A linear map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfies $f(av + bw) = af(v) + bf(w)$ for any $a, b \in \mathbb{R}$ and $v, w \in \mathbb{R}^m$.

It’s enough to know where the map sends coordinate vectors. We can write this information in the columns of a matrix.

Map 1, $\mathbb{R}^2 \rightarrow \mathbb{R}$: forget vertical coordinate
Although these maps look different, that’s just a matter of \textbf{which coordinate is first and which is second.}

If we label our axes differently, they have the same matrix.
Although these maps look different, that’s just a matter of which coordinate is first and which is second.

If we label our axes differently, they have the same matrix.
This map seems fundamentally different, but...

if we pick the right coordinates, it’s still “keep one piece of information and forget the other.”
Change of basis

Map 3, $\mathbb{R}^2 \rightarrow \mathbb{R}$: add (original) coordinates

This map seems fundamentally different, but...

if we pick the right coordinates, it’s still “keep one piece of information and forget the other.”
A theorem on change of basis

Theorem

Let \( f : \mathbb{R}^m \to \mathbb{R}^n \) be a linear map. Then it is always possible to choose coordinates for \( \mathbb{R}^m \) and \( \mathbb{R}^n \) such that the matrix of \( f \) looks like

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 
\end{pmatrix}
\]

Given any map, we can choose tailored coordinates, from whose perspective the map is “keep some information and forget the rest.”

What’s important is rank — the number of 1’s in the matrix.
Directions for generalization

- Direction 1: place restrictions on coordinates (like fixing angles and lengths)

- Not the subject of this talk, but leads to tools like the singular value decomposition.

- Direction 2: consider multiple maps at once

  \[ \mathbb{R}^m \xrightarrow{f} \mathbb{R}^n \xrightarrow{g} \mathbb{R}^p \]

- Can we choose coordinates of $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^p$ such that $f$ and $g$ both have nice matrices? What does “nice” mean in this context?
- What are the fundamentally different ways this diagram can behave?
A **quiver** is a diagram consisting of vertices and arrows between them.

A **representation** of a quiver is an assignment of a vector space to every vertex and a linear map to every arrow.
Two representations of a quiver are **isomorphic** if we can pick coordinates for each one so that they consist of the same matrices.

\[ \mathbb{R}^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \rightarrow \mathbb{R} \cong \mathbb{R}^2 \begin{pmatrix} 0 & 1 \end{pmatrix} \rightarrow \mathbb{R} \cong \mathbb{R}^2 \begin{pmatrix} 1 & 1 \end{pmatrix} \rightarrow \mathbb{R} \]

The **direct sum** of two representations:

\[ \mathbb{R}^{m_1} \begin{array}{c} f_1 \\ \oplus \\ f_2 \end{array} \rightarrow \mathbb{R}^{n_1} \begin{array}{c} g_1 \\ \rightarrow \mathbb{R}^{p_1} \\ \mathbb{R}^{m_2} \begin{array}{c} f_2 \\ \rightarrow \mathbb{R}^{n_2} \\ \rightarrow \mathbb{R}^{p_2} \end{array} \]

\[ \cong \begin{pmatrix} \mathbb{R}^{m_1} \\ \mathbb{R}^{m_2} \end{pmatrix} \begin{array}{c} (f_1 \ 0) \\ (0 \ f_2) \end{array} \rightarrow \begin{pmatrix} \mathbb{R}^{n_1} \\ \mathbb{R}^{n_2} \end{pmatrix} \begin{array}{c} (g_1 \ 0) \\ (0 \ g_2) \end{array} \rightarrow \begin{pmatrix} \mathbb{R}^{p_1} \\ \mathbb{R}^{p_2} \end{pmatrix} \]
Examples of direct sum

\[(\mathbb{R} \xrightarrow{1} \mathbb{R}) \oplus (\mathbb{R} \xrightarrow{1} \mathbb{R}) \oplus (\mathbb{R} \xrightarrow{0} 0) \cong \mathbb{R}^3 \xrightarrow{(1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0)} \mathbb{R}^2\]

- In general, any matrix of the form
  \[
  \begin{pmatrix}
    1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
    0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 0 & 0 & \cdots & 0
  \end{pmatrix}
  \]

breaks down as a direct sum of \(\mathbb{R} \xrightarrow{1} \mathbb{R}, \mathbb{R} \xrightarrow{0} 0,\) and \(0 \xrightarrow{0} \mathbb{R}.\)
A representation is **indecomposable** if it isn’t isomorphic to a direct sum of smaller ones.

With this new language, we can restate the theorem we started with.

**Theorem**

*The only indecomposable representations of the quiver* 

\[
\bullet \rightarrow \bullet
\]

*are*

\[
\mathbb{R} \overset{1}{\rightarrow} \mathbb{R} \quad \mathbb{R} \overset{0}{\rightarrow} 0 \quad 0 \overset{0}{\rightarrow} \mathbb{R}
\]

so every representation is isomorphic to a direct sum of these.
Our original question becomes: for each quiver, can we classify the indecomposable representations?

We can capture the key features of $\bullet \to \bullet$ with the following property:

**Definition**

A quiver is **finite type** if it has finitely many indecomposable representations.

For a representation, the analogue of “rank” is “how many copies of each indecomposable show up?”
Similarly to $\bullet \rightarrow \bullet$, we can interpret a direct sum of these as "keep some information, forget the rest" at each step.
There are 120 indecomposables, with spaces of dimension up to 6.
Gabriel’s theorem

A quiver is finite type if and only if, ignoring the direction of arrows, it has one of these shapes:

- $A_n$: \[ \cdots \]
- $D_n$: \[ \cdots \]
- $E_6$: \[ \]
- $E_7$: \[ \]
- $E_8$: \[ \]

These diagrams — the **simply laced Dynkin diagrams** — are ubiquitous and important.
Systems of mirrors

- A **hyperplane** in $\mathbb{R}^n$ is a subspace of dimension $n - 1$ through the origin.

**Definition**

A **closed system of mirrors** is a finite collection $\mathcal{H}$ of hyperplanes such that:

- For $H_1, H_2 \in \mathcal{H}$, the reflection of $H_1$ through $H_2$ is also in $\mathcal{H}$. 
Notating a system of mirrors

- Knowing the mirrors bordering a single region (walls) is enough.
- We record the angle between each pair of walls.

Angle between…

1 and 2 \( \frac{\pi}{4} \)
2 and 3 \( \frac{\pi}{3} \)
1 and 3 \( \frac{\pi}{2} \)
Now draw a diagram with a vertex for each wall.

For each pair of walls:
- If the angle between is $\pi/2$, do nothing.
- If the angle between is $\pi/3$, draw an edge between them.
- If the angle between is $\pi/m$, $m > 3$, draw an edge between them and label it with $m$.

Angle between...

1 and 2 $\pi/4$
2 and 3 $\pi/3$
1 and 3 $\pi/2$
What are the closed systems of mirrors?

**Theorem**

The closed systems of mirrors correspond to these diagrams.

- $A_n$: \[ \cdots \]
- $E_8$: \[ \cdots \]
- $B_n/C_n$: \[ 4 \cdots \]
- $D_n$: \[ \cdots \]
- $E_6$: \[ \cdots \]
- $E_7$: \[ \cdots \]
- $F_4$: \[ 4 \]
- $H_3$: \[ 5 \]
- $H_4$: \[ 5 \]
- $I_2(m)$: \[ m \]

The unlabeled ones look familiar!
The **dimension vector** of a quiver representation records the dimension at each vertex.

\[
\begin{align*}
\text{R} & \rightarrow \mathbb{R}^2 \\
(1) & \rightarrow (1,2,1,1)
\end{align*}
\]

Choose a region of a system of mirrors, and for each mirror pick the normal vector pointing towards that region. These are **root vectors**.
Theorem

There is a change of basis taking the dimension vectors of indecomposable representations of a finite type quiver to the root vectors of the system of mirrors with the same diagram.

Or, in other words...

Theorem

Indecomposable representations correspond precisely with mirrors.