathrm{D}^j(A)^\mathrm{T}$, $\mathrm{D}^j(A^\dagger)=\mathrm{D}^j(A)^\dagger$, $\mathrm{D}^j(A^{-1})=\mathrm{D}^j(A)^{-1}$. The general finite-dimensional, irreducible representation of $\mathrm{L}_+^\uparrow$, labeled by two half integers (j',j), is given by the direct product $\mathrm{D}^{j'}(A)\otimes\mathrm{D}^j(A)^{-1\dagger}$.

Now we define $(2j+1)$-component spinors, whose indices take the values $-j,-j+1,\ldots,j$, with

$$\zeta_\alpha\,,\quad \zeta_{\dot\alpha}\,,\quad \zeta^\alpha,\quad \zeta^{\dot\alpha}$$

transforming respectively according to $\mathrm{D}^j(A)$, $\mathrm{D}^j(A)^*$, $\mathrm{D}^j(A)^{-1\mathrm{T}}$, and $\mathrm{D}^j(A)^{-1\dagger}$. The metric symbol becomes

$$\begin{aligned}&[\,j\,]^{\alpha\beta}\equiv\mathrm{D}^j(\varepsilon)^{\alpha\beta}=[\,j\,]^{\dot\alpha\dot\beta}=(-1)^{j-\alpha}\delta_\alpha{}^{-\beta},\\ &\{j\}_{\alpha\beta}\equiv\mathrm{D}^j(\varepsilon^{-1})_{\alpha\beta}=\{j\}_{\dot\alpha\dot\beta}=(-1)^{2j}[\,j\,]^{\alpha\beta}.\end{aligned} \tag{13}$$

Then, as before, $\zeta_\alpha\eta^\alpha$ is an invariant form, with

$$\zeta_\alpha\eta^\alpha=(-1)^{2j}\,\zeta^\alpha\eta_\alpha\,.$$

In order to emphasize that they are invariant spinors [12], we write the CG coefficients in the form

$$\begin{aligned}\mathrm{C}(j_1j_2j_3;\alpha_1\alpha_2\alpha_3)&\equiv[\,j_3j_1j_2\,]_{\alpha_3}{}^{\alpha_1\alpha_2}=[\,j_3j_1j_2\,]_{\dot\alpha_3}{}^{\dot\alpha_1\dot\alpha_2}\\ &=(-1)^{2j_3}[\,j_3j_1j_2\,]^{\alpha_3}{}_{\alpha_1\alpha_2}\end{aligned} \tag{14}$$

Our discussion of generalized Dirac matrices is essentially a discussion of various relations betwen spinors and irreducible tensors. The building blocks of the construction are a special class of generalized Pauli matrices, which we review in the rest of this section. The particular matrices considered here correspond to irreducible tensors of the minimum rank, which are therefore traceless. They are special cases of general constructions given elsewhere [13].

First, for the representation (j,j), we get $(2j{+}1)\times(2j{+}1)$ matrices, defined inductively by successive application of CG coefficients:

$$\begin{aligned}\sigma^{\mu_1\cdots\mu_{2j}}(jj)_{\alpha\dot\beta} &= [\,j, j-\tfrac{1}{2}, \tfrac{1}{2}\,]_\alpha{}^{\alpha_1\alpha_2}\,[\,j, j-\tfrac{1}{2}, \tfrac{1}{2}\,]_{\dot\beta}{}^{\dot\beta_1\dot\beta_2} \\ &\qquad\times \sigma^{\mu_1\cdots\mu_{2j-1}}\left(j-\tfrac{1}{2}, j-\tfrac{1}{2}\right)_{\alpha_1\dot\beta_1}\,\sigma^{\mu_{2j}}{}_{\alpha_2\dot\beta_2}\,,\end{aligned} \tag{15}$$

and from the analogous constructon using $\widetilde\sigma$ we get

$$\widetilde\sigma^{\mu_1\cdots\mu_{2j}}(jj)^{\dot\alpha\beta} = \sigma_{\mu_1\cdots\mu_{2j}}(jj)_{\alpha\dot\beta}\,. \tag{16}$$

These matrices are Hermitean, and it is straightforward to show [13] that they are symmetric and traceless, respectively, in the interchange and contraction of any pair of four-vector indices. They satisfy transformation laws precisely analogous to (6a) and (7).

We introduce the notation $(\mu)_n = \mu_1\cdots\mu_n$ for the indices of a tensor that is symmetric under all permutations of indices. When the number of indices is determined from the context, we usually drop the label n and write $(\mu) = \mu_1\cdots\mu_n$.

From the orthogonality relations for the 2×2 Pauli matrices (11a) and for the CG coefficients, it follows that

$$\sigma_{(\mu)}(jj)_{\alpha\dot\beta}\,\widetilde\sigma^{(\mu)}(jj)^{\dot\beta'\alpha'} = 2^{2j}\,\delta_\alpha{}^{\alpha'}\,\delta_{\dot\beta}{}^{\dot\beta'}. \tag{17}$$

The "inverse" expression

$$\begin{aligned}\mathcal{P}(jj)^{(\mu):(\nu)} &\equiv 2^{-2j}\,\sigma^{(\mu)}(jj)_{\alpha\dot\beta}\,\widetilde\sigma^{(\nu)}(jj)^{\dot\beta\alpha} \\ &= 2^{-2j}\,\mathrm{Tr}\left[\sigma^{(\mu)}(jj)\,\widetilde\sigma^{(\nu)}(jj)\right]\end{aligned} \tag{18}$$

is the projection operator for the symmetric and traceless part of a $2j$-th rank tensor [13].

For representations (j',j), with $j'\neq j$, there are in general several ways to construct tensors of minimum rank. Here, we choose for convenience a particular symmetry scheme. As in the case $j'=j$, we do not prove the symmetries in question because they are well known, and explicit proofs in terms of symmetries of the CG coefficients can be found elsewhere [13].

Consider the representation $(1,0)$. This corresponds to a second-rank, self-dual tensor:

$$\begin{aligned}\sigma^{\mu\nu}(1)_\alpha &\equiv [\,1\tfrac{1}{2}\tfrac{1}{2}\,]_\alpha{}^{\beta_1\beta_2}\,[\,0\tfrac{1}{2}\tfrac{1}{2}\,]_0{}^{\dot\gamma_1\dot\gamma_2}\,\sigma^\mu{}_{\beta_1\dot\gamma_1}\,\sigma^\nu{}_{\beta_2\dot\gamma_2} \\ &= \frac{i}{2}\,\epsilon^{\mu\nu}{}_{\lambda\rho}\,\sigma^{\lambda\rho}(1)_\alpha\,, \qquad \epsilon_{0123} = -1\,,\end{aligned} \tag{19}$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the alternating symbol. For $(0,1)$, we have an antiselfdual tensor,

$$\widetilde{\sigma}^{\mu\nu}(1)^{\dot{\alpha}} = \sigma_{\mu\nu}(1)_{\alpha} = -\frac{i}{2}\,\epsilon^{\mu\nu}{}_{\lambda\rho}\,\widetilde{\sigma}^{\lambda\rho}(1)^{\dot{\alpha}}\,. \tag{20}$$

For $(j,0)$ and $(0,j)$, respectively, with j an integer, we define inductively

$$\begin{aligned} &\sigma^{\mu_1\nu_1\cdots\mu_j\nu_j}(j)_{\alpha} = [\,j, j{-}1, 1\,]_{\alpha}{}^{\beta\gamma}\,\sigma^{\mu_1\nu_1\cdots\mu_{j-1}\nu_{j-1}}(j{-}1)_{\beta}\,\sigma^{\mu_j\nu_j}(1)_{\gamma} \\ &\widetilde{\sigma}^{\mu_1\nu_1\cdots\mu_j\nu_j}(j)^{\dot{\alpha}} = \sigma_{\mu_1\nu_1\cdots\mu_j\nu_j}(j)_{\alpha}\,. \end{aligned} \tag{21}$$

We extend our convention for labeling tensor indices as follows: Write $(\mu\nu)_n = \mu_1\nu_1\cdots\mu_n\nu_n$ for the indices of a tensor that is antisymmetric under any interchange $\mu_i\nu_i \leftrightarrow \nu_i\mu_i$, and symmetric under permutations of pairs $\mu_i\nu_i \leftrightarrow \mu_j\nu_j$. Again, we usually write $(\mu\nu)$ instead of $(\mu\nu)_n$.

We say that a tensor is, respectively, selfdual or antiselfdual (with respect to $(\mu\nu)$) if it is selfdual or antiselfdual in each pair $\mu_i\nu_i$ of $(\mu\nu)$.

In particular, $\sigma^{(\mu\nu)}(j)$ and $\widetilde{\sigma}^{(\mu\nu)}(j)$ are traceless tensors, with the indicated symmetries, that are, respectively, selfdual and antiselfdual with respect to $(\mu\nu)$.

Again, we get orthogonality relations,

$$\begin{aligned} &\sigma^{(\mu\nu)}(j)_{\alpha}\,\sigma_{(\mu\nu)}(j)^{\beta} = 2^{2j}\,\delta_{\alpha}{}^{\beta}, \\ &\widetilde{\sigma}^{(\mu\nu)}(j)_{\dot{\alpha}}\,\widetilde{\sigma}_{(\mu\nu)}(j)^{\dot{\beta}} = 2^{2j}\,\delta_{\dot{\alpha}}{}^{\dot{\beta}}, \\ &\sigma^{(\mu\nu)}(j)_{\alpha}\,\widetilde{\sigma}_{(\mu\nu)}(j)^{\dot{\beta}} = 0\,, \end{aligned} \tag{22}$$

and projection operators

$$\begin{aligned} \mathcal{P}(j)^{(\mu\nu):(\lambda\rho)} &\equiv 2^{-2j}\,\sigma^{(\mu\nu)}(j)_{\alpha}\,\sigma^{(\lambda\rho)}(j)^{\alpha} \\ &= \mathcal{P}(j)^{(\lambda\rho):(\mu\nu)} \\ &= \left[\widetilde{\mathcal{P}}(j)^{(\mu\nu):(\lambda\rho)}\right]^{*} \equiv 2^{-2j}\,\left[\widetilde{\sigma}^{(\mu\nu)}(j)_{\dot{\alpha}}\,\widetilde{\sigma}^{(\lambda\rho)}(j)^{\dot{\alpha}}\right]^{*} \\ &= \left[\mathcal{P}(j)_{(\mu\nu):(\lambda\rho)}\right]^{*}. \end{aligned} \tag{23}$$

To form tensors for any (j',j), where $j'{+}j$ is an integer, define

$$\begin{aligned} j' \geq j:\ \sigma^{(\mu\nu)_{\Delta}(\lambda)_{2j}}(j'j)_{\alpha\dot{\beta}} &= [\,j', \Delta, j\,]_{\alpha}{}^{\alpha_1\alpha_2} \\ &\quad\times \sigma^{(\mu\nu)}(\Delta)_{\alpha_1}\,\sigma^{(\lambda)}(jj)_{\alpha_2\dot{\beta}}\,, \end{aligned} \tag{24a}$$

$$\Delta \equiv |j'{-}j|\,,$$

$$j' \leq j:\ \sigma^{(\mu\nu)_{\Delta}(\lambda)_{2j'}}(j'j)_{\alpha\dot{\beta}} = \left[\sigma^{(\mu\nu)_{\Delta}(\lambda)_{2j'}}(jj')_{\beta\dot{\alpha}}\right]^{*}. \tag{24b}$$

In the same way as before, we construct (or define)

$$\widetilde{\sigma}^{(\mu\nu)(\lambda)}(j'j)^{\dot{\alpha}\beta} = \sigma_{(\mu\nu)(\lambda)}(j'j)_{\alpha\dot{\beta}}\,, \tag{25}$$

which gives the rectangular matrix relations (square when $j' = j$)

$$\begin{aligned} &\sigma^{(\mu\nu)(\lambda)}(j'j)^{\dagger} = \sigma^{(\mu\nu)(\lambda)}(jj')\,, \\ &\sigma^{(\mu\nu)(\lambda)}(j'j)^{*} = \{j'\}\,\widetilde{\sigma}^{(\mu\nu)(\lambda)}(j'j)\,\{j\}^{\mathrm{T}}. \end{aligned} \tag{26}$$

Additional properties of $\sigma^{(\mu\nu)(\lambda)}(j'j)$ and $\widetilde{\sigma}^{(\mu\nu)(\lambda)}(j'j)$, aside from the symmetries indicated by the notation, are as follows:

(i) $\sigma(j'j)$ and $\widetilde{\sigma}(j'j)$ are traceless tensors.

(ii) For $j' > j$, $\sigma(j'j)$ is selfdual with respect to $(\mu\nu)$ and $\widetilde{\sigma}(j'j)$ is antiselfdual. For $j' < j$, the reverse is true. Moreover, $\sigma(j0) = \sigma(j)$ and $\widetilde{\sigma}(j0) = \widetilde{\sigma}(j)$.

(iii) For $j' = j$, we get the same matrices defined earlier.

Finally, we have orthogonality relations and projection operators, which include as special cases those given before:

$$\begin{aligned} &\sigma^{(\mu\nu)(\lambda)}(j'j)_{\alpha\dot{\beta}}\,\widetilde{\sigma}_{(\mu\nu)(\lambda)}(jj')^{\dot{\beta}'\alpha'} = 2^{2M}\,\delta_{\alpha}{}^{\alpha'}\,\delta_{\dot{\beta}}{}^{\dot{\beta}'}, \\ &\qquad\qquad\qquad\qquad M \equiv \max[j',j]\,; \end{aligned} \tag{27}$$

$$\begin{aligned} \mathcal{P}(j'j)^{(\mu\nu)(\lambda):(\rho\sigma)(\tau)} &\equiv 2^{-2M}\ \mathrm{Tr}\left[\sigma^{(\mu\nu)(\lambda)}(j'j)\ \widetilde{\sigma}^{(\rho\sigma)(\tau)}(jj')\right] \\ &= \left[\widetilde{\mathcal{P}}(j'j)^{(\mu\nu)(\lambda):(\rho\sigma)(\tau)}\right]^{*} \equiv 2^{-2M}\ \mathrm{Tr}\left[\widetilde{\sigma}^{(\mu\nu)(\lambda)}(j'j)\ \sigma^{(\rho\sigma)(\tau)}(jj')\right]^{*} \\ &= \left[\mathcal{P}(j'j)_{(\mu\nu)(\lambda):(\rho\sigma)(\tau)}\right]^{*} \\ &= \mathcal{P}(jj')_{(\mu\nu)(\lambda):(\rho\sigma)(\tau)} \\ &= \mathcal{P}(j'j)^{(\rho\sigma)(\tau):(\mu\nu)(\lambda)}\,; \end{aligned} \tag{28}$$

$$\mathcal{P}(j'j)^{(\mu\nu)(\lambda):(\rho\sigma)(\tau)}\ \mathcal{P}(j'j)_{(\rho\sigma)(\tau)}{}^{(\mu'\nu')(\lambda')} = \mathcal{P}(j'j)^{(\mu\nu)(\lambda):(\mu'\nu')(\lambda'))}. \tag{29}$$

The tensors $\mathcal{P}(j'j)$ are invariant, by construction, and thus they are combinations of the metric symbol, $g_{\mu\nu}$, and the alternating symbol, $\epsilon_{\mu\nu\lambda\rho}$. Note that the operation "$\,\widetilde{}\,$" on $\mathcal{P}$ corresponds to space inversion, and that for such invariant tensors it is the same as complex conjugation. Thus, the tensor part of $\mathcal{P}(j'j)$ is real, and the pseudotensor part pure imaginary.

III. The $2(2j+1)$-Component Formalism

It is most convenient to start from the Dirac equation in the van der Waerden representation [14]. Then the 4×4 Dirac matrices are written in terms of the 2×2 Pauli matrices, (II.2) and (II.3),

$$\gamma_{\mu} = \begin{pmatrix} 0 & \sigma_{\mu} \\ \widetilde{\sigma}_{\mu} & 0 \end{pmatrix}, \tag{1}$$

with the Dirac equation for a free particle of mass m and spin $\frac{1}{2}$ being given by

$$\begin{aligned} i\gamma^\mu \partial_\mu \psi(x) \equiv i\gamma\cdot\partial\, \psi(x) = m\psi(x)\,, \\ \partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,. \end{aligned} \tag{2}$$

By using the formal definition (II.12) for $\mathrm{D}^j(a\cdot\sigma)$ with an operator argument, $a\cdot\sigma$, this can be written

$$\begin{pmatrix} 0 & \mathrm{D}^{\frac{1}{2}}(i\sigma\cdot\partial) \\ \mathrm{D}^{\frac{1}{2}}(i\widetilde{\sigma}\cdot\partial) & 0 \end{pmatrix} \psi(x) = m\psi(x)\,. \tag{3}$$

The generalization for any half-integer spin and nonzero mass is immediate [7,8]:

$$\begin{pmatrix} 0 & \mathrm{D}^{j}(i\sigma\cdot\partial) \\ \mathrm{D}^{j}(i\widetilde{\sigma}\cdot\partial) & 0 \end{pmatrix} \psi(x) = m^{2j}\psi(x)\,, \tag{4a}$$

$$-\Box\psi(x) \equiv -\partial_\mu\partial^\mu\psi(x) = m^2\psi(x)\,. \tag{4b}$$

We refer to this pair of equations as “the wave equation”.

The Klein-Gordon equation, (4b), is needed as a subsidiary condition because (4a) is an equation of order $2j$, and thus requires extra boundary conditions. The subsidiary condition guarantees the uniqueness of the mass, and that $\psi(x)$ has the usual Fourier decomposition in terms of positive and negative energy, plane-wave solutions.

Although (4b) is not automatically satisfied by the solutions of (4a) if $j > \frac{1}{2}$, it is easy to see that the solutions of (4a) identically satisfy

$$i^{4j}\,\Box^{2j}\psi(x) = m^{4j}\psi(x)\,. \tag{5}$$

This follows by applying the operator on the left side of (4a) to both sides of that equation and then making use of the identity

$$\mathrm{D}^j(a\cdot\sigma)\,\mathrm{D}^j(a\cdot\widetilde{\sigma}) = \mathrm{D}^j(a\cdot\sigma\, a\cdot\widetilde{\sigma}) = \mathrm{D}^j(a\cdot a) = (a\cdot a)^{2j}, \tag{6}$$

which holds whenever the components commute,

$$[\, a_\mu, a_\nu\,] = 0 = [\, a_\mu, \sigma_\nu\,]\;. \tag{7}$$

The analogy with the Dirac equation in the usual form (2) is emphasized when we use the definitions (II.15) and (II.16) to write (4a) as

$$i^{2j}\begin{pmatrix} 0 & \sigma_{\mu_1\cdots\mu_{2j}}(jj) \\ \widetilde{\sigma}_{\mu_1\cdots\mu_{2j}}(jj) & 0 \end{pmatrix} \partial^{\mu_1}\cdots\partial^{\mu_{2j}}\psi(x) = m^{2j}\psi(x)\,. \tag{8}$$

and then define

$$\gamma_{(\mu)}(jj) = \begin{pmatrix} 0 & \sigma_{\mu_1\cdots\mu_{2j}}(jj) \\ \widetilde{\sigma}_{\mu_1\cdots\mu_{2j}}(jj) & 0 \end{pmatrix}. \tag{9}$$

Because $\sigma(jj)$ and $\widetilde{\sigma}(jj)$ are completely symmetric and traceless in their tensor indices, so is $\gamma(jj)$. The label (jj) is written with two j's because later we shall consider matrices that are not square, but rectangular. This label is often suppressed when the context is clear. Then (4a) becomes

$$i^{2j}\,\gamma^{(\mu)}\partial_{(\mu)}\psi(x) = m^{2j}\psi(x)\,, \tag{10}$$

where we use the notation

$$\partial^{(\mu)} \equiv \partial^{\mu_1}\cdots\partial^{\mu_{2j}}. \tag{11}$$

The matrices $\gamma_{(\mu)}$ are the same as those arrived at by Weinberg, except for a factor $-i^{2j}$. The algebra generated by these matrices is studied in detail in Section IV.

In order to set the context for that discussion, I shall list the basic properties of the solutions of the spin-j wave equation. For the most part, this amounts to substituting j for $\frac{1}{2}$ in the appropriate places in the Dirac theory. Some of these points are developed in more detail in the papers of Joos [7] and Weinberg [8].

A. Lorentz Covariance

The wave equation is so constructed that the spinor indices of its solutions transform according to $(j,0)\oplus(0,j)$.

Thus, if we define

$$\mathrm{S}^j(A) = \begin{pmatrix} \mathrm{D}^j(A) & 0 \\ 0 & \mathrm{D}^j(A)^{-1\dagger} \end{pmatrix}, \qquad A \in \mathrm{SL}(2,\mathrm{C})\,, \tag{12}$$

then the transformation laws for $\sigma(jj)$ and $\widetilde{\sigma}(jj)$ imply that

$$\mathrm{S}^j(A)\,\gamma^{(\mu)}\,\mathrm{S}^j(A)^{-1} = \Lambda_{(\nu)}{}^{(\mu)}(A)\,\gamma^{(\nu)} \equiv \Lambda_{\nu_1}{}^{\mu_1}\cdots\Lambda_{\nu_{2j}}{}^{\mu_{2j}}\gamma^{\nu_1\cdots\nu_{2j}}. \tag{13}$$

From this, it follows that if $\psi(x)$ is a solution of (4a) and (4b), then so is

$$[U(a,A)\psi](x) \equiv \mathrm{S}^j(A)\,\psi[\Lambda^{-1}(x-a)]\,, \tag{14}$$

where a is an arbitrary real four-vector, and $\Lambda = \Lambda(A)$. It is clear that the operators $U(a,A)$ defined in this way form a representation of the connected part of the Poincaré group, $\mathrm{P}_+^\uparrow$, with the multiplication law

$$U(a,A)\,U(a',A') = U[a+\Lambda(A)a', AA']\,.$$

We shall see that these operators form a unitary representation of $\mathrm{P}_+^\uparrow$ in a Hilbert space, and we shall see how to extend the set to include the discrete transformations: space inversion (P), time inversion (T), and total inversion (Y).

The analog of the Dirac adjoint spinor is defined with the help of the matrix

$$B = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \gamma_{0\cdots 0}(jj) = \gamma_{(0)}\,, \tag{15}$$

which has the properties

$$B = B^\dagger, \qquad B^{-1}\gamma_{(\mu)}B = {\gamma_{(\mu)}}^\dagger, \qquad \det B = 1\,, \tag{16}$$

from (15), (9), and the fact that $\sigma(jj)$ and $\widetilde{\sigma}(jj)$ are Hermitean. As in the $j = \frac{1}{2}$ case, it is only in the van der Waerden representation of $\gamma(jj)$ that $B = \gamma_{(0)}$. The matrix $\gamma_{(0)}$ has the properties

$${\gamma_{(0)}}^2 = I, \qquad \gamma_{(0)}\gamma_{(\mu)}\gamma_{(0)} = \gamma^{(\mu)}, \tag{17}$$

from the definitions.

The adjoint spinor $\overline{\psi}(x)$ is formed in the usual way,

$$\overline{\psi}(x) = \psi(x)^\dagger B\,; \tag{18}$$

and because of (16), it satisfies

$$(-i)^{2j}\,\partial^{(\mu)}\overline{\psi}(x)\,\gamma_{(\mu)} = m^{2j}\,\overline{\psi}(x)\,. \tag{19}$$

From the definitions (12) and (15) we get

$$B^{-1}\mathrm{S}^j(A)^\dagger B = \mathrm{S}^j(A)^{-1}. \tag{20}$$

Thus $\overline{\psi}(x)$ transforms under $\mathrm{P}_+^\uparrow$ according to

$$\overline{[U(a,A)\psi]}(x) = \overline{\psi}[\Lambda^{-1}(x-a)]\,\mathrm{S}^j(A)^{-1}. \tag{21}$$

For the discrete transformations, we again take over the discussion from the Dirac theory by introducing, in addition to B given above, a set of $2(2j+1)\times 2(2j+1)$ matrices, γ_5, K, C, R, M, having the following effects on $\gamma_{(\mu)}$:

$$\gamma_5\,\gamma_{(\mu)}\,{\gamma_5}^{-1} = -\gamma_{(\mu)}\,, \tag{22a}$$

$$K\,{\gamma_{(\mu)}}^{\mathrm{T}}\,K^{-1} = \gamma_{(\mu)}\,, \tag{22b}$$

$$C\,{\gamma_{(\mu)}}^{\mathrm{T}}\,C^{-1} = (-1)^{2j}\,\gamma_{(\mu)}\,, \tag{22c}$$

$$R\,\gamma_{(\mu)}\,R^{-1} = \gamma^{(\mu)}, \tag{22d}$$

$$M\,{\gamma_{(\mu)}}^{*}\,M^{-1} = (-1)^{2j}\,\gamma_{(\mu)}\,. \tag{22e}$$

In the Dirac theory, these matrices are essentially determined by (16) and (22), once a representation of γ_μ is chosen. The same is most likely true here. Although we have so far considered only a special representation for $\gamma_{(\mu)}$, we shall indicate later that the generalized γ matrices, for give j, satisfy an anticommutation relation that appears to determine an algebra, with only one irreducible representation for each j, unique up to a similarity transformation. If this is true, then, for any $\gamma_{(\mu)}$ related to our special choice by a similarity transformation, the matrices in (16) and (22) can be chosen to have certain properties. These are listed below, along with the specific expressions for the matrices in the generalized van der Waerden representation.[4]

$$B = B^{\dagger} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{23a}$$

$$\gamma_5 = {\gamma_5}^{-1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{23b}$$

$$K = (-1)^{2j} K^{\mathrm{T}} = \begin{pmatrix} [\,j\,] & 0 \\ 0 & [\,j\,] \end{pmatrix}, \tag{23c}$$

$$R = R^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \gamma_{(0)}, \tag{23d}$$

$$C = (-1)^{2j} C^{\mathrm{T}} = \begin{pmatrix} [\,j\,] & 0 \\ 0 & (-1)^{2j}[\,j\,] \end{pmatrix} = \begin{cases} \gamma_5 K, \text{ fermions} \\ K, \text{ bosons} \end{cases} \tag{23e}$$

$$M = B^{-1} C^{*-1} = \begin{pmatrix} 0 & [\,j\,] \\ (-1)^{2j}[\,j\,] & 0 \end{pmatrix}, \tag{23f}$$

$$\det B = (-1)^{2j} \det \gamma_5 = \det K = \det R = \det C = \det M = 1\,.$$

In these equations, $[\,j\,]$ is the metric symbol defined in (II.13).

The proof that these properties can be satisfied with a general representation of $\gamma_{(\mu)}$, assuming that up to an equivalence there is only one that is irreducible, is not given here, because it is so similar to the proof for $j = \frac{1}{2}$. In any event, they are easily verified for our specific representation.

Using these matrices, we define the transformation of $\psi(x)$ under P, T, Y, and charge conjugation C just as for spin $\frac{1}{2}$,[5]

$$\psi_P(x) = R^{-1}\psi(\tilde{x})\,, \qquad \tilde{x}^{\mu} \equiv x_{\mu} \tag{24a}$$

$$\psi_T(x) = K^{\mathrm{T}} R^{\mathrm{T}}\, \overline{\psi}^{\,\mathrm{T}}(-\tilde{x})\,, \tag{24b}$$

$$\psi_Y(x) = K^{\mathrm{T}}\, \overline{\psi}^{\,\mathrm{T}}(-x)\,, \tag{24c}$$

$$\psi_C(x) = C^{\mathrm{T}}\, \overline{\psi}^{\,\mathrm{T}}(x)\,. \tag{24d}$$

[4] Our conventions are slightly different from those of Weinberg in [8].

[5] We currently prefer the convention where R^{-1}, R^{T}, K^{T}, and C^{T} are replaced by R, K, and C in these laws, which is numerically the same for R, and differs by a factor $(-1)^{2j}$ for K and C.

It is easy to verify that if $\psi(x)$ is a solution of the wave equation then so are $\psi_{P,T,Y,C}(x)$, and that P, T, and Y leave the spaces of positive and negative energy solutions invariant, whle C interchanges the two spaces.

The adjoint solutions can be calculated from these, with the aid of such identities as

$$\begin{aligned} &B^{\mathrm{T}}C^{\dagger}B = (-1)^{2j}C^{\mathrm{T}}, \\ &B^{\mathrm{T}}K^{\dagger}B = K^{-1}, \end{aligned} \tag{25}$$

and it is also a simple matter to write down the transformation laws of bilinear covariants, which look exactly the same as in the case of $j = \frac{1}{2}$.

B. Plane-Wave Solutions

The space of solutions of the wave equation can be constructed from superpositions of plane-wave solutions, in just the same way as for $j = \frac{1}{2}$. Here, we write down the plane-wave solutions explicitly.

By virtue of the Klein-Gordon equation, the positive and negative energy solutions, $\psi_{\pm}$, respectively, take the form

$$\begin{aligned} &\psi_{\pm}(x) = e^{\mp ik\cdot x}\, u_{\pm}(k)\,, \qquad k_0 = +(m^2 + \mathbf{k}^2)^{\frac{1}{2}}, \\ &\overline{\psi}_{\pm}(x) = e^{\pm ik\cdot x}\, \overline{u}_{\pm}(k)\,, \end{aligned} \tag{26}$$

where $u_{\pm}(k)$ and $\overline{u}_{\pm}(k)$ satisfy

$$\begin{aligned} &k^{(\mu)}\gamma_{(\mu)}u_{\pm}(k) = \begin{pmatrix} 0 & \mathrm{D}^j(k\cdot\sigma) \\ \mathrm{D}^j(k\cdot\widetilde{\sigma}) & 0 \end{pmatrix} u_{\pm}(k) = (\pm 1)^{2j}\, m^{2j}\, u_{\pm}(k)\,, \\ &\overline{u}_{\pm}(k)\,\gamma_{(\mu)}k^{(\mu)} = (\pm 1)^{2j}\, m^{2j}\, \overline{u}_{\pm}(k)\,. \end{aligned} \tag{27}$$

The functions $u_{\pm}(k)$ are labeled by the physical energy-momentum, k.

It is easy to write down the solutions, once a direction for the spin axis in a particular rest frame of the particle is chosen. For each k, we specify a rest frame, the "standard frame", by the requirement that k be obtained from it by a pure velocity transformation, or "boost" as Weinberg [8] calls it. The element of $\mathrm{SL}(2,\mathrm{C})$ that corresponds to this transformation (up to a sign) is the Hermitean, positive-definite square root

$$(k\cdot\sigma/m)^{\frac{1}{2}} = (m + k\cdot\sigma)/[2m(m + k^0)]^{\frac{1}{2}}. \tag{28}$$

To each spin direction in the standard frame, represented by a unit three-vector $\mathbf{e}(k)$, we assign an element $U(k) \in \mathrm{SU}(2,\mathrm{C})$ corresponding to a rotation from the 3-direction to $\mathbf{e}$:

$$\mathbf{e}(k)\cdot\boldsymbol{\sigma} = U(k)\sigma_3 U(k)^{\dagger}. \tag{29}$$

This assignment is of course determined only up to rotations about the 3-direction.

The solutions of (27) are then

$$
\begin{aligned}
u_\pm(k) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{D}^j[A(k)] \\ (\pm 1)^{2j}\mathrm{D}^j[A(k)]^{-1\dagger} \end{pmatrix}, \\
\overline{u}_\pm(k) &= \frac{1}{\sqrt{2}} \left((\pm 1)^{2j}\mathrm{D}^j[A(k)]^{-1},\, \mathrm{D}^j[A(k)]^\dagger \right), \\
A(k) &\equiv (k\cdot\sigma/m)^{\frac{1}{2}}\, U(k)\,.
\end{aligned} \tag{30}
$$

The right-hand, column index of the $2(2j{+}1)\times(2j{+}1)$ matrix $u_\pm(k)$ we call the "spin index", and the left-hand, row index we call the "bispinor index". The bispinor index transforms like the direct sum of a spin-j spinor with a lower undotted index and a spin-j spinor with an upper dotted index, from (14). The spin index transforms according to a spin-j representation of the little group of the vector $(m,0,0,0)$, in the language of the Wigner [15] representation theory for $\mathrm{P}_+^\uparrow$.

Finally, we note the orthogonality relations

$$
\begin{aligned}
\overline{u}_\pm(k)\, u_\pm(k) &= (\pm 1)^{2j} I\,, \\
\overline{u}_\pm(k)\, u_\mp(k) &= \left(\tfrac{1}{2} + (\pm 1)^{2j}\tfrac{1}{2}\right) I\,,
\end{aligned} \tag{31}
$$

where I is the $(2j{+}1)$-dimensional unit matrix. Note that for fermions, the positive and negative energy solutions are orthogonal in the bispinor space, whereas for bosons, $u_+(k) = u_-(k)$.

C. Conserved Currents and the Scalar Products

All that we have to do in order to make a Hilbert space from the space of solutions is to introduce a suitable, positive-definite, scalar product, the other axioms for a Hilbert space being trivially satisfied. This can be done with the help of the following "conserved current density".[6] Let ϕ and ψ denote solutions of the wave equation, and define

$$
\begin{aligned}
J_\mu(x:\phi,\psi) = (i)^{2j-1}\Big\{ &\overline{\phi}(x)\,\gamma_{\mu\mu_1\cdots\mu_{2j-1}}\,\partial^{\mu_1}\cdots\partial^{\mu_{2j-1}}\psi(x) \\
&- \left[\partial^{\mu_1}\overline{\phi}(x)\right]\,\gamma_{\mu\mu_1\cdots\mu_{2j-1}}\,\partial^{\mu_2}\cdots\partial^{\mu_{2j-1}}\psi(x) + \cdots \\
&+ (-1)^{2j+1}\left[\partial^{\mu_1}\cdots\partial^{\mu_{2j-1}}\overline{\phi}(x)\right]\,\gamma_{\mu\mu_1\cdots\mu_{2j-1}}\,\psi(x)\Big\}.
\end{aligned} \tag{32}
$$

Because of the wave equation,

$$
\partial_\mu J^\mu(x:\phi,\psi) = 0\,. \tag{33}
$$

[6] This expression has also been considered by S. Weinberg (private communication).

It is straightforward to see that J_μ transforms as a vector under $\mathrm{P}_+^\uparrow$:

$$J_\mu[x : U(a, A)\phi, U(a, A)\psi] = \Lambda_\mu{}^\nu \, J_\nu[\Lambda^{-1}(x - a) : \phi, \psi] \,, \tag{34}$$

and for the discrete transformations:

$$\begin{aligned} J_\mu(x : \phi_P, \psi_P) &= J^\mu(\tilde{x} : \phi, \psi) = \widetilde{J}_\mu(\tilde{x} : \phi, \psi) \,, \\ J_\mu(x : \phi_T, \psi_T) &= J^\mu(-\tilde{x} : \psi, \phi) = \widetilde{J}_\mu(-\tilde{x} : \psi, \phi) \,, \\ J_\mu(x : \phi_Y, \psi_Y) &= J_\mu(-x : \psi, \phi) \,, \\ J_\mu(x : \phi_C, \psi_C) &= (-1)^{2j+1} J_\mu(x : \psi, \phi) \,. \end{aligned} \tag{35}$$

Note that when $j > \frac{1}{2}$, $J_0(x : \psi, \psi)$ is not positive definite, so that it cannot be used as a probability density as in the Dirac theory. But as usual that poses no problem when one goes over to a field theory, where it may represent a charge density.

Consider the bilinear form

$$\langle \phi, \psi \rangle = \int \mathrm{d}^3 x \, J_0(x : \phi, \psi) = \langle \psi, \phi \rangle^* \,. \tag{36}$$

This form is constant in time, from (33). Moreover, from (34), $U(a, A)$ is unitary within the bilinear form, as well as P, while T and Y are antiunitary. Charge conjugation is antiunitary for fermions; and up to a sign, which will be eliminated shortly, it is antiunitary for bosons. (In the second quantized theory, charge conjugation becomes unitary, as usual [8].)

The above bilinear form is not positive definite, but it is trivially converted into one that is. For this purpose, we use the plane-wave decomposition of the solutions ψ and ϕ. Thus, for example,

$$\begin{aligned} \psi(x) &= \psi_+(x) + \psi_-(x) \,, \\ \psi_\pm(x) &= \int \frac{\mathrm{d}^3 k}{k_0} \, u_\pm(k) \, f_\pm(k) \, e^{\mp i k \cdot x}, \\ k_0 &= (m^2 + \mathbf{k}^2)^{\frac{1}{2}} \,, \end{aligned} \tag{37}$$

where $(\pm)$ indicate, respectively, positive and negative energy parts, and where we use the matrix notation described before for the momentum space solutions $u_\pm$, so that $f_\pm$ are $(2j+1)$-dimensional column vectors. Appropriate integrability properties are taken for granted.

Then it is straightforward to show that the bilinear form has the following properties:

$$\begin{aligned} (\phi_+, \psi_-) &= (\phi_-, \psi_+) \,, \\ (\pm 1)^{2j+1} \, (\psi_\pm, \psi_\pm) &\geq 0 \,, \end{aligned} \tag{38}$$

with equality in the latter expression holding if and only if $\psi_\pm$ vanishes. Thus, we can define a positive-definite, scalar product:

$$\langle\phi,\psi\rangle = (\phi_+,\psi_+) + (\pm 1)^{2j+1}(\phi_-,\psi_-)\,. \tag{39}$$

Within this scalar product, $U(a,A)$ and P are unitary; and T, Y, and C are antiunitary.

Finally, note that, as one expects [2], the choice of a conserved current density is by no means unique. In our case, we could just as well have chosen

$$J'_\mu(x:\phi,\psi) = i\left[\overline{\phi}\,\partial_\mu\psi - (\partial_\mu\overline{\phi})\psi\right]. \tag{40}$$

Then J_μ and J'_μ have the same transformation laws, both are divergenceless, and both lead to the same bilinear form. We leave the verification to the interested reader.

IV. The Generalized Dirac Algebra

Here we begin to study the algebras generated by the matrices $\gamma_{(\mu)}$. Our general procedure is first to classify and construct a complete set of $2(2j'+1)\times 2(2j+1)$ matrices according to spins, and then to study the various algebraic relations among the basis elements.

The basic point of the discussion, which we use again and again, is that the complete set of generalized Dirac matrices is nothing but a collection of CG coefficients, expressed partly in Cartesian form, for the reduction of a direct product of representations irreducible under $\mathrm{L}^\uparrow$ into a direct sum of such representations. This fact makes it possible to compute all matrices in the complete set, and all traces of products of such matrices, in terms of the ordinary CG coefficients for finite-dimensional representations of $\mathrm{L}^\uparrow_+$.

The objects to be studied are $2(2j'+1)\times 2(2j+1)$ matrices H, transforming under $\mathrm{L}^\uparrow$ according to

$$\mathrm{S}^{j'}(A)\,H\,\mathrm{S}^{j}(A)^{-1} = H_A\,, \qquad j'+j = \text{integer}\,. \tag{1}$$

We restrict $j'+j$ to be an integer because otherwise H cannot be represented as a sum of tensors. If we write out the bispinor indices of H as H_{ab}, then the index a transforms according to $\mathrm{S}^{j'}(A)$, and b according to $\mathrm{S}^{j}(A)^{-1\mathrm{T}}$. From the definitions of B and C, (III.15) and (III.23e), and from (II.10), we have that

$$CB\,\mathrm{S}^{j}(A)^{-1\mathrm{T}}\,B^{-1}C^{-1} = \mathrm{S}^{j}(A)\,, \tag{2}$$

so that $\mathrm{S}^{j}(A)^{-1\mathrm{T}}$ is unitary-equivalent to $\mathrm{S}^{j}(A)$. Thus, H transforms according to the representation $[(j',0)\oplus(0,j')]\otimes[(j,0)\oplus(0,j)]$. By a Clebsch-Gordan analysis, we get from this representation the following invariant subspaces, irreducible under $\mathrm{L}^\uparrow_+$:

$$[(j'+j,0)\oplus(0,j'+j)]\;\oplus\;[(j'+j-1,0)\oplus(0,j'+j-1)]$$
$$\oplus\cdots\oplus\;[(|j'-j|,0)\oplus(0,|j'-j|)]\;\oplus\;[(j',j)\oplus(j,j')]\;.$$

Of course, when $j' = j$, the subspaces $(0,0) \oplus (0,0)$ and $(j,j) \oplus (j,j)$ are no longer irreducible under $\mathrm{L}^{\uparrow}$, but rather each subspace $(0,0)$ and (j,j) is separately irreducible.

Our first task is to construct matrices spanning each of these irreducible subspaces. Before doing that we recall how the classification works in the familiar case $j' = j = \frac{1}{2}$. Then we have the following correspondence:

(i) scalar: $I \leftrightarrow (0,0)$;

(ii) vector: $\gamma_\mu \leftrightarrow (\frac{1}{2}, \frac{1}{2})$;

(iii) antisymmetric tensor (or pseudotensor):
$\sigma_{\mu\nu}$(or $\gamma_5\sigma_{\mu\nu}$) $\leftrightarrow (1,0) \oplus (0,1)$;

(iv) pseudovector: $\gamma_5\gamma_\mu \leftrightarrow (\frac{1}{2}, \frac{1}{2})$;

(v) pseudoscalar: $\gamma_5 \leftrightarrow (0,0)$;

In the following we shall see how this classification generalizes for any j' and j, with $j'+j =$ integer.

A. Construction of a Complete Set

Because of the spinor character of the diagonal and off-diagonal blocks of H, indicated schematically below, it is easy to guess how the invariant subspaces are to be characterized.

$$H = \begin{pmatrix} X_\alpha{}^\beta & Y_{\alpha\dot\beta} \\ Y'^{\dot\alpha\beta} & X'^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{3}$$

The first natural division is between the space of H's with only diagonal blocks nonvanishing, $\mathcal{H}_\mathcal{D}$, and the space of H's with only off-diagonal blocks nonvanishing, $\mathcal{H}_\mathcal{O}$. Then $\mathcal{H}_\mathcal{D}$ and $\mathcal{H}_\mathcal{O}$ are clearly invariant subspaces under $\mathrm{L}^{\uparrow}$, from the form (III.12) of $\mathrm{S}^j(A)$ and the form (III.23d) of $\gamma_{(0)}$, which represents space inversion.

To construct a complete set, we shall use the generalized Pauli matrices given in Section II, which we normalize for convenience as follows:

$$\begin{aligned} \rho(j'j) &\equiv 2^{-M}\,\sigma(j'j)\,, \\ \widetilde{\rho}(j'j) &\equiv 2^{-M}\,\widetilde{\sigma}(j'j)\,, \\ M &= \max[j',j]\,. \end{aligned} \tag{4}$$

Consider first the space $\mathcal{H}_\mathcal{O}$. From the spinor character of the off-diagonal blocks in (3), $\mathcal{H}_\mathcal{O}$ corresponds to the representation $(j',j) \oplus (j,j')$. To get a complete set of matrices in $\mathcal{H}_\mathcal{O}$, note that the matrices $\rho^{(\mu\nu)(\lambda)}(j'j)$, defined by (II.24) and (4) above, span the upper right-hand block, because of the orthogonality relation (II.27), and that the matrices $\widetilde{\rho}^{(\mu\nu)(\lambda)}(j'j)$ span the lower left-hand block, for the same reason. Thus, the following are a complete set in $\mathcal{H}_{\mathcal{O}}$:

$$
\begin{aligned}
\kappa^{(\mu\nu)(\lambda)}(j'j) &\equiv \begin{pmatrix} 0 & \rho^{(\mu\nu)(\lambda)}(j'j) \\ \widetilde{\rho}^{(\mu\nu)(\lambda)}(j'j) & 0 \end{pmatrix}, \\
\gamma_5(j')\,\kappa^{(\mu\nu)(\lambda)}(j'j) &\equiv \begin{pmatrix} 0 & \rho^{(\mu\nu)(\lambda)}(j'j) \\ -\widetilde{\rho}^{(\mu\nu)(\lambda)}(j'j) & 0 \end{pmatrix} \\
&= -\kappa^{(\mu\nu)(\lambda)}(j'j)\,\gamma_5(j)\,,
\end{aligned}
\tag{5}
$$

where $\gamma_5(j')$ and $\gamma_5(j)$ are the square matrices of appropriate dimension defined in (III.23b).

From the transformation laws of ρ and $\widetilde{\rho}$ under $\mathrm{L}_+^\uparrow$, and the space inversion relation (II.25) between ρ and $\widetilde{\rho}$, which gives

$$
\begin{aligned}
\gamma_{(0)}(j')\,\kappa_{(\mu\nu)(\lambda)}(j'j)\,\gamma_{(0)}(j) &= \kappa^{(\mu\nu)(\lambda)}(j'j)\,, \\
\gamma_{(0)}(j')\,\gamma_5(j')\,\kappa_{(\mu\nu)(\lambda)}(j'j)\,\gamma_{(0)}(j) &= -\kappa^{(\mu\nu)(\lambda)}(j'j)\,,
\end{aligned}
\tag{6}
$$

it follows that each of the sets of matrices $\kappa^{(\mu\nu)(\lambda)}$ and $\gamma_5\kappa^{(\mu\nu)(\lambda)}$ spans an invariant subspace with respect to $\mathrm{L}^\uparrow$, the first forming a tensor and the second a pseudotensor with respect to space inversion, P. Now $\mathcal{H}_{\mathcal{O}}$ corresponds to $(j', j) \oplus (j, j')$, so that if $j' \neq j$ the $\gamma_5\kappa^{(\mu\nu)(\lambda)}$ are not linearly independent from the $\kappa^{(\mu\nu)(\lambda)}$, because there is only one invariant subspace under $\mathrm{L}^\uparrow$. But if $j' = j$, there are two invariant subspaces; and the two sets of matrices are therefore independent.

Indeed this follows from our construction. When $j' \neq j$, $\rho^{(\mu\nu)(\lambda)}(j'j)$ is selfdual ($j' > j$) or antiselfdual ($j' < j$) with respect to $(\mu\nu)$, while the reverse is true of $\widetilde{\rho}(j'j)$. Thus,

$$
\begin{aligned}
\gamma_5(j')\kappa^{(\mu\nu)(\lambda)}(j'j) &= \pm\frac{i}{2}\,\epsilon^{\mu_1\nu_1}{}_{\sigma\tau}\;\kappa^{\sigma\tau\mu_2\nu_2\ldots\mu_\Delta\nu_\Delta(\lambda)}(j'j)\,, \\
&(\pm),\ \text{resp:}\ \ j' > j,\ j' < j\,; \qquad \Delta \equiv |j'{-}j|\,.
\end{aligned}
\tag{6a}
$$

From the properties of ρ and $\widetilde{\rho}$, $\kappa^{(\mu\nu)(\lambda)}$ and $\gamma_5\kappa^{(\mu\nu)(\lambda)}$ are traceless in their tensor indices, with the symmetries indicated by the notation introduced in Section II. Of course $\kappa(jj) = 2^{-j}\gamma(jj)$.

To complete our set of matrices, we must still span the space $\mathcal{H}_{\mathcal{D}}$ of matrices with diagonal blocks, X and X', having spinor indices $X_\alpha{}^\beta$ and $X'^{\dot\alpha}{}_{\dot\beta}$. Keeping in mind the earlier decomposition of H into irreducible parts, we reduce X and

X', making use of the completeness of the CG coefficients:

$$
\begin{aligned}
X_\alpha{}^\beta &= \sum_J (-1)^{2j'} \, [\, Jj'j\,]^\gamma{}_\alpha{}^\beta \, [\, Jj'j\,]_\gamma{}^{\alpha'}{}_{\beta'} \, X_{\alpha'}{}^{\beta'} , \\
&\equiv \sum_J X(J)_\gamma \, [\, Jj'j\,]^\gamma{}_\alpha{}^\beta , \\
X'^{\dot\alpha}{}_{\dot\beta} &= \sum_J X'(J)^{\dot\gamma} \, [\, Jj'j\,]_{\dot\gamma}{}^{\dot\alpha}{}_{\dot\beta} \, .
\end{aligned}
\tag{7}
$$

Now the spinors $\rho^{(\mu\nu)}(J) = 2^{-J}\sigma^{(\mu\nu)}(J)$ and $\widetilde{\rho}^{(\mu\nu)}(J)$ constructed in (II.21) span the $(2J{+}1)$-dimensional spaces corresponding to $X(J)$ and $X'(J)$, because of the orthogonality relations (II.22). Thus, if we define $(2j'{+}1){\times}(2j{+}1)$ matrices

$$
\begin{aligned}
\eta^{(\mu\nu)}(J:j'j)_\alpha{}^\beta &\equiv \rho^{(\mu\nu)}(J)_\gamma \, [\, Jj'j\,]^\gamma{}_\alpha{}^\beta , \\
\widetilde{\eta}^{(\mu\nu)}(J:j'j)^{\dot\alpha}{}_{\dot\beta} &\equiv \widetilde{\rho}^{(\mu\nu)}(J)^{\dot\gamma} \, [\, Jj'j\,]_{\dot\gamma}{}^{\dot\alpha}{}_{\dot\beta} \\
&= \eta_{(\mu\nu)}(J:j'j)_\alpha{}^\beta ,
\end{aligned}
\tag{8}
$$

we get a complete set in $\mathcal{H}_{\mathcal{D}}$ by choosing

$$
\tau^{(\mu\nu)}(J:j'j) = \begin{pmatrix} \eta^{(\mu\nu)}(J:j'j) & 0 \\ 0 & \widetilde{\eta}^{(\mu\nu)}(J:j'j) \end{pmatrix} ,
\tag{9a}
$$

and

$$
\begin{aligned}
\gamma_5(j')\,\tau^{(\mu\nu)}(J:j'j) &= \tau^{(\mu\nu)}(J:j'j)\,\gamma_5(j) \\
&= \begin{pmatrix} \eta^{(\mu\nu)}(J:j'j) & 0 \\ 0 & -\widetilde{\eta}^{(\mu\nu)}(J:j'j) \end{pmatrix} ,
\end{aligned}
\tag{9b}
$$

with $J = |j'{-}j|,\ |j'{-}j|{+}1,\ \ldots,\ j'{+}j$.

As before, $\tau(J:j'j)$ and $\gamma_5\tau(J:j'j)$ are, respectively, tensors and pseudotensors under $\mathrm{L}^\uparrow$. Both sets of matrices, for a given $J \neq 0$, span invariant subspaces under $\mathrm{L}^\uparrow$, which are at least $2(2j{+}1)$-dimensional, from the original decomposition of H; and hence the sets $\tau^{(\mu\nu)}(J:j'j)$ and $\gamma_5(j')\tau^{(\mu\nu)}(J:j'j)$ cannot be linearly independent. In fact, from Section II, the upper blocks are selfdual with respect to $(\mu\nu)$ and the lower blocks antiselfdual, so we have the relation

$$
\begin{aligned}
\gamma_5(j')\,\tau^{(\mu\nu)}(J:j'j) &= \frac{i}{2}\,\epsilon^{\mu_1\nu_1}{}_{\sigma\tau}\,\tau^{\sigma\tau\mu_2\nu_2\cdots\mu_J\nu_J}(J:j'j) , \\
& J \neq 0 .
\end{aligned}
\tag{10}
$$

From the properties of η and $\widetilde{\eta}$, defined in (8), the $\tau^{(\mu\nu)}(J:j'j)$ are traceless tensors with the indicated symmetry.

Thus, our final complete sets can be chosen as follows, with the symmetries indicated by the notation of Section II, and all tensors having vanishing trace:

Case I: $j' = j$

(i) scalar: $I \leftrightarrow (0,0)$;

(ii) $2j$-th rank tensor: $\kappa^{(\mu)}(jj) \leftrightarrow (j,j)$;

(iii) $2J$-th rank tensor (or pseudotensor):
$\tau^{(\mu\nu)}(J\!:\!jj)$ [or $\gamma_5(j)\,\tau^{(\mu\nu)}(J\!:\!jj)$] $\leftrightarrow (J,0)\oplus(0,J)$;
$J = 1,\ldots,2j$;

(iv) $2j$-th rank pseudotensor: $\gamma_5(j)\,\kappa^{(\mu)}(jj) \leftrightarrow (j,j)$;

(v) pseudoscalar: $\gamma_5(j) \leftrightarrow (0,0)$.

Case II: $j' \neq j$

(i) $2M$-th rank tensor (or pseudotensor), $M = \max[j',j]$:
$\kappa^{(\mu\nu)(\lambda)}(j'j)$ [or $\gamma_5(j)\,\kappa^{(\mu\nu)(\lambda)}(j'j)$] $\leftrightarrow (j',j)\oplus(j,j')$;

(ii) $2J$-th rank tensor (or pseudotensor):
$\tau^{(\mu\nu)}(J\!:\!j'j)$ [or $\gamma_5(j')\,\tau^{(\mu\nu)}(J\!:\!j'j)$] $\leftrightarrow (J,0)\oplus(0,J)$;
$J = |j'-j|,\ldots,j'+j$.

B. Infinitesimal Generators

It is hardly surprising that the components of the second rank, antisymmetric tensor $\tau^{(\mu\nu)}(1\!:\!jj)$ are the infinitesimal generators of the representation $(j,0)\oplus(0,j)$ of $\mathrm{L}^{\uparrow}$, except for a normalization.

Take the standard conventions for the Hermitian matrices $\mathbf{J}(j)$ that are the infinitesimal generators of the $(2j+1)$-dimensional representation of the rotation group [11]. The six infinitesimal generators for the representations $(j,0)$ and $(0,j)$ of $\mathrm{L}^{\uparrow}_{+}$ are $\mathbf{J}(j)$ and $\mathbf{K}(j) = \mp i\mathbf{J}(j)$, respectively. These form the usual antisymmetric tensor $M^{\mu\nu}$, with

$$\begin{aligned} &M_{ij} = \epsilon_{ijk}\,J_k\,, \qquad i,j,k = 1,2,3\,, \qquad \epsilon_{123} = +1\,, \\ &M^{i0} = K_i\,. \end{aligned} \tag{11}$$

A slightly tedious, but straightforward calculation gives

$$\begin{aligned} &M^{\mu\nu}(j,0) = \zeta(j)\,\eta^{\mu\nu}(1\!:\!jj)\,, \\ &M^{\mu\nu}(0,j) = \zeta(j)\,\widetilde{\eta}^{\mu\nu}(1\!:\!jj)\,, \\ &\qquad \zeta(j) \equiv 2i[(2j+1)j(j+1)/3]^{\frac{1}{2}}\,. \end{aligned} \tag{12}$$

The generators $\mathcal{M}^{\mu\nu}(j)$ for $(j,0)\oplus(0,j)$ in the van der Waerden representation are then

$$\mathcal{M}^{\mu\nu}(j) = \begin{pmatrix} M^{\mu\nu}(j,0) & 0 \\ 0 & M^{\mu\nu}(0,j) \end{pmatrix}. \tag{13}$$

C. Representations of $2(2j'+1)\times 2(2j+1)$ Matrices

Traces are particularly easy to study in the van der Waerden representation. The most general traces that we need in this section are those of products of two matrices.

Traces of (square) matrices in $\mathcal{H}_{\mathcal{O}}$ vanish trivially. This means that the trace of any product of matrices with an odd number of factors from $\mathcal{H}_{\mathcal{O}}$, that is an odd number of factors of the form $\kappa(j'j)$ or $\gamma_5\kappa(j'j)$, must vanish. Thus in particular

$$\begin{aligned}
&\mathrm{Tr}\,\kappa^{(\mu)}(jj) = \mathrm{Tr}\left[\gamma_5(j)\,\kappa^{(\mu)}(jj)\right] = 0\,,\\
&\mathrm{Tr}\left[\kappa^{(\mu\nu)(\lambda)}(j'j)\,\tau^{(\sigma\tau)}(J:jj')\right] = 0\,,\\
&\mathrm{Tr}\left[\gamma_5(j')\,\kappa^{(\mu\nu)(\lambda)}(j'j)\,\tau^{(\sigma\tau)}(J:jj')\right] = 0\,.
\end{aligned} \tag{14}$$

Furthermore, because $[\,Jjj\,]_\alpha{}^\beta{}_\beta = 0$, and the expression (III.23b) for γ_5,

$$\begin{aligned}
&\mathrm{Tr}\,\tau^{(\mu\nu)}(J:jj) = \mathrm{Tr}\left[\gamma_5(j)\,\tau^{(\mu\nu)}(J:jj)\right] = 0\,, \qquad J\neq 0\,,\\
&\qquad\mathrm{Tr}\,\gamma_5(j) = 0\,.
\end{aligned} \tag{15}$$

Finally, we consider traces of products of matrices in the same class:

$$\begin{aligned}
&\mathrm{Tr}\left[\tau^{(\mu\nu)}(J:j'j)\,\tau^{(\mu'\nu')}(J':jj')\right]\\
&\qquad = \mathrm{Tr}\left[\gamma_5(j')\,\tau^{(\mu\nu)}(J:j'j)\,\gamma_5(j)\,\tau^{(\mu'\nu')}(J':jj')\right]\\
&\qquad = (-1)^{J+j'-j}\,\delta_{JJ'}\left[\mathcal{P}(J)^{(\mu\nu):(\mu'\nu')} + \widetilde{\mathcal{P}}(J)^{(\mu\nu):(\mu'\nu')}\right]\\
&\qquad \equiv (-1)^{J+j'-j}\,\delta_{JJ'}\,T(J)^{(\mu\nu):(\mu'\nu')}, \qquad J\neq 0\,,
\end{aligned} \tag{16}$$

where $\mathcal{P}(J)$ and $\widetilde{\mathcal{P}}(J)$ are the projection operators for the selfdual and antiselfdual tensors corresponding to $(J,0)$ and $(0,J)$, defined in (II.23). This result is immediate from the definitions. (The factor $(-1)^{J+j'-j}$ comes partly from the symmetry of the CG coefficients under the interchange of j' and j, partly from their orthogonality relations.) Note that $T(J)$ is an invariant *tensor* under $\mathrm{L}^\uparrow$, and that it is a projection operator, corresponding to the space of traceless tensors with symmetry $(\mu\nu)$.

The last case is calculated in the same way:

$$\begin{aligned}
&\mathrm{Tr}\left[\kappa^{(\mu\nu)(\lambda)}(j'j)\,\kappa^{(\mu'\nu')(\lambda')}(jj')\right]\\
&\qquad = -\,\mathrm{Tr}\left[\gamma_5(j')\,\kappa^{(\mu\nu)(\lambda)}(j'j)\,\gamma_5(j)\,\kappa^{(\mu'\nu')(\lambda')}(jj')\right]\\
&\qquad = \mathcal{P}(j'j)^{(\mu\nu)(\lambda):(\mu'\nu')(\lambda')} + \widetilde{\mathcal{P}}(j'j)^{(\mu\nu)(\lambda):(\mu'\nu')(\lambda')}\\
&\qquad \equiv T(j'j)^{(\mu\nu)(\lambda):(\mu'\nu')(\lambda')}.
\end{aligned} \tag{17}$$

Again, $T(j'j)$ is an invariant tensor, and for $j' \neq j$ a projection operator for traceless tensors with the indicated symmetry, while for $j' = j$, $T(jj)/2$ is the projection operator, because $\mathcal{P}(jj) = \widetilde{\mathcal{P}}(jj)$, from (II.28).

Note that

$$T(j'j) = T(jj')\,. \tag{18}$$

We have already seen that the generalized Dirac matrices form a complete set. With the traces above, we can give an explicit representation for any $2(2j'+1)\times 2(2j+1)$ matrix, just as for the spin-$\frac{1}{2}$ case, as a sum of terms with definite symmetries:

$$\begin{aligned} H(j'j) &= \frac{\delta_{j'j}}{2(2j+1)}\left\{\mathrm{Tr}[H(j'j)]\, I + \mathrm{Tr}\left[H(j'j)\,\gamma_5(j)\right]\,\gamma_5(j)\right\} \\ &\quad + \sum_{J\neq 0}(-1)^{J+j'+j}\ \mathrm{Tr}\left[H(j'j)\,\tau_{(\mu\nu)}(J:jj')\right]\,\tau^{(\mu\nu)}(J:j'j) \\ &\quad + \zeta(jj')\ \mathrm{Tr}\left[H(j'j)\,\kappa_{(\mu\nu)(\lambda)}(jj')\right]\,\kappa^{(\mu\nu)(\lambda)}(j'j) \\ &\quad - \zeta(jj')\ \mathrm{Tr}\left[H(j'j)\,\gamma_5(j)\,\kappa_{(\mu\nu)(\lambda)}(jj')\right]\,\gamma_5(j')\,\kappa^{(\mu\nu)(\lambda)}(j'j)\,, \\ \zeta(jj') &\equiv \begin{cases} 1 & j\neq j', \\ \frac{1}{2} & j = j'. \end{cases} \end{aligned} \tag{19}$$

To prove this relation, multiply both sides by one of the basis elements and take the trace, using the results described above.

D. γ-Algebra: Threefold Traces

Now that we know how to represent any matrix in terms of the basis above, we can give formulas that represent any of the basis elements as linear combinations of products of γ matrices. To do this we need certain threefold traces. For completeness, we compute all such traces, in terms of CG coefficients.

Consider the following quantities:

$$\begin{aligned} &\mathrm{C}(Jj'j[j''])_{(\sigma\tau)}{}^{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)} \\ &\qquad \equiv \rho_{(\sigma\tau)}(J)_\alpha\,[\,Jj'j\,]^{\alpha}{}_{\beta}{}^{\gamma}\left[\rho^{(\mu\nu)(\lambda)}(jj'')\,\widetilde{\rho}^{(\mu'\nu')(\lambda')}(j''j)\right]_{\gamma}{}^{\beta}\,; \end{aligned} \tag{20}$$

and define $\widetilde{\mathrm{C}}(Jj'j[j''])$ by the usual rule of respectively raising and lowering all lower and upper tensor indices. Of course $j''+j'$ and $j''+j$ are integers.

These quantities are just CG coefficients $[\,Jj'j\,]$ in various Cartesian representations, labeled by j'', with orthogonality relations

$$\begin{aligned} &\mathrm{C}(Jj'j[j''])_{(\sigma\tau)}{}^{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)}\,\mathrm{C}(J'j'j[j''])_{(\sigma'\tau'):(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)} \\ &\qquad = \delta_{JJ'}\,(2j''+1)\,\mathcal{P}(J)_{(\sigma\tau):(\sigma'\tau')}\,. \end{aligned} \tag{21}$$

Similar relations hold for $\widetilde{\mathrm{C}}$; and in addition, C and $\widetilde{\mathrm{C}}$ are orthogonal to each other:

$$\mathrm{C}(Jj'j[j''])_{(\sigma\tau)}{}^{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)}\,\widetilde{\mathrm{C}}(J'j'j[j''])_{(\sigma'\tau'):(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)} = 0\,. \tag{22}$$

These equations follow from the defintions, from orthogonality relations such as (II.27), and from the properties of the CG coefficients.

When J, j', and j are all integers, the special case $j''=0$ in the above leads to Cartesian representations of $[\,Jj'j\,]$ in terms of selfdual and antiselfdual tensors:

$$\mathrm{C}(Jj'j)_{(\sigma\tau)}{}^{(\mu'\nu'):(\mu\nu)} \equiv \rho_{(\sigma\tau)}(J)_\alpha\,[\,Jj'j\,]^{\alpha\beta\gamma}\,\rho^{(\mu'\nu')}(j')_\beta\,\rho^{(\mu\nu)}(j)_\gamma\,, \qquad J,j',j = \text{integers}\,, \tag{23}$$

with an analogous expression for $\widetilde{\mathrm{C}}(Jj'j)$. These quantities satisfy (21) and (22) with $j''=0$ in (21).

The threefold traces of basis matrices become:

$$\mathrm{Tr}\left[\tau_{(\sigma\tau)}(J:j'j)\,\kappa^{(\mu\nu)(\lambda)}(jj'')\,\kappa^{(\mu'\nu')(\lambda')}(j''j)\right] = \mathrm{C}(Jj'j[j''])_{(\sigma\tau)}{}^{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)} + \widetilde{\mathrm{C}}(Jj'j[j''])_{\cdots}{}^{\cdots}\,, \tag{24a}$$

$$\mathrm{Tr}\left[\gamma_5(j')\,\tau(J:j'j)\,\kappa(jj'')\,\kappa(j''j')\right] = \mathrm{C}(Jj'j[j'']) - \widetilde{\mathrm{C}}(Jj'j[j''])\,, \tag{24b}$$

$$\mathrm{Tr}\left[\tau^{(\mu\nu)}(J:j'j)\,\tau^{(\mu'\nu')}(J':jj'')\,\tau^{(\mu''\nu'')}(J'':j''j')\right] = \mathrm{C}(JJ'J'')^{(\mu\nu):(\mu'\nu'):(\mu''\nu'')} + \widetilde{\mathrm{C}}(JJ'J'')^{\cdots}\,, \tag{24c}$$

$$\mathrm{Tr}\left[\gamma_5(j')\,\tau(J:j'j)\,\tau(J':jj'')\,\tau(J'':j''j')\right] = \mathrm{C}(JJ'J'') - \widetilde{\mathrm{C}}(JJ'J'')\,, \tag{24d}$$

where we have occasionally suppressed tensor indices. Those traces without a factor γ_5 are of course (real) tensors, the tensor parts of $\mathrm{C}(Jj'j[j''])$ and $\mathrm{C}(JJ'J'')$, while those having a factor γ_5 are the corresponding (pure imaginary) pseudotensor parts. Traces with more factors γ_5 can be written as one of the above. The remaining threefold traces have an odd number of factors κ, and hence vanish.

In effect, we get an expression for $\tau(J:j'j)$ in terms of $\kappa(j'j'')\kappa(j''j)$ by inverting (24a). Note that the matrices η defined in (8) satisfy

$$\eta^{(\sigma\tau)}(J:j'j) = (2j''+1)^{-1}\,\mathrm{C}(Jj'j[j''])^{(\sigma\tau)}{}_{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)} \times \rho^{(\mu'\nu')(\lambda')}(j'j'')\,\widetilde{\rho}^{(\mu\nu)(\lambda)}(j''j)\,, \tag{25}$$

from the definition of C and the orthogonality properties of ρ and $\widetilde{\rho}$. By similar arguments,

$$\widetilde{\mathrm{C}}(Jj'j[j''])^{(\sigma\tau)}{}_{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)} \times \rho^{(\mu'\nu')(\lambda')}(j'j'')\,\widetilde{\rho}^{(\mu\nu)(\lambda)}(j''j) = 0\,, \qquad J \neq 0\,. \tag{26}$$

Analogous equations hold for $\widetilde{\eta}$. Thus, from the definition of τ in terms of η and $\widetilde{\eta}$, Eq. (9a), we get for $J \neq 0$:

$$\tau^{(\sigma\tau)}(J : j'j) = \frac{1}{2j''+1}\Big[\mathrm{C}(Jj'j[j''])^{(\sigma\tau)}{}_{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)} + \widetilde{\mathrm{C}}(Jj'j[j''])^{(\sigma\tau)}{}_{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)}\Big] \times \kappa^{(\mu'\nu')(\lambda')}(j'j'')\,\kappa^{(\mu\nu)(\lambda)}(j''j)\,, \tag{27a}$$

$$\gamma_5(j')\,\tau^{(\sigma\tau)}(J : j'j) = \frac{1}{2j''+1}\Big[\mathrm{C}(Jj'j[j''])^{(\sigma\tau)}{}_{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)} - \widetilde{\mathrm{C}}(Jj'j[j''])^{(\sigma\tau)}{}_{(\mu'\nu')(\lambda'):(\mu\nu)(\lambda)}\Big] \times \kappa^{(\mu'\nu')(\lambda')}(j'j'')\,\kappa^{(\mu\nu)(\lambda)}(j''j)\,. \tag{27b}$$

These equations say that $\tau(J)$ and $\gamma_5\tau(J)$, for $J \neq 0$, are generated by $\kappa(j'j'')\kappa(j''j)$ for any j'' with $j''{+}j'$ and $j''{+}j$ being integers. The only remaining basis matrices are I and γ_5. From the orthogonality relations (II.27) we get for the $2(2j{+}1)$-dimensional unit matrix

$$I(j) = (2j'+1)^{-1}\,\kappa_{(\mu\nu)(\lambda)}(jj')\,\kappa^{(\mu\nu)(\lambda)}(j'j)\,, \tag{28}$$

and from similar arguments, using properties of the CG coefficients, we get

$$\gamma_5(j) = (-1)^{1-j'+j}\,\frac{2j+1}{3}\,\frac{i}{2}\,\epsilon_{\mu\nu\lambda\rho}\,\tau^{\mu\nu}(1 : jj')\,\tau^{\lambda\rho}(1 : j'j)\,. \tag{29}$$

Thus all basis elements are generated by products of $\kappa(j'j)$ matrices, which can be chosen with any row and column dimensions that yield the row and column dimensions desired for the basis elements.

E. Abstract Characterization of the Algebras

We restrict our attention to the case $j' = j$, so that the $\kappa^{(\mu)}(jj)$ are square matrices that generate the algebra of $2(2j{+}1){\times}2(2j{+}1)$ matrices. The Cartesian CG coefficients defined in (20) can be used to define projection operators, for $J \neq 0$,

$$T(Jjj)^{(\lambda)(\rho):(\lambda')(\rho')} \equiv \mathrm{C}(Jjj[j])^{(\mu\nu):(\lambda):(\rho)}\,\mathrm{C}(Jjj[j])_{(\mu\nu)}{}^{(\lambda'):(\rho')} + \widetilde{\mathrm{C}}(Jjj[j])^{(\mu\nu):(\lambda):(\rho)}\,\widetilde{\mathrm{C}}(Jjj[j])_{(\mu\nu)}{}^{(\lambda'):(\rho')}. \tag{30}$$

These operators are pure tensors under parity, from the definition of $\widetilde{\mathrm{C}}$; and hence they can be expanded as (real) combinations of the metric symbol $g_{\mu\nu}$.

A straightforward calculation gives the identity:

$$\sum_{J=1}^{2j} T(Jjj)^{(\mu)(\nu)}{}_{(\mu')(\nu')}\,\kappa^{(\mu')}(jj)\,\kappa^{(\nu')}(jj) + g(j)^{(\mu)(\nu)}\,I \\ = (2j{+}1)\,\kappa^{(\mu)}(jj)\,\kappa^{(\nu)}(jj)\,, \tag{31}$$

where $g(j)$ is the metric symbol for the space of symmetric and traceless tensors of rank $2j$, *i.e.*,

$$g(j)^{(\mu)(\nu)} = \mathrm{Tr}\,\rho^{(\mu)}(jj)\,\widetilde{\rho}^{(\nu)}(jj)\,. \tag{32}$$

This quantity is proportional to $\mathrm{C}(0jj[j])$.

The familiar symmetry of the CG coefficients,

$$[\,Jjj\,]_{\alpha}{}^{\beta\gamma} = (-1)^{J-2j}\,[\,Jjj\,]_{\alpha}{}^{\gamma\beta},$$

and the definition (20), imply that

$$\mathrm{C}(Jjj[j])^{(\mu\nu)}{}_{(\lambda):(\rho)} = (-1)^{J}\,\mathrm{C}(Jjj[j])^{(\mu\nu)}{}_{(\rho):(\lambda)}\,, \tag{33}$$

with a similar expression for $\widetilde{\mathrm{C}}$. In particular

$$g(j)^{(\mu)(\nu)} = g(j)^{(\nu)(\mu)}. \tag{34}$$

These symmetries imply that (31) is equivalent to the expression

$$(2j{+}1)\left[\kappa^{(\mu)}(jj)\,\kappa^{(\nu)}(jj) + \kappa^{(\nu)}(jj)\,\kappa^{(\mu)}(jj)\right] \\ = 2g(j)^{(\mu)(\nu)}\,I + 2\sum_{J=1}^{2J\le 2j} T(2J,jj)^{(\mu)(\nu)}{}_{(\mu')(\nu')}\,\kappa^{(\mu')}(jj)\,\kappa^{(\nu')}(jj)\,. \tag{35}$$

For $j=\frac{1}{2}$, this reduces to the usual Dirac anticommutation relation.

We have arrived at this equation by means of our special representation for $\kappa(jj)$. Instead, we could have postulated this anticommutation relation as fundamental and asked for the irreducible representations of $\kappa(jj)$. We do not as yet have a complete proof (for $j>\frac{1}{2}$) that all irreducible representations are equivalent to the $2(2j{+}1)$-dimensional representation constructed in previous sections, but it seems plausible. It does not seem appropriate, however, to emphasize this possibility very much, because of our view of the wave equation as merely a convenient summary of covariance properties rather than as a fundamental equation of motion. We present it only as a perhaps interesting side remark.

V. Plane-Wave Matrix Elements

To complete our study, we present a few remarks on the representation of plane-wave matrix elements, which illustrate the fact that the usual concepts from the Dirac theory go over without change. We continue to use the van der Waerden representation.

Because we want to use wave functions with different masses and spins, we introduce the notation $u^j_\eta(k)$, where $\eta = (\pm)^{2j}$, for the spin-j, plane-wave, $\pm$ energy solutions with physical four-momentum k defined earlier in (III.30). The label η gives a consistent notation because, for bosons, $u_+ = u_-$.

Consider the $2(2j+1)$-dimensional square matrices

$$\Lambda^j_\eta(k) \equiv \eta\, u^j_\eta(k)\, \overline{u}^j_\eta(k)\,. \tag{1}$$

From the orthogonality properties (III.31), it follows that

$$\Lambda^j_\eta(k)\, \Lambda^j_\eta(k) = \Lambda^j_\eta(k)\,, \tag{2}$$

and that for fermions,

$$\Lambda^j_\pm(k)\, \Lambda^j_\mp(k) = 0\,. \qquad \textit{(fermions)} \tag{3}$$

From the completeness of the u's, it follows that for any positive or negative energy, momentum space solution of the wave equation $v^j_\eta(k)$, we get

$$\begin{aligned} \Lambda^j_\eta(k)\, v^j_\eta(k) &= v^j_\eta(k)\,, \\ \overline{v}^j_\eta(k)\, \Lambda^j_\eta(k) &= \overline{v}^j_\eta(k)\,, \end{aligned} \tag{4}$$

and for fermions

$$\Lambda^j_\eta(k)\, v^j_{-\eta}(k) = \overline{v}^j_{-\eta}(k)\, \Lambda^j_\eta(k) = 0\,. \qquad \textit{(fermions)} \tag{5}$$

In other words, for fermions $\Lambda^j_\pm(k)$ are respectively positive and negative energy projection operators for plane-wave solutions. For bosons, because $u_+ = u_-$, we have only one plane-wave projection operator.

It is clear from what we have just said that $\Lambda^j_\eta(k)$ does not depend on any special choice of the spin axis in the definition of $u^j_\eta(k)$. In fact, from the explicit forms for the solutions (III.30) we have

$$\Lambda^j_\eta(k) = \frac{1}{2}\begin{pmatrix} I & \eta\, \mathrm{D}^j\!\left(\dfrac{k\cdot\sigma}{m}\right) \\ \eta\, \mathrm{D}^j\!\left(\dfrac{k\cdot\widetilde{\sigma}}{m}\right) & I \end{pmatrix}. \tag{6}$$

For fermions, this implies that

$$\Lambda^j_+(k) + \Lambda^j_-(k) = I\,, \qquad \textit{(fermions)} \tag{7}$$

whereas for bosons

$$\Lambda^j_+(k) + \gamma_5\, \Lambda^j_-(k)\, \gamma_5 = I\,. \qquad \textit{(bosons)} \tag{8}$$

In the latter case, $\gamma_5 \Lambda^j_+(k) \gamma_5$ is the projection operator for plane-wave solutions of momentum k corresponding to the equation

$$(i)^{2j}\, \gamma_{(\mu)} \partial^{(\mu)}\, \psi(x) = -m^{2j}\, \psi(x)\,, \tag{9}$$

an equation that we have no occasion to consider at present.

Except for a normalization, $\Lambda^j_\eta(k)$ are the same matrices that occur in Weinberg's expressions for spin-j propagators [8].

These projection operators can be used to simplify the representation of $(2j'{+}1) \times (2j{+}1)$ plane-wave matrix elements of the form

$$\begin{aligned} R &= \overline{u}^{j'}_{\eta'}(k')\, H\, u^j_\eta(k) \\ &= \overline{u}^{j'}_{\eta'}(k')\, \Lambda^{j'}_{\eta'}(k')\, H\, \Lambda^j_\eta(k)\, u^j_\eta(k)\,, \end{aligned} \tag{10}$$

where H is a $2(2j'{+}1) {\times} 2(2j{+}1)$ matrix, which we assume to transform covariantly according to (IV.1), and $j'{+}j$ is an integer.

If the blocks of H are written as in (IV.3), the following relations are easy to verify:

$$H\, \Lambda^j_\eta(k) = \begin{pmatrix} X + \eta\, Y\, \mathrm{D}^j\!\left(\dfrac{k{\cdot}\widetilde{\sigma}}{m}\right) & 0 \\ 0 & \eta\, Y'\, \mathrm{D}^j\!\left(\dfrac{k{\cdot}\sigma}{m}\right) + X' \end{pmatrix}, \tag{11}$$

$$= \begin{pmatrix} 0 & \eta\, X\, \mathrm{D}^j\!\left(\dfrac{k{\cdot}\sigma}{m}\right) + Y \\ Y' + \eta\, X'\, \mathrm{D}^j\!\left(\dfrac{k{\cdot}\widetilde{\sigma}}{m}\right) & 0 \end{pmatrix} \Lambda^j_\eta(k)\,, \tag{12}$$

$$\Lambda^{j'}_{\eta'}(k')\, H = \Lambda^{j'}_{\eta'}(k') \begin{pmatrix} X + \eta'\, \mathrm{D}^{j'}\!\left(\dfrac{k'{\cdot}\sigma}{m'}\right) Y' & 0 \\ 0 & \eta'\, \mathrm{D}^{j'}\!\left(\dfrac{k'{\cdot}\widetilde{\sigma}}{m'}\right) Y + X' \end{pmatrix}, \tag{13}$$

$$= \Lambda^{j'}_{\eta'}(k') \begin{pmatrix} 0 & Y + \eta'\, \mathrm{D}^{j'}\!\left(\dfrac{k'{\cdot}\sigma}{m'}\right) X' \\ \eta'\, \mathrm{D}^{j'}\!\left(\dfrac{k'{\cdot}\widetilde{\sigma}}{m'}\right) X + Y' & 0 \end{pmatrix}. \tag{14}$$

These relations can also be derived by taking note of the spinor character of the various blocks and observing that $\mathrm{D}^j(k{\cdot}\sigma/m)$ and $\mathrm{D}^j(k{\cdot}\widetilde{\sigma}/m)$ are "metric" symbols for converting dotted into undotted or undotted into dotted indices [7, 16].

Since H is a $2(2j'+1) \times 2(2j+1)$ matrix, it can be expanded in terms of the basis set constructed in previous sections. The effect of the relations above is that plane-wave matrix elements of H can be represented in terms of basis matrices with only off-diagonal blocks nonzero, or alternatively, with only diagonal blocks nonzero, that is, in terms of $\kappa(j'j)$ (and $\gamma_5\kappa(jj)$, if $j'=j$), or alternatively in terms of the set $\tau(J:j'j)$ (plus I and γ_5 if $j'=j$).

The generalization of this procedure to plane-wave matrix elements with arbitrary numbers of incoming and outgoing particles is of course straightforward.

Finally, we indicate how to represent any S-matrix element for the scattering of particles in plane-wave states as an expression similar to (10). For simplicity, we consider the case where one incoming particle and one outgoing particle have nonzero spin, but the generalization will be clear.

To do this, we use the spinor representation [7] for scattering amplitudes, which is the same as the M-function representation of Stapp [16]. In this representation, the spin index of the particle, which transforms under $\mathrm{P}_+^\uparrow$ according to Wigner rotations [15] (representations of the little group of the momentum vector in the rest frame), is converted into a covariant spinor index. For plane-wave solutions of the wave equation, this amounts to defining new solutions,

$$\begin{aligned} w^j_\eta(k) &\equiv u^j_\eta(k)\,\mathrm{D}^j[A(k)]^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathrm{D}^j\!\left(\dfrac{k\cdot\sigma}{m}\right)\\ \eta\, I\end{pmatrix}\\ \overline{w}^j_\eta(k) &= \mathrm{D}^j[A(k)]\,\overline{u}^j_\eta(k) = \frac{1}{\sqrt{2}}\left(\eta\, I,\ \mathrm{D}^j\!\left(\frac{k\cdot\sigma}{m}\right)\right), \end{aligned} \tag{15}$$

where $A(k)$ is defined by (III.30).

Now let $M^{j'j}_{\alpha\dot\beta}$ be any spinor function, where α is a spinor index of spin j', and $\dot\beta$ an index of spin j. Then is it easy to see that

$$\begin{aligned} M^{j'j} &= \overline{w}^{j'}_{\eta'}(k')\begin{pmatrix}\eta'\,M^{j'j}\,\mathrm{D}^j\!\left(\dfrac{k\cdot\widetilde{\sigma}}{m}\right) & 0\\ 0 & \eta\mathrm{D}^{j'}\!\left(\dfrac{k'\cdot\widetilde{\sigma}}{m'}\right)\,M^{j'j}\end{pmatrix} w^j_\eta(k)\,,\\ &= \overline{w}^{j'}_{\eta'}(k')\begin{pmatrix}0 & \eta\eta'\,M^{j'j}\\ \mathrm{D}^{j'}\!\left(\dfrac{k'\cdot\widetilde{\sigma}}{m'}\right)\,M^{j'j}\,\mathrm{D}^j\!\left(\dfrac{k\cdot\sigma}{m}\right) & 0\end{pmatrix} w^j_\eta(k)\,. \end{aligned} \tag{16}$$

From these expressions it is straightforward to work out the effects of the discrete symmetries on the scattering amplitudes in the spinor representation, by making use of (III.24a)–(III.24d). The interested reader should consult the papers of Joos [7] and Stapp [16] for the results.

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