Bounds for the cutoff, Euclidean $:\Phi^4:$ perturbation expansion

David N. Williams
Randall Laboratory of Physics, The University of Michigan, Ann Arbor, Michigan 48109
(Received 2 November 1978)

We study the unrenormalized perturbation expansion of the Euclidean, massive $\lambda{:}\Phi^4{:}/4!$ field theory in $d \geq 1$ space-time dimensions, with a volume cutoff, and with the free propagator regulated by an α-parameter cutoff in case $d \geq 2$. In the formal expansion of the Schwinger n-point function, $S(x_1,\ldots,x_n) = \sum_{l=0}^{\infty}(-\lambda)^l S_l(x_1,\ldots,x_n)$, we show that $0 \leq S_l(x_1,\ldots,x_n) \leq A^l(l!)^{-1}\Pi_{i=1}^{l}[i+(n-5)/4]^2 \times cS_0(x_1,\ldots,x_n)$. The constant A diverges as the volume cutoff is removed, and, in $d \geq 2$ dimensions, as the ultraviolet cutoff is removed. We also give finite bounds for no volume cutoff, at the expense of lowering the mass in the free field and multiplying by an extra factor $l!$. We give analogous bounds for the connected n-point function in terms of the tree approximation. The method is combinatoric, once we establish x-space bounds on two basic diagrams. These follow from some properties of Bessel potentials of α cutoffs, which we believe to be new.

I. DISCUSSION OF RESULTS

Lately, we have seen the development of elegant techniques for studying the large-order behavior of the perturbation expansion of various quantum field theories.[1-5] In essence, certain classical methods of asymptotic estimation have been made available in the path-space version of quantum field theory. The natural question of whether some of the classical techniques for summation of divergent series may also be taken over remains under intense investigation.

We derive upper bounds on the n-point Schwinger function in lth order for the massive, Euclidean $\lambda{:}\Phi^4{:}/4!$ field theory in $d \geq 1$ dimensions, with volume and momentum cutoffs, by a direct, combinatoric method. The bounds have a structure similar to that of the asymptotic behavior obtained from saddle points,[2,5] but there are enough differences to prevent an exact comparison.

Our theory continues perhaps more directly on the older line started by Hurst,[6] who counted the number of Feynman graphs for polynomial interactions of bosons and fermions, by Riddell,[7] who counted Feynman graphs in QED (by a method which works more generally), and by Caianiello,[8] Buccafurri and Caianiello,[9] and Yennie and Gartenhaus[10] (with all reviewed in Caianiello[11]), who studied the radius of convergence of the perturbation expansion for volume- and momentum-cutoff interactions of bosons and fermions, at most linear in the boson field, with and without a fermion mode cutoff. These estimates[8-10] did not remove the effects of disconnected vacuum bubbles, and treated the x dependence in the Minkowski-space n-point function by taking the supremum; they took advantage of fermionic cancellations among graphs to get a nonzero radius of convergence.

Subsequently, Glimm[12] proved bounds for Minkowski-space regulated graphs for bosons, which treat the external x dependence in the L^2-norm sense. His bounds were important in the development of the constructive field-theory program for the $:\Phi^4:$ interaction in three space-time dimensions. Hepp[13] gives a discussion of Glimm's bounds and their proof.

The last result on the same line as ours, of which we are aware, is a set of path-integral L^p bounds on cutoff, polynomial interactions, both Euclidean and Minkowski, due originally to Glimm and Jaffe,[14,15] and with developments by a number of people that can be traced from the discussion of Theorems I.17, I.22, and V.2 in Simon.[16] We adapt that result to our situation later in this section in Remark vii. These bounds contain rather detailed x-space information, in principle, but do include the effects of vacuum bubbles.[17]

Our bounds take advantage of the *non*cancellation of Euclidean Feynman graphs for scalar bosons (because of their positivity), remove the effects of vacuum bubbles (as do the saddle-point results), treat connected graphs as well, and bound the external x-space behavior by that of the free field, or by the connected tree approximation, in the case of connected graphs.

To describe our results, let

$$\Delta_\alpha(x) = (2\pi)^{-d}\int (k^2+\mu^2)^{-1} \times e^{-\alpha(k^2+\mu^2)} e^{ik\cdot x} dk\,, \tag{1.1}$$

$$x, k \in \mathrm{R}^d, \quad \mu^2, \alpha > 0\,,$$

be the α-regulated, free Euclidean propagator. Let $d\mu_\alpha(\varphi)$ be the normalized, Gaussian path space measure with zero mean [on paths $\varphi \in \mathrm{Re}\,\mathcal{S}'(\mathrm{R}^d)$, for example] whose two-point function is

$$\Delta_\alpha(x_1 - x_2) = \int \varphi(x_1)\varphi(x_2)d\mu_\alpha(\varphi).$$

The n-point function of the cutoff theory is

$$\begin{aligned} &S(x_1, \ldots, x_n) = N^{-1}\int \varphi(x_1)\cdots\varphi(x_n)e^{-\lambda U}d\mu_\alpha\,, \\ &N \equiv \int e^{-\lambda U}d\mu_\alpha\,, \\ &U \equiv (4!)^{-1}\int :\varphi^4(y): g(y)dy\,, \\ &0 \leqslant g \in L^1(\mathrm{R}^d)\cap L^\infty(\mathrm{R}^d). \end{aligned} \tag{1.2}$$

The function g in the interaction is the volume cutoff, and the normal ordering is relative to the free measure $d\mu_\alpha$.

The α-cutoff theory is similar in spirit to the Gaussian propagator model discussed from the saddle-point method by Bervillier, Drouffe, Zinn-Justin, and Godrèche.[5]

Our main result is the following:

Theorem. Let

$$S(x_1, \ldots, x_n) = \sum_{l=0}^\infty (-\lambda)^l S_l(x_1, \ldots, x_n)$$

be the formal perturbation expansion of Eq. (1.2), which includes volume and α cutoffs. For $d=1$, put $\alpha=0$. Let S^C and S_l^C be the corresponding connected parts. Let $l \geqslant 1$.

(a) Then

$$0 \leqslant S_l(x_1, \ldots, x_n) \leqslant A^l B_l c S_0(x_1, \ldots, x_n)\,,$$

$$A \equiv a\gamma(\alpha)^2\|g\|_1\,,$$

$$B_l \equiv (l!)^{-1}\prod_{i=1}^{l}[i+(n-5)/4]^2\,,$$

where a and c are cutoff-independent constants, defined in Lemma 8, close to unity for n large, and where $\gamma(\alpha)$, defined in Lemma 5, diverges at $\alpha=0$ for $d \geqslant 2$.

(b) Let $l_n \equiv n/2-2$ be the order of the tree diagrams in the connected n-point function. Then

$$0 \leqslant S^C_{l+l_n}(x_1, \ldots, x_n)$$

$$\leqslant A_T^l B_{T,l} c_T S^C_{l_n}(x_1, \ldots, x_n)\,,$$

$$A_T \equiv a_T\gamma(\alpha)^2\|g\|_1\,,$$

$$B_{T,l} = l_n!(l+l_n)!^{-1}\prod_{i=1}^{l}[i+l_n+(n-5)/4]^2\,,$$

where a_T and c_T are cutoff-independent constants close to two and unity, respectively, for n large, defined in Lemma 10.

(c) Let $\mu^2 > l\epsilon^2 > 0$. Let S_{l,ϵ^2} be the lth-order n-point function in which the mass squared is replaced by $\mu^2-\epsilon^2$. Then

$$S_l(x_1, \ldots, x_n) \leqslant A_\epsilon^l B_l c_\infty S_{0,l\epsilon^2}(x_1, \ldots, x_n)\,,$$

$$A_\epsilon \equiv a_\infty\epsilon^{-2}\gamma(\alpha)\|g\|_\infty\,,$$

$$S^C_{l+l_n}(x_1, \ldots, x_n) \leqslant A^l_{T,\epsilon}B_{T,l}c_{T,\infty}S^C_{l_n,l\epsilon^2}(x_1, \ldots, x_n)\,,$$

$$A_{T,\epsilon} = a_{T,\infty}\epsilon^{-2}\gamma(\alpha)\|g\|_\infty\,,$$

where a_∞, c_∞, and $a_{T,\infty}$, $c_{T,\infty}$ are cutoff-independent constants, defined in Lemma 13, which for n large are near 1, 1, 2, and 1, respectively. Only in $d=1$ and 2 dimensions does γ have a mass dependence; in those cases it is evaluated at squared mass $\mu^2 - l\epsilon^2$, as discussed in Sec. III 3.

Remarks. (i) Although our model differs in some respects from the scalar field models considered up to now by saddle-point methods,[2,5] e.g., the presence and/or form of cutoffs and the normal ordering of the interaction, the constant in the bound in part (a) can be replaced by

$$l!\,a'^l l^{(n-5)/2}c'[\gamma(\alpha)^2\|g\|_1]^l\,,$$

where a' is of order unity and c' is exponentially decreasing in n, for $l \geqslant 2$. We omit the proof. The cutoff-independent part of this constant has a structure similar to that of the asymptotic behavior at large l from saddle points,[2,5] with differences in detail.

(ii) In all parts of the theorem, one may put $\alpha=0$ in S_0 and $S^C_{l_n}$ on the right-hand side, because it is a fact, to be reviewed later, that $\Delta_\alpha(x) \leqslant \Delta_0(x)$. Of course, for $d=1$ we already have $\alpha=0$. The volume cutoff in $S^C_{l_n}$ may also be removed (there is none in S_0).

(iii) For $d>2$, $\gamma(\alpha)$ diverges at $\alpha=0$ like $\Delta_\alpha(0)$, a little worse for $d=2$. The constants in the bounds in parts (a)–(c) then diverge. This is unlike the large order behavior from saddle points, where the constant analogous to A is a cutoff-independent number. The fact that we have a divergent ultraviolet dependence even for $d=2$, which ultimately requires no ultraviolet renormalization, is a suggestion that our particular, free field bounds cannot be extended to the renormalized, no-cutoff perturbation expansion.

(iv) In parts (a) and (b), $\|g\|_1$, and hence A and A_T, diverge when the volume cutoff is removed, $g\equiv 1$, certainly unlike the saddle points. In part (c), we may put $g\equiv 1 = \|g\|_\infty$. The price is the nonuniform l dependence in $S_{0,l\epsilon^2}$. By choosing $l\epsilon^2 = \epsilon_0^2$ fixed, we may translate this into a factor $(l/\epsilon_0^2)^l$, coming from A_ϵ and $A_{T,\epsilon}$, which effectively supplies an extra factor $l!$.

(v) These distortions in structure from the saddle-point results are possibly related to the fact that the x dependence in our bounds is carried by the free field in S_0, or by the tree approximation.

The x dependence in the saddle-point results is carried by a summation over classical solutions (at negative mass squared). It is already a bit of a surprise in the totally cutoff theory that Feynman graphs of all orders for the n-point function would have the same exponential times power order at large x as the free field. The exponential part is not surprising, but one could have expected powers of x to pile up with the order in λ. Indeed, we expect that is exactly what happens when the volume cutoff is removed; we can prove that by an example in one dimension with $\alpha = 0$, where the free two-point function is a simple exponential (we omit the example). Such a pile up is compensated by decreasing the exponential fall off, in part (c) of the theorem, through a decrease in mass. Although the same one-dimensional example shows that a free field bound with the original mass is generally impossible for no volume cutoff, the argument we use to prove part (c) is a considerable overestimation; and it remains open whether it could be improved enough to remove the extra $l!$ in the constant, with some x-dependent factor uniform in l.

(vi) Part (a) of the theorem could be a natural starting point for investigations of resummation methods to recover the exact, totally cutoff theory. We do not pursue that question; but let us review a few conclusions about the cutoff theory that can be drawn from existing, Euclidean path-space techniques. They are an elementary adaptation of results of Glimm, Jaffe, and Spencer[15] in two dimensions; and we omit the proofs. The combination of volume and momentum cutoffs makes the Euclidean path integral as well behaved as usual. It can be shown that $\exp(-\lambda U)$ belongs to $L^\infty(d\mu_\alpha)$ for $\mathrm{Re}\,\lambda \geq 0$; i.e., it is uniformly bounded for configurations φ in the support of $d\mu_\alpha$. Thus, the interacting measure $N^{-1}\exp(-\lambda U)d\mu_\alpha$ is absolutely continuous relative to $d\mu_\alpha$. Furthermore, the interacting measure is meromorphic in $\mathrm{Re}\lambda > 0$, with no poles on the positive real axis, when integrated with functions that belong to $L^1(d\mu_\alpha) \cap L^1(|U|d\mu_\alpha)$. In particular, the smeared Schwinger functions, and perhaps even the unsmeared ones, inherit the above analyticity. Finally, one can also prove infinite differentiability of the smeared Schwinger functions at $\mathrm{Re}\lambda = 0$, as long as the denominator N in (1.2) does not vanish (it does not vanish at $\lambda = 0$).

(vii) There is a Fock space estimate in Simon,[16] theorem V.2, which is a precursor of the $l!$ and cutoff behavior in part (a) of our theorem, and whose proof can easily be imitated to fit our particular choice of cutoffs. We state the following without proof:

L^p bounds on cutoff interactions. Let $g \in L^1(\mathrm{R}^d)$. Then for any $2 \leq p < \infty$,

$$\left\| \int :\varphi(y)^m: g(y)dy \right\|_{L^p(d\mu_\alpha)} \leq (m!)^{1/2}\Delta_\alpha(0)^{m/2}(p-1)^{m/2}\|g\|_1 .$$

Theorem V.2[16] is stated for $g \in L^q(\mathrm{R}^2)$, $1 < q \leq 2$, with no ultraviolet cutoff. The values $1 < q \leq 2$ can also be covered here. Taking the pth power of this estimate and putting $m = 4$ and $p = l$ gives a bound for integrals of the form $(l!)^{-1}\int FU^l d\mu_\alpha$, where $F \in L^\infty(d\mu_\alpha)$, having the same no-cutoff divergence (for $d \neq 2$) and $l!$ structure as in part (a) of our theorem. A volume divergence is to be expected in the L^p bounds, because disconnected, vacuum bubbles are included. Aside from the removal of such graphs in our Schwinger function bound, the function being integrated is

$$F = \varphi(x_1) \cdots \varphi(x_n) \notin L^\infty(d\mu_\alpha) .$$

In Sec. II, we discuss several estimates for certain combinations of Bessel potentials of α cutoffs, leading to x-space bounds on two particular Feynman graphs which serve as our combinatoric inputs. Among these, we feel that the proofs of Lemmas 2 and 3, for $d > 2$, are rather efficient and the results fairly sharp. Lemma 1, for $d = 2$, gets a seemingly less efficient proof in the Appendix, and yields an $\alpha = 0$ divergence which possibly could be improved.

The combinatorics for our main theorem are handled in Sec. III, by an inductive argument that bounds S_l by S_{l-1} and S_{l-2}. The ideas involved here, the classification of interaction vertices according to their modes of "dissolution" and the counting of those modes, are key ideas in this paper, and are quite simple to apply for $:\Phi^4:$. They can certainly be applied to other polynomial interactions; the input estimates in Sec. II certainly extend to more complicated vertices; the main complication would be in the classification and counting problem, about which we are optimistic.

II. BASIC ESTIMATES

Everything in this section flows from particular properties of certain Bessel potentials, which we now develop.

A. Bessel potentials for α cutoffs

Following the terminology of Calderón,[18] but not his normalizations, we define Bessel potentials for the Fourier transform of an α cutoff by

$$B_{\alpha,z}(x) = (2\pi)^{-d}\int (k^2+\mu^2)^{-z/2} \times e^{-\alpha(k^2+\mu^2)}e^{ik\cdot x}dk , \qquad (2.1)$$

$\mathrm{Re}z > 0$.

The Bessel potentials of the δ function correspond to the no-cutoff limit $\alpha=0$. They are known to belong to $L^1(\mathbb{R}^d)$, for any $\mathrm{Re}z>0$; indeed the no-cutoff case may be written explicitly in terms of Hankel functions, Eq. (2.4). Their exponential behavior at $x=\infty$ is reviewed in Eq. (2.5), and they have integrable singularities at the origin.

Our estimates with $\alpha\neq 0$ rely (in a somewhat hidden fashion) on having the same large x behavior as the $\alpha=0$ case. The role of the cutoff seems to be to preserve this behavior and also to regulate the singularity at $x=0$. In other words, our discussion perhaps makes essential use of the special feature of the α cutoff that it is analytic in a region larger than $|\mathrm{Im}k|\leq\mu$, to help it preserve the large x behavior.

Much of the discussion centers on the cutoff Euclidean propagator, for which we use the notation

$$\Delta_\alpha(x)=B_{\alpha,2}(x),$$

but we do need information about $B_{\alpha,z}$ for $z=4$ and 6.

The following integral representation are convenient: for $\mathrm{Re}z>0$,

$$B_{\alpha,z}(x)=\mu^{d-z}(4\pi)^{-d/2}\Gamma(z/2)^{-1}\int_{\mu^2\alpha}^{\infty}e^{-(\mu^2x^2/4t+t)}t^{-d/2}(t-\mu^2\alpha)^{-1+z/2}dt\,, \tag{2.2}$$

$$B_{\alpha,z}(x)=\alpha^{(z-d)/2}(4\pi)^{-d/2}\Gamma(z/2)^{-1}\int_{1}^{\infty}e^{-(x^2/4\alpha t+\mu^2\alpha t)}t^{-d/2}(t-1)^{-1+z/2}dt\,. \tag{2.3}$$

The two are related by a change of scale. They result from the usual Γ function and Gaussian tricks. There is a change of orders of integration involved, but it can be shown that (2.2), (2.3), and (2.1) define the same function in $L^1(\mathbb{R}^d)$ in the sense of distributions, for $\mathrm{Re}z>0$ and $\alpha\geq 0$. For $\alpha>0$, $B_{\alpha,z}\in\mathcal{S}(\mathbb{R}^d)$.

The $\alpha=0$ limit may be taken directly in (2.2), and yields, with the help of a table of Laplace transforms[19]:

$$B_{0,z}(x)=2^{1-z/2}\mu^{d-z}(2\pi)^{-d/2}\Gamma(z/2)^{-1}\times|\mu x|^{(z-d)/2}K_{(d-z)/2}(|\mu x|)\,. \tag{2.4}$$

The large-x behavior is thus (Peirce,[20] 783-85)

$$B_{0,z}=\pi^{(1-d)/2}2^{(1-d-z)/2}\mu^{d-z}\Gamma(z/2)^{-1}\times|\mu x|^{(z-d-1)/2}e^{-|\mu x|}[1+O(|\mu x|^{-1})]\,. \tag{2.5}$$

B. Estimates for $B_{\alpha,z}$

One of our key estimates is an upper bound on $\sup_y\Delta_\alpha(x-y)\Delta_\alpha(y)$ proportional to $\Delta_\alpha(x)$. After a little calculation, one sees that

$$\sup_{y\in\mathbb{R}^d}\Delta_\alpha(x-y)\Delta_\alpha(y)=\sup_{0<\lambda<1}\Delta_\alpha(\lambda x)\Delta_\alpha[(1-\lambda)x]\,. \tag{2.6}$$

This results from the fact that $\Delta_\alpha(x-y)\Delta_\alpha(y)$ is smooth, and zero at infinity in y, so its gradient must vanish at the maximum.

The large-x behavior is an essential reason that the above maximum is proportional to $\Delta_\alpha(x)$, as the proofs for $d=1$ and 2 illustrate.

For $d=1$, no α cutoff is necessary, because for $d=1$,

$$\Delta_0(x)=(2\mu)^{-1}e^{-|\mu x|}\,, \tag{2.7}$$

which is continuous at $x=0$. Thus,

$$\sup_{y\in\mathbb{R}^d}\Delta_0(x-y)\Delta_0(y)=(2\mu)^{-1}\Delta_0(x)\,, \tag{2.8}$$

from (2.6). Note that the coefficient on the right-hand side of (2.8) is $\Delta_0(0)$; $\Delta_\alpha(0)\Delta_\alpha(x)$ is a lower bound for the supremum in (2.6), which is achieved in this case.

For $d=z=2$, we follow a method of proof that could also be used for any d. Unfortunately, the method does not give optimum α control for $d>2$; and we do not know whether it is optimum for $d=2$.

Lemma 1. Let $d=2$, and $\alpha>0$. Then there exist positive constants, independent of α and x, such that when α is sufficiently small,

(a) $0<C_1(1+|\mu x|)^{-1/2}e^{-|\mu x|}\leq\Delta_\alpha(x)\leq C_2\Delta_\alpha(0)(1+|\mu x|)^{-1/2}e^{-|\mu x|}$,

(b) $\sup_{y\in\mathbb{R}^2}\Delta_\alpha(x-y)\Delta_\alpha(y)\leq C_2^{\,2}C_1^{\,-1}\Delta_\alpha(0)^2\Delta_\alpha(x)$.

Proof. (a) The proof of the upper and lower bounds of $\Delta_\alpha(x)$ is given in the Appendix, where the sizes of C_1 and C_2 are also estimated.

(b) The bound on the supremum follows by substituting part (a) into (2.6). The product of the exponential factors is independent of λ; and the product of power factors is maximum at the endpoints, $\lambda=0$ and 1. □

The anomalous feature of the bound on the supremum for $d=2$ is that $\Delta_\alpha(0)$ is squared. It is true that $\Delta_\alpha(0) = -(4\pi)^{-1}\ln\mu^2\alpha$ for small α when $d=2$; but we do not know whether the square of the logarithm is nonoptimal.

For $d>2$, we get fairly sharp α control, because of the following:

Lemma 2. Let $d>2$, $\alpha>0$. Then for any $p>1$,

$$\Delta_\alpha(x/p) \le p^{(d-2)/2p}\Delta_\alpha(x)^{1/p}C_\alpha{}^{(p-1)/p},$$
$$C_\alpha \equiv \mu^{d-2}(2\pi)^{-d/2}(d-2)^{-1}(2\mu^2\alpha)^{1-d/2}.$$

Proof. Think of the representation (2.2) as an integral of the exponential factor times a factor unity, and apply the Hölder inequality to obtain

$$\Delta_\alpha(x) \le \mu^{d-2}(4\pi)^{-d/2}\left(\int_{\mu^2\alpha}^{\infty} e^{-p(\mu^2x^2/4t+t)}t^{-d/2}dt\right)^{1/p} \times\left(\int_{\mu^2\alpha}^{\infty} t^{-d/2}dt\right)^{(p-1)/p}.$$

The result follows by choosing $s=pt$ as a new variable in the first integral, then lowering its lower limit from $s=\mu^2\alpha p$ to $s=\mu^2\alpha$, then replacing x by x/p on both sides of the inequality. □

Note that $C_\alpha > \Delta_\alpha(0)$, because the integral defining C_α results from that defining $\Delta_\alpha(0)$ by dropping an exponential factor less than unity. Nevertheless, for $d>2$ C_α is the leading approximation to $\Delta_\alpha(0)$ near $\alpha=0$. Indeed it is straightforward to compute that

$$\Delta_\alpha(0) = \mu^{d-2}(4\pi)^{-d/2}(d/2-1)^{-1} \times[(\mu^2\alpha)^{1-d/2} - \eta(\alpha)], \tag{2.9}$$

where for small α,

$$\begin{aligned}&\eta(\alpha) \cong \Gamma(2-d/2), \quad 2<d<4\\ &\eta(\alpha) \cong -\ln\mu^2\alpha, \quad d=4\\ &\eta(\alpha) \cong (1-d/2)(2-d/2)(\mu^2\alpha)^{2-d/2}, \quad d>4.\end{aligned} \tag{2.10}$$

The sharpness of the bound in Lemma 2 turns out to reside in that fact.

Lemma 3. Let $d>2$, $\alpha>0$. Then

$$\sup_{0<\lambda<1} \Delta_\alpha(\lambda x)\Delta_\alpha[(1-\lambda)x] \le 2^{(d-2)/2}\Delta_\alpha(0)\Delta_\alpha(x)[1+\epsilon_d(\alpha)],$$

where $\epsilon_d(\alpha)>0$ and $\epsilon_d(\alpha)\to 0$ as $\alpha\to 0$.

Proof. Let $\lambda=p^{-1}$ and $1-\lambda=q^{-1}$, and apply Lemma 2 for p and q to the factors on the left-hand side of the above. The factor $2^{(d-2)/2}$ results from

$$\sup_{0<\lambda<1} \lambda^{-\lambda}(1-\lambda)^{-(1-\lambda)} = 2,$$

and the factor $\Delta_\alpha(0)(1+\epsilon) = C_\alpha$ from the discussion preceding the lemma. □

The estimate in Lemma 3 is optimal in the sense that for α small and d near 2 the upper bound is near the lower bound $\Delta_\alpha(0)\Delta_\alpha(x)$ for the supremum. The actual supremum must be larger than this lower bound, because when $\alpha>0$ one can show that the slope of the function of λ on the left-hand side is positive at $\lambda=0$ and negative at $\lambda=1$.

So far, all results have been for $z=2$. We use the following lemma in case $z_1=2$ and $z_2=4, 6$ to remove the volume cutoff in our bound on the n-point function by the free field n-point function, at the expense of decreasing the mass in the free field.

Lemma 4. Let $B_{\alpha,z,\epsilon}(x)$ be $B_{\alpha,z}(x)$ evaluated at mass squared $\mu^2-\epsilon^2$. Let $\alpha>0$, $0<\epsilon^2<\mu^2$, $z_1\le z_2$, with z_1 and z_2 real. Then

$$B_{\alpha,z_2}(x) \le [(z_2-z_1)/2\epsilon^2]^{(z_2-z_1)/2}B_{\alpha,z_1,\epsilon}(x) \times\Gamma(z_1/2)\Gamma(z_2/2)^{-1}.$$

Proof. The result follows from the representation (2.3) by bringing out a factor $(t-1)^{(z_2-z_1)/2}\exp(-\epsilon^2\alpha t)$ in the integrand and replacing it by its maximum at $t=1+(z_2-z_1)/2\epsilon^2\alpha$. If $z_1=z_2$, it is clear from (2.3) that $B_{\alpha,z}<B_{\alpha,z,\epsilon}$. □

C. Bounds on Feynman graphs

Consider the x-space, Euclidean Feynman graph Γ_1 in Fig. 1(a). We define

$$\Gamma_1(x_1,\ldots,x_4) \equiv \int\prod_{i=1}^{4}\Delta_\alpha(x_i-y)g(y)dy. \tag{2.11}$$

The next lemma bounds Γ_1 by the graph F_1 in Fig. 1(b), which has the interaction "dissolved" in one of three possible ways:

$$F_1(x_1,\ldots,x_4) \equiv \Delta_\alpha(x_1-x_2)\Delta_\alpha(x_3-x_4). \tag{2.12}$$

We use the notation $F_{1,\epsilon}$ for F_1 evaluated at mass squared $\mu^2-\epsilon^2$.

Lemma 5. The following are pointwise bounds at any value of $x_1,\ldots,x_4$:

(a) $0\le\Gamma_1\le\gamma(\alpha)^2\|g\|_1F_1$,

where

$$\gamma(\alpha) = (2\mu)^{-1},$$

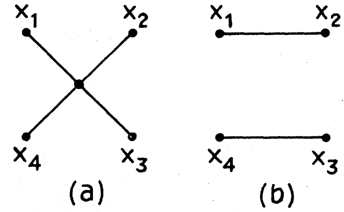

FIG. 1. Basic tree graph and its bound.

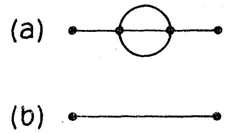

FIG. 2. Basic self-energy graph and its bound.

for $d=1$ and $\alpha=0$,

$$\gamma(\alpha)=C_2^2C_1^{-1}\Delta_\alpha(0)^2,$$

defined for $d=2$, $0<\alpha$ small, as in Lemma 1,

$$\gamma(\alpha)=2^{(d-2)/2}C_\alpha,$$

defined for $d>2$, $\alpha>0$, as in Lemma 2.

(b) Let $0<\epsilon^2<\mu^2$, and define $\gamma(\alpha)$ as in (a). Then

$$0\leqslant\Gamma_1\leqslant\epsilon^{-2}\gamma(\alpha)\|g\|_\infty F_{1,\epsilon}.$$

Proof. (a) The first estimate follows from (2.8) and Lemmas 1.b and 3, keeping in mind that $\Delta_\alpha(0)[1+\epsilon_d(\alpha)]=C_\alpha$, upon replacing $\Delta_\alpha(x_1-y)\Delta_\alpha(x_2-y)$ and $\Delta_\alpha(x_3-y)\Delta_\alpha(x_4-y)$ in the integral that defines Γ_1 by their suprema. The positivity of g and Δ_α is also used.

(b) Note that

$$\begin{aligned}\Gamma_1^2\leqslant&\int\Delta_\alpha(x_1-y_1)^2\Delta_\alpha(x_2-y_1)^2g(y_1)dy_1\\&\times\int\Delta_\alpha(x_3-y_2)^2\Delta_\alpha(x_4-y_2)^2g(y_2)dy_2\\\leqslant&\|g\|_\infty^2\sup_{y_1}\Delta_\alpha(x_1-y_1)\Delta_\alpha(x_2-y_1)\\&\times\sup_{y_2}\Delta_\alpha(x_3-y_2)\Delta_\alpha(x_4-y_2)\Delta_\alpha*\Delta_\alpha(x_1-x_2)\\&\times\Delta_\alpha*\Delta_\alpha(x_3-x_4).\end{aligned}\tag{2.13}$$

The suprema give $\gamma(\alpha)^2F_1$. Then we put $F_1<F_{1,\epsilon}$, because $B_{\alpha,2}<B_{\alpha,2,\epsilon}$, from (2.3). The convolutions may be estimated from Lemma 4, because

$$B_{\alpha_1,z_1}*B_{\alpha_2,z_2}=B_{\alpha_1+\alpha_2,z_1+z_2}.\tag{2.14}$$

Here, we have $z_1+z_2=4$,

$$B_{2\alpha,4}\leqslant\epsilon^{-2}B_{2\alpha,2,\epsilon}\leqslant\epsilon^{-2}B_{\alpha,2,\epsilon}.\tag{2.15}$$

The last bound follows from the fact that $B_{\alpha,2}$ is monotone decreasing in α, according to (2.2). Thus, the convolutions give $\epsilon^{-4}F_{1,\epsilon}$. □

We want to bound one more graph, Γ_2, shown in Fig. 2(a), by $F_2(x_1,x_2)=\Delta_\alpha(x_1-x_2)$, shown in Fig. 2(b). Here

$$\begin{aligned}\Gamma_2(x_1,x_2)\equiv&\int\Delta_\alpha(x_1-y_1)\Delta_\alpha(y_1-y_2)^3\Delta_\alpha(y_2-x_2)\\&\times g(y_1)g(y_2)dy_1dy_2.\end{aligned}\tag{2.16}$$

Lemma 6. Define $\gamma(\alpha)$ as in Lemma 5:

(a) $0\leqslant\Gamma_2\leqslant\|g\|_1^2\Delta_\alpha(0)^2\gamma(\alpha)^2F_2$,

(b) $0\leqslant\Gamma_2\leqslant 2\epsilon^{-4}\|g\|_\infty^2\Delta_\alpha(0)^2F_{2,\epsilon}$.

Proof. (a) Use $\Delta_\alpha(y_1-y_2)^2\leqslant\Delta_\alpha(0)^2$ in the integral (2.16) for Γ_2, and then use

$$\sup_{y_2}\sup_{y_1}\Delta_\alpha(x_1-y_1)\Delta_\alpha(y_1-y_2)\Delta_\alpha(y_2-x_2)\leqslant\gamma(\alpha)^2\Delta_\alpha(x_1-x_2).$$

(b) Note that

$$\Gamma_2\leqslant\Delta_\alpha(0)^2\|g\|_\infty^2\Delta_\alpha*\Delta_\alpha*\Delta_\alpha(x_1-x_2)$$

and

$$\begin{aligned}\Delta_\alpha*\Delta_\alpha*\Delta_\alpha&=B_{3\alpha,6}\\&\leqslant 2\epsilon^{-4}B_{3\alpha,2,\epsilon}\\&\leqslant 2\epsilon^{-4}B_{\alpha,2,\epsilon},\end{aligned}\tag{2.17}$$

where we used Lemma 4, and monotonicity in α for $z=2$. □

III. COMBINATORICS

The lth order term in the expansion of $S(x_1,\ldots,x_n)$ has the form $l!S_l=G_l$, where G_l is the sum of all x-space Feynman graphs with n external and l internal vertices constructed as follows:

$:\Phi^4:/4!$ *graph rules.* (i) Every vertex, represented by a dot, is labeled by a position in R^d, the external vertices by $x_1,\ldots,x_n$, the internal vertices by $y_1,\ldots,y_l$. Graphs which are topologically equivalent after labeling are identified.

(ii) All lines have distinct vertices on the two ends. One line ends on each external vertex, and four lines on each internal vertex.

(iii) Each line corresponds to an x-space propagator $\Delta_\alpha(x_i-y_j)$ or $\Delta_\alpha(y_i-y_j)$, with $\alpha=0$ in case $d=1$, and otherwise $\alpha>0$, the same for all lines. There is a volume cutoff factor $g(y_i)$ for each internal vertex. Each y_i argument is integrated; no x_i argument is integrated.

(iv) The effect of the $(4!)^{-1}$ in the interaction is to assign to each graph a weight factor $(s!)^{-1}$ for each set of s multiple lines between the same two vertices.

(v) All graphs with a vacuum bubble are deleted.

Note that, because $\Delta_\alpha>0$ and $g\geqslant 0$, every graph is positive, pointwise in $x_1,\ldots,x_n$; note that n must be even; and note that we have imposed, to conform with the normal ordering of the interaction, that no tadpole graphs, such as in Fig. 3, occur. In that figure, we have introduced the graphical notation that short, perpendicular bars isolate parts of a subgraph.

Our strategy is to bound the sum of graphs of order l by the sum of graphs of order $l-1$ and

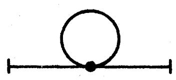

FIG. 3. Tadpole.

$l-2$, with the help of Lemmas 5 and 6. Two processes are involved. The first is that of "dissolving" the lth internal vertex of an lth order graph according to each among the three ways shown in Fig. 4 which gives a graph of order $l-1$ in the allowed class (no vacuum bubbles, no tadpoles). The original graph is then bounded, because of Lemma 5.a, up to weight factors for multiple lines, by $\gamma(\alpha)^2\|g\|_1/r$ times the sum of the $r=1$, 2, or 3 graphs into which it dissolves. To save words, we shall carry the discussion through initially only for the $\|g\|_1$ case. At the end, we shall describe the changes to get $\|g\|_\infty$ bounds.

The second process handles the exceptional case where the lth internal vertex is an internal vertex of a self-energy subgraph like that in Fig. 2(a). Any way of dissolving such a vertex produces a tadpole. These graphs are treated with the help of Lemma 6.a by replacing the self-energy subgraph with a free Δ_α line, so that the original graph is bounded by $\Delta_\alpha(0)^2\gamma(\alpha)^2\|g\|_1{}^2$ times a graph of order $l-2$, up to weight factors for multiple lines.

We need to be sure that the last internal vertex of every graph may be treated in one of these two ways, and to classify and count the lower-order graphs that result. To do that, it is helpful to think in terms of the inverse to the process of dissolving a vertex, the process of pinching together two distinct lines of a graph to create a new internal vertex.

First, we discuss the reduction when all allowed, connected and disconnected graphs are included. Then we treat separately the further exceptional cases that arise from the restriction to connected graphs only.

A. Graphs with no connectedness restriction

As notation, let (l) denote a typical graph of order l.

First, let us handle the tadpoles. An allowed (l) graph has none, by assumption. To see which vertices can dissolve into a tadpole, we take an $(l-1)$ graph with a tadpole and pinch it. We do not pinch a line with itself, because that produces a tadpole in the (l) graph, where it is absent. The only possible pinches are shown in Fig. 5. The first of these gives a self-energy graph, which we treat separately. If the second of these is not a self-energy graph, then neither of the upper two barred ends connects, without an intervening vertex, to either of the lower two. In that case, each of the other two ways of dissolving the pinched vertex produces an allowed graph: we get no tadpole, because there is another vertex beyond each of the upper bars; we get no vacuum bubble, because neither dissolution disconnects, so that the result is still connected to an external vertex, since the (l) graph has no vacuum bubbles.

Conclusion. In an allowed graph, any nonself-energy internal vertex which dissolves to produce a tadpole also dissolves in two other, allowed ways.

Next, consider vertices which dissolve to produce a vacuum bubble. We may assume the bubble to be connected, because a dissolution cannot add

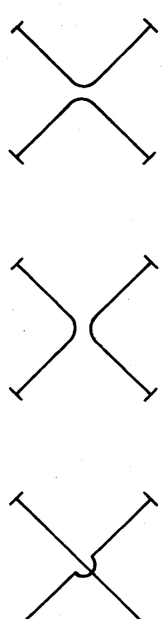

FIG. 4. Dissolutions of a vertex.

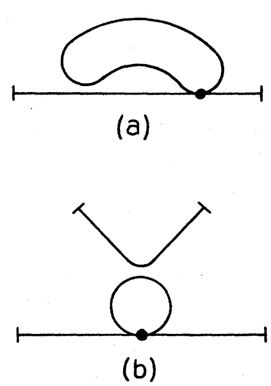

FIG. 5. Tadpole pinches.

more than one disconnected piece. If we pinch a line from the bubble with a line from the rest of the graph, as in Fig. 6, we see that each of the other two ways of dissolving the pinch produces no disconnection, hence no vacuum bubble. The pinch cannot be a self-energy vertex, because they do not disconnect on dissolution. The pinch therefore cannot produce a tadpole on dissolution, for the conclusion above would then require it to dissolve in two other allowed ways, and we know that one of the other ways is a disallowed vacuum bubble.

Conclusion. In an allowed graph, any internal vertex which dissolves to produce a vacuum bubble also dissolves in two other allowed ways.

Combining the conclusions, we have proved the following:

Dissolution Lemma. Every nonself-energy internal vertex of an allowed graph dissolves in at least two ways to produce an allowed graph.

To begin estimating, write

$$G_l = G_{l,1} + G_{l,2},$$

where $G_{l,1}$ is the sum of allowed (l) graphs where the last internal vertex is not a self-energy vertex, and $G_{l,2}$ is the rest. We first want to estimate how many times each $(l-1)$ graph occurs when we dissolve the lth internal vertex of each graph in $G_{l,1}$ in every allowed way. Until further notice, we disregard weight factors for multiple lines. Then the number of times a given $(l-1)$ graph occurs is exactly the number of distinct pinches of two lines of the graph. (Note that pinches of distinct lines never produce tadpoles or vacuum bubbles.) All pinches of distinct pairs of lines produce distinct graphs, except when one of the lines belongs to a multiplet between the same pair of vertices. We overestimate the occurrences of an $(l-1)$ graph by pinching all pairs of its lines. Since the number of lines in $(l-1)$ graphs is $2(l-1)+n/2$, each such graph results no more than $[2(l-1)+n/2][2(l-1)+n/2-1]/2$ times from dissolutions of the lth internal vertex of graphs in $G_{l,1}$.

Now we should like to insert the correct weights for multiple lines, as in graph rule (iv). We claim that the overcounting of $(l-1)$ graphs from pinches of multiple lines gives an overestimation of correctly weighted $(l-1)$ graphs, when the residual weight factors from (l) graphs are taken into account. To see that, we divide the possibilities into two cases:

(i) The two lines in $(l-1)$ whose pinch gives the vertex dissolved in (l) do not belong to the same multiplet. We suppose that the first of these lines in $(l-1)$ belongs to a multiplet of s_1 lines, and the second to a multiplet of s_2 lines. We want these multiplets to carry a weight $(s_1!s_2!)^{-1}$. The (l) graph is related to the $(l-1)$ graph by $s_1 s_2$ pinches, so if we count all pinches and insert the correct weights, we get a factor $[(s_1-1)!(s_2-1)!]^{-1}$ as a candidate for the weighted number of $(l-1)$ graphs resulting from the corresponding dissolution of (l). Now if s_1, s_2 is larger than one, then the s_1, s_2 multiplet in $(l-1)$ must have corresponded to an s_1-1, s_2-1 multiplet in (l), which carried a weight $[(s_1-1)!(s_2-1)!]^{-1}$. Upon dissolving (l) into $(l-1)$, this weight may be traded for the combination of number of pinches and $(l-1)$ weights above, which so far is an exact count and not an overestimation. It becomes an overestimation in those cases where (l) has multiple lines not present in $(l-1)$. The only possibility consistent with our assumption that the two lines being pinched do not lie between the same vertices is shown in Fig. 7(a); note that all triplets in (l) also belong to $(l-1)$, because we are not dissolving a self-energy vertex. In this case (l) carries a weight $\frac{1}{2}$, which remains after dissolution, so that our counting candidate above overestimates these dissolutions by a factor of two.

(ii) In the remaining case, the two lines in $(l-1)$ whose pinch gives the dissolved vertex belong to the same multiplet. The only possibility is shown in Fig. 7(b), where two of the barred lines on the outer two vertices may actually be the same line. The dissolution now gives a multiplet with $s=2$ or 3 lines. There are now $\binom{s}{2}$ pinches, and the correct weight for the multiplet in $(l-1)$ is $(s!)^{-1}$, so counting all pinches with the correct weight gives

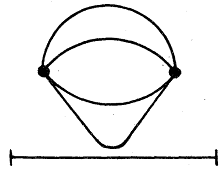

FIG. 6. Typical vacuum bubble pinch.

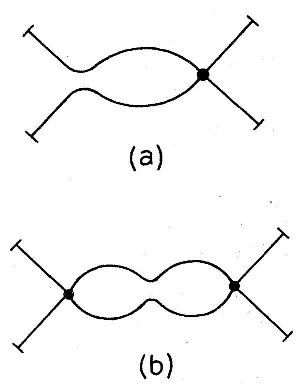

FIG. 7. Multiple lines in (l) but not in $(l-1)$.

a factor $[2(s-2)!]^{-1}$. This is larger than the weight that carries over from dissolution, which is $[4(s-2)!]^{-1}$.

Thus, we conclude that the number of pinches of all pairs of lines of a correctly weighted $(l-1)$ graph overestimates its occurrences by dissolution of correctly weighted (l) graphs. Combining this with Lemma 5.a and the Dissolution Lemma, we get the bound

$$\begin{aligned} &G_{l,1} \leq c_1(l,n)\kappa G_{l-1}\,, \\ &\kappa \equiv \gamma(\alpha)^2 \| g\|_1\,, \qquad (3.1) \\ &c_1(l,n) \equiv (l-1+n/4)[l-1+(n-2)/4]\,. \end{aligned}$$

That leaves the graphs in $G_{l,2}$, where the lth vertex is a self-energy vertex. Consider the subset of these graphs where the $(l-1)$th and lth vertices are part of the same self-energy subgraph. Replacing the subgraph by a single line gives an allowed $(l-2)$ graph, with a weight factor $(3!)^{-1}$ left over. We need to absorb a factor 2^{-1} of this to correct the weight in the $(l-2)$ graph, in case the replacement converts a singlet in (l) into a doublet in $(l-2)$, and a factor 3^{-1} in case we convert a doublet into a triplet. However, it emerges that we shall overcount these $(l-2)$ occurrences by factors of 2 and 3, respectively, so that we get to keep out all of the $(3!)^{-1}$. A given $(l-2)$ graph occurs twice as many times as it has lines, into which a self-energy graph can be inserted with two orientations, except that insertions into different lines of a multiplet are not distinct. Thus, a precise count of correctly weighted, $(l-2)$ occurrences from replacements of a self-energy graph on the last two internal vertices is $2[2(l-2)+n/2]/3!$. Taking into account that the lth vertex may be in the same self-energy subgraph as the 1st through $(l-1)$th internal vertex, and Lemma 6.a, we find

$$\begin{aligned} &G_{l,2} \leq c_2(l,n)\eta^2\kappa^2 G_{l-2}\,, \\ &\eta \equiv \Delta_\alpha(0)/\gamma(\alpha)\,, \qquad (3.2) \\ &c_2(l,n) = 2(l-1)(l-2+n/4)/3\,, \end{aligned}$$

and we have proved the following lemma:

Lemma 7. For $l \geq 1$ and $n \geq 2$, the sum of allowed spatial and α cutoff graphs obeys the bound

$$\begin{aligned} 0 &\leq G_l(x_1, \ldots, x_n) \\ &\leq c_1(l,n)\kappa G_{l-1}(x_1, \ldots, x_n) \\ &\quad + c_2(l,n)\eta^2\kappa^2 G_{l-2}(x_1, \ldots, x_n)\,, \end{aligned}$$

where c_1, c_2, κ, and η are defined in Eqs. (3.1) and (3.2). This bound may be interpreted for $l=1$ by putting $G_{-1}=0$.

To solve the recursion in Lemma 7 for an upper bound on G_l, we make the *Ansatz* for $l \geq 1$:

$$G_l \leq \prod_{i=1}^{l} [i+(n-5)/4]^2 (a\kappa)^l c G_0\,. \qquad (3.3)$$

It is easy to check that the *Ansatz* is obeyed for $l=1$ if $ca \geq 1$, and for $l=2$ if

$$1+128n(n-1)^{-2}(n+3)^{-2}\eta^2/3 \leq a^2 c\,. \qquad (3.4)$$

As a general induction step, for $l \geq 3$, we insert the *Ansatz* for $l-1$ and $l-2$ into the bound in Lemma 7, and demand that the result be no larger than the *Ansatz* for l. Dividing out the *Ansatz* for l, noting that

$$(l-1+n/4)[l-1+(n-2)/4][l-1+(n-1)/4]^{-2} \leq 1\,,$$

and defining

$$\begin{aligned} \beta \equiv \sup_{l\geq 3} &\, 2(l-1)(l-2+n/4)[l-1+(n-1)/4]^{-2} \\ &\times [l-2+(n-1)/4]^{-2}/3\,, \qquad (3.5) \end{aligned}$$

we find that a sufficient condition for the *Ansatz* in the induction step is

$$a^{-1}+\eta^2\beta a^{-2} \leq 1\,.$$

This may be solved to give the value $a \geq 1$ listed in Lemma 8, where we collect the results.

Lemma 8. A sufficient condition for the validity of the *Ansatz* (3.3) for $l \geq 1$ and even $n \geq 2$ is

$$\begin{aligned} &a = [1+(1+4\eta^2\beta)^{1/2}]/2\,, \\ &c = \max\{a^{-1}, a^{-2}[1+128n(n-1)^{-2}(n+3)^{-2}\eta^2/3]\}\,. \end{aligned}$$

The constant β is defined in (3.5), and η, defined in (3.2), obeys $\eta \leq 1$ for all α, with the possible exception of $d=2$ dimensions. At $\alpha=0$

$$\begin{aligned} \eta(0) &= 1, \quad d=1 \\ &= 0, \quad d=2 \\ &= 2^{-(d-2)/2}, \quad d>2\,. \end{aligned}$$

The exceptional value of $\eta(0)$ in two dimensions reflects the possibly nonoptimal estimate in Lemma 1. Note that

$$\beta \leq 32(n+3)^{-2}/3\,; \qquad (3.6)$$

and that β also has a lower bound that is $O(n^{-2})$ for n large; so that, for large n, a becomes just a little larger than, and c just a little smaller than, unity. For smaller values of n, the values of a and c can be improved by requiring the *Ansatz* only above a larger value of l, and treating the smaller values of l explicitly by Lemma 7.

We have now proved part (a) of the theorem in Sec. I.

B. Connected graphs

Let G_l^C denote the set of all connected Feynman graphs with l internal and n external vertices. We want to modify our dissolution procedure so that we get allowed graphs of lower order that remain connected. The lowest connected order corresponds to tree graphs. Certain vertices dissolve only into disconnected graphs, namely the "tree vertices," shown in Fig. 8, which connect to four disconnected subgraphs. To handle that situation, we decompose

$$G_l^C = G_{l,L}^C + G_{l,T}^C ,$$

where $G_{l,T}^C$ is the subsum of all graphs where the lth (internal) vertex is a tree vertex, and $G_{l,L}^C$ is the rest, where the lth vertex is a "loop vertex," lying on some closed loop.

We estimate $G_{l,T}^C$ as follows. Unless l is at the connected tree order (no loops), $l = n/2 - 1$, every graph in $G_{l,T}^C$ has some loop vertex. For each of these graphs, we permute the labels of the last internal vertex and the loop vertex which is closest to last. This converts $G_{l,T}^C$ into the subsum of all graphs where the last internal vertex is a loop vertex, and the next to last is a tree vertex. Therefore, $G_{l,T}^C < G_{l,L}^C$ (positivity of cutoff, Euclidean graphs), and we get for

$$l > n/2 - 1: \ \ G_l^C < 2G_{l,L}^C . \tag{3.7}$$

Note that this bound appears optimal for large l.

Next, write

$$G_{l,L}^C = G_{l,L,1}^C + G_{l,L,2}^C$$

where the first term on the right-hand side is the sum of graphs whose last internal vertex is a nonself-energy loop vertex, and the second is the sum of graphs whose last vertex is a self-energy vertex. The self-energy vertices are treated just as before; replacing a self-energy graph by a line preserves connectedness, as does the inverse operation of inserting a self-energy graph in a line. Thus, we find, as in (3.2),

$$G_{l,L,2}^C \leq c_2(l,n)\eta^2\kappa^2 G_{l-2}^C . \tag{3.8}$$

A nonself-energy loop vertex may be dissolved in at least two ways that leave the graph connected, shown in Fig. 9. Of course this produces no disconnected vacuum bubbles. Going back to our earlier discussion of nonself-energy dissolutions that produce a tadpole, as in Fig. 5(b), we see that the

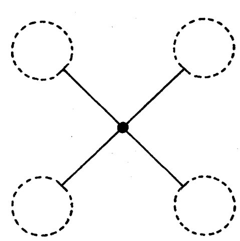

FIG. 8. Tree vertex.

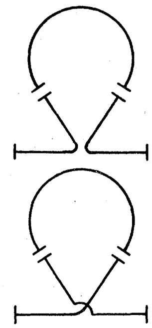

FIG. 9. Connected dissolutions of a nonself-energy, loop vertex.

remaining two dissolutions are not only allowed, but preserve connectedness. Thus we have proved the following:

Connected Dissolution Lemma. Every nonself-energy, internal loop vertex of an allowed, connected graph dissolves in at least two ways to produce an allowed, connected graph.

Since the inverse operation of pinching pairs of lines of an allowed, connected graph produces an allowed, connected graph whose last vertex is a nonself-energy, loop vertex (from previous discussion), exactly the same counting procedure for dissolutions, with the same maximum weight, applies as before, and we find

$$G_{l,L,1}^C \leq c_1(l,n)\kappa G_{l-1}^C . \tag{3.9}$$

Collecting estimates, we have proved the following:

Lemma 9. Let $l > n/2 - 1$, $n \geq 2$. Then

$$\begin{aligned} 0 &\leq G_l{}^C(x_1, \ldots, x_n) \\ &\leq 2c_1(l,n)\kappa G_{l-1}^C(x_1, \ldots, x_n) \\ &\quad + 2c_2(l,n)\eta^2\kappa^2 G_{l-2}^C(x_1, \ldots, x_n) , \end{aligned}$$

where c_1, c_2, κ, and η are defined in Eqs. (3.1) and (3.2), and where it is understood that $G^C{}_{n/2-2} = 0$.

The recursion in Lemma 9 is the same as that in Lemma 7, except for the overall factor of 2 and the fact that the recursion stops at the tree level.

To solve it for a bound on $G_l{}^C$, we proceed as before, but with the *Ansatz* for $l \geq 1$:

$$\begin{aligned} G^C{}_{l+l_n} &\leq \prod_{i=1}^{l} [i + l_n + (n-5)/4]^2 (a_T\kappa)^l c_T G_{l_n}{}^C , \\ l_n &\equiv n/2 - 2 , \end{aligned} \tag{3.10}$$

where the subscript T denotes "truncated" or "connected." Note that we should expect a_T to be roughly twice the parameter a in Lemma 8, because of the factor two in the recursion. An analysis like the previous one gives the following:

Lemma 10. A sufficient condition for the *Ansatz* (3.10) for $l \geq 1$ and even $n \geq 2$ is

$$a_T = 1 + (1+\eta^2\beta_T)^{1/2},$$
$$c_T = \max\{2a_T^{-1},\ 4a_T^{-2}(1+(l_n+1)(l_n+n/4)[l_n+(n+3)/4]^{-2}[l_n+(n-1)/4]^{-2}/3)\},$$
$$\beta_T \equiv \sup_{l\geq 3} 4(l+l_n-1)[l+l_n+(n-8)/4][l+l_n+(n-5)/4]^{-2}[l+l_n+(n-9)/4]^{-2}/3.$$

The constant η is defined in Eq. (3.2).

Just as in Lemma 8, the constants here are uniformly bounded in the cutoff and in n, and approach the values $a_T = 2+$, $c_T = 1-$ as $n\to\infty$. And a_T and c_T may also be improved for small n by enforcing the *Ansatz* only above some larger l.

We have now proved part (b) of the theorem in Sec. I.

C. Removing the volume cutoff

We turn now to the application of Lemmas 5.b and 6.b, to replace $\|g\|_1$ by $\|g\|_\infty$ in our bounds, so that we can remove the volume cutoff. The translation of the recursions in Lemmas 7 and 9 is quite easy; we may simply replace the constants κ and η appearing there by analogous constants depending on ϵ derived from Lemms 5.b and 6.b, while at the same time replacing G_{l-1}, G_{l-2}, G^C_{l-1}, and G^C_{l-2} by G's with mass squared reduced by ϵ^2.

The latter replacements overestimate the bounds more than one perhaps would like; they replace the mass squared in *every* line of the lower-order graphs by $\mu^2-\epsilon^2$, not just in those lines affected by a single vertex dissolution or by a single self-energy replacement. Unfortunately, we have not found a way to use the recursion that efficiently keeps track of which lines are necessarily affected at each step.

To solve the recursions by *Ansätze* analogous to (3.3) and (3.10), we take advantage of the fact that the ϵ's in the $l-1$ and $l-2$ terms may be chosen independently, and write the following:

Lemma 12. Let $\mu^2 > 2\epsilon^2 > 0$. The sums of allowed, spatial and α cutoff graphs and their connected subsums obey the bounds

$$\begin{aligned}
l\geq 1:\quad 0 &\leq G_l(x_1,\ldots x_n)\\
&\leq c_1(l,n)\kappa_\epsilon G_{l-1,\epsilon^2}(x_1,\ldots,x_n)\\
&\quad + c_2(l,n)(\eta^2/2)\kappa_\epsilon^2 G_{l-2,2\epsilon^2}(x_1,\ldots,x_n),\\
0 &\leq G^C_{l+l_n}(x_1,\ldots,x_n)\\
&\leq 2c_1(l+l_n,n)\kappa_\epsilon G^C_{l+l_n-1,\epsilon^2}(x_1,\ldots,x_n)\\
&\quad + 2c_2(l+l_n,n)(\eta^2/2)\\
&\quad \times \kappa_\epsilon^2 G^C_{l+l_n-2,2\epsilon^2}(x_1,\ldots,x_n),\\
\kappa_\epsilon &\equiv \epsilon^{-2}\gamma(\alpha)\|g\|_\infty.
\end{aligned}$$

Before formulating *Ansätze* to bound these recursions, we need to mention the mass dependence of κ_ϵ and η. Except when $d=1$ or 2, $\gamma(\alpha)$, defined in Lemmas 2 and 5, is mass independent, so κ_ϵ is mass independent, for fixed ϵ. Then $\eta = \Delta_\alpha(0)/\gamma(\alpha) \leq 2^{-(d-2)/2} < 1$ has only a weak mass dependence, which we may handle by replacing η by unity in our bounds. In case $d=1$ and $\alpha=0$, $\gamma=(2\mu)^{-1}$, while $\eta=1$. In case $d=2$, $\gamma(\alpha)$ is weakly mass dependent, going like $(\ln\mu^2\alpha)^2$ for $\mu^2\alpha$ small, and η may again be replaced by unity for α not too large (Lemma 8). In both cases where γ has a mass dependence, it is monotone decreasing.

The *Ansätze* become, for $l\geq 1$ and $\mu^2 > l\epsilon^2 > 0$,

$$G_l \leq \prod_{i=1}^{l}[i+(n-5)/4]^2(a_\infty\kappa_\epsilon)^l c_\infty G_{0,l\epsilon^2}, \tag{3.11}$$

$$G^C_{l+l_n} \leq \prod_{i=1}^{l}[i+l_n+(n-5)/4]^2(a_{T,\infty}\kappa_\epsilon)^l c_{T,\infty} G^C_{l_n,l\epsilon^2}, \tag{3.12}$$

with the understanding that for $d=1$ or 2, the γ in the definition of κ_ϵ is evaluated at the same mass squared, $\mu^2-l\epsilon^2$, as that appearing in G_0 and $G_{l_n}{}^C$. Then it is not hard to show the following:

Lemma 13. Sufficient conditions for the validity of the *Ansätze* (3.11) and (3.12) result by defining a_∞, c_∞ through the replacement of η^2 by $\frac{1}{2}$ in the definition of a, c in Lemma 8, and $a_{T,\infty}$, $c_{T,\infty}$ through the same replacement in the definition of a_T, c_T in Lemma 10.

The qualitative remarks about the approach of a_∞, c_∞, $a_{T,\infty}$, $c_{T,\infty}$ to 1+, 1−, 2+, 1− as n grows remain valid.

That completes the proof of all parts of our main theorem. □

ACKNOWLEDGMENT

It is a pleasure to thank Paul Federbush for helpful conversations.

APPENDIX: PROOF OF LEMMA 1.a

By a change of variables in Eq. (2.3), we get the following integral representation:

$d=z=2, \quad 2\mu\alpha \leq |x| \Rightarrow$

$$\Delta_\alpha(x) = (4\pi)^{-1}\left[\left(\int_1^\infty + \int_1^u\right) e^{-|\mu x| t}(t^2-1)^{-1/2}dt\right], \tag{A1}$$

$u \equiv x/4\mu\alpha + \mu\alpha/x.$

Putting $\alpha=0$, we get for all x

$$\Delta_0(x) = (2\pi)^{-1}\int_1^\infty e^{-|\mu x| t}(t^2-1)^{-1/2}dt = (2\pi)^{-1}K_0(|\mu x|), \tag{A2}$$

a standard representation for K_0; and we find

$$2\mu\alpha \leq |x| \Rightarrow \Delta_0(x)/2 \leq \Delta_\alpha(x) \leq \Delta_0(x), \tag{A3}$$

where the lower bound comes from dropping the second integral in (A1) and the upper bound from monotonicity in α when $z=2$, shown by Eq. (2.2).

For all x, we bound K_0 above by

$$K_0(|\mu x|) = |\mu x|^{-1/2}e^{-|\mu x|}\int_0^\infty e^{-t}[t(2+t/|\mu x|)]^{-1/2}dt \leq |2\mu x|^{-1/2}e^{-|\mu x|}\Gamma(1/2),$$

while for $|\mu x| \geq 1$, we can bound it below by

$$K_0(|\mu x|) \geq |\mu x|^{-1/2}e^{-|\mu x|} \times \int_0^1 e^{-t}[t(2+t/|\mu x|)]^{-1/2}dt \geq |\mu x|^{-1/2}e^{-|\mu x|}\int_0^1 e^{-t}(3t)^{-1/2}dt = |\mu x|^{-1/2}e^{-|\mu x|}(\pi/3)^{1/2}\,\mathrm{erf}(1).$$

Collecting these results, we get upper and lower bounds for larger values of x,

$$|\mu x| \geq \max\{1, 2\mu^2\alpha\} \Rightarrow (4\pi)^{-1}(\pi/3)^{1/2}\,\mathrm{erf}(1) \leq |\mu x|^{1/2}e^{|\mu x|}\Delta_\alpha(x) \leq (8\pi)^{-1/2}. \tag{A4}$$

For $|\mu x| \leq 1$, we use monotonicity in x, from Eq. (2.2),

$$\Delta_\alpha(\mu^{-1}) \leq \Delta_\alpha(x) \leq \Delta_\alpha(0).$$

In the lower bound, when $2\mu^2\alpha \leq 1$, we have $2\mu\alpha \leq \mu^{-1}$, so we may use the representation (A1) to conclude that

$$2\mu^2\alpha \leq 1 \text{ and } |\mu x| \leq 1 \Rightarrow (4\pi)^{-1}K_0(1) \leq \Delta_\alpha(x) \leq \Delta_\alpha(0). \tag{A5}$$

It is straightforward to combine these results to obtain

$$2\mu^2\alpha \leq 1 \Rightarrow$$

$$C_1 \leq (1+|\mu x|)^{1/2}e^{|\mu x|}\Delta_\alpha(x) \leq C_2',$$

$$C_1 \equiv (4\pi)^{-1}\min\{K_0(1), (\pi/3)^{1/2}\,\mathrm{erf}(1)\}, \tag{A6}$$

$$C_2' \equiv 2^{1/2}\max\{e\Delta_\alpha(0), (8\pi)^{-1/2}\}.$$

For small α, $\Delta_\alpha(0)$ diverges logarithmically; and then the constants in Lemma 1.a are given by Eq. (A6) and

$$C_2 = 2^{1/2}e. \quad \square \tag{A7}$$

[1]L. N. Lipatov, Tbilisi Conference on High Energy Physics, 1976, paper T4 (438); Zh. Eksp. Teor. Fiz. 72, 411 (1977) [Sov. Phys.—JETP 45, 216 (1977)].

[2]E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D 15, 1544 (1977); 15, 1558 (1977).

[3]G. Parisi, Phys. Lett. 66B, 167 (1977); 66B, 382 (1977); 68B, 361 (1977).

[4]C. Itzykson, G. Parisi, and J. B. Zuber, Phys. Rev. Lett. 38, 306 (1977); Phys. Rev. D 16, 996 (1977).

[5]C. Bervillier, J. M. Drouffe, J. Zinn-Justin, and C. Godrèche, Phys. Rev. D 17, 2144 (1978).

[6]C. A. Hurst, Proc. R. Soc. London A214, 44 (1952).

[7]R. J. Riddell, Jr., Phys. Rev. 91, 1243 (1953).

[8]E. R. Caianiello, Nuovo Cimento 3, 223 (1956).

[9]A. Buccafurri and E. R. Caianiello, Nuovo Cimento 8, 170 (1958).

[10]D. R. Yennie and S. Gartenhaus, Nuovo Cimento 9, 59 (1958).

[11]E. R. Caianiello, Nuovo Cimento Suppl. 9, 569 (1958).

[12]J. Glimm, Commun. Math. Phys. 10, 1 (1968).

[13]K. Hepp, *Théorie de la Renormalisation* (Springer, New York, 1969), Théorème 4.5.

[14]J. Glimm and A. Jaffe, in *Statistical Mechanics and Quantum Field Theory*, 1970 Les Houches lectures, edited by C. DeWitt and R. Stora (Gordon and Breach, New York, 1971), Proposition 2.1.2, p. 19.

[15]J. Glimm, A. Jaffe, and T. Spencer, in *Constructive Quantum Field Theory*, Proceedings of the International School of Mathematical Physics "Ettore Majorana," Erice, Italy, 1973, edited by G. Velo and A. Wightman (Springer, New York, 1973), p. 205.

[16]B. Simon, *The $P(\phi)_2$ Euclidean (Quantum) Field Theory* (Princeton Univ. Press, Princeton, 1974).

[17]In this brief literature survey, we have essentially restricted ourselves to results that bound the sum of all graphs of a given order. We have thereby neglected an important portion of the literature; for example, the extensive studies of the large-p behavior of particular classes of Feynman graphs.

[18]A. P. Calderón, in *Proceedings of the Symposium on Pure Mathematics* (American Math. Soc., Providence, R.I., 1961), Vol. 33, p. 4.

[19]W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea, New York, 1949).

[20]B. O. Peirce and R. M. Foster, *A Short Table of Integrals* (Ginn and Company, Boston, Mass., 1956), 4th edition. (783–85 refers to formula numbers.)