Representations of quivers and Lie algebras

Day 5: How to prove Ringel's theorem

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1. Last time. . .
2. φ is well-defined
3. φ is surjective
4. φ is injective

Last time...

- We can reconstruct a Lie algebra from its root system.

Theorem

Suppose $\mathfrak{g}$ is a simple Lie algebra with root system Φ and simple roots $\alpha_1, \ldots, \alpha_n$. Then $\mathfrak{g}$ is isomorphic to the Lie algebra with generators

$$x_1, \ldots, x_n, y_1, \ldots, y_n, h_1, \ldots, h_n$$

and the following relations (where $R(\alpha_j, \alpha_i) = 2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$, in the simply laced case just $\langle \alpha_j, \alpha_i \rangle$):

(1) $[h_i, h_j] = 0$ *for all* i, j.

(2) $[x_i, y_i] = h_i$, $[x_i, y_j] = 0$ *for all* $i \neq j$.

(3) $[h_i, x_j] = R(\alpha_j, \alpha_i) x_j$, $[h_i, y_j] = -R(\alpha_j, \alpha_i) y_j$ *for all* i, j

(4) $\boxed{\mathrm{ad}(x_i)^{-R(\alpha_j, \alpha_i)+1}(x_j) = 0, \ \mathrm{ad}(y_i)^{-R(\alpha_j, \alpha_i)+1}(y_j) = 0 \text{ for all } i, j.}$

Last time...

- Fix a quiver Q and a **finite field** $\mathbb{F}_q$.
- For representations V, M_1, M_2 of Q over $\mathbb{F}_q$, define

$$F^V_{M_1 M_2} = \#\{\text{subrepresentations } W \subset V \mid W \cong M_2, V/W \cong M_1\}.$$

Definition

The **Ringel-Hall algebra** $H(Q, \mathbb{F}_q)$ is a $\mathbb{C}$-algebra with:

- a basis indexed by isomorphism classes of representations of Q over $\mathbb{F}_q$.
- multiplication defined by

$$[M_1] \cdot [M_2] := \sum_{[L]} F^L_{M_1 M_2} [L]$$

Last time...

- $H(Q, \mathbb{F}_q)$ is associative, but not commutative. For example, if $Q = 1 \to 2$:
  - $[S_1] \cdot [S_2] = [\mathbb{F}_q \xrightarrow{1} \mathbb{F}_q] + [\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q]$
  - $[S_2] \cdot [S_1] = [\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q] = [S_1 \oplus S_2]$
- Which finite field should we pick to recover part of the Lie algebra? None of them.
- Instead, we use this result to define a universal Ringel-Hall algebra:

Theorem (Ringel)

For any Dynkin quiver Q and representations V, M_1, M_2 of Q, the structure constant $F^V_{M_1 M_2}$ is a polynomial in q.

- Specializing $q = 1$ gives the Ringel-Hall algebra we're interested in, $H(Q)$.

Last time...

- The Ringel-Hall algebra is associative, but we want to recover a Lie algebra. The link:

Definition

The **universal enveloping algebra**, $U(\mathfrak{g})$, of a Lie algebra $\mathfrak{g}$ is the quotient of the tensor algebra of $\mathfrak{g}$ by the relation

$$g \otimes h - h \otimes g = [g, h]$$

for all $g, h \in \mathfrak{g}$.

- This turns representation theory of Lie algebras into that of associative algebras.

Last time...

Theorem (Ringel)

- *Let Q be a quiver.*
- *Let $\mathfrak{g}$ be the Lie algebra whose Dynkin diagram is the underlying undirected graph.*
- *Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the decomposition into negative, Cartan, and positive parts.*
- *Let $x_1, \ldots, x_n \in \mathfrak{n}_+$ be elements from the simple root spaces.*
- *Let $S_1, \ldots, S_n$ be the simple representations*

Then there is an isomorphism

$$\varphi : U(\mathfrak{n}_+) \to H(Q)$$

sending $x_i \mapsto [S_i]$.

Back to the Serre relations

Corollary (to Serre relations)

Let $\mathfrak{g}$ be a simply laced, simple Lie algebra with root system Φ, Dynkin diagram G, and simple roots $\alpha_1, \ldots, \alpha_n$. Then the positive part $\mathfrak{n}_+$ is isomorphic to the Lie algebra with generators $x_1, \ldots, x_n$ and relations

$$[x_i, x_j] = 0 \text{ if } i \text{ and } j \text{ are not adjacent in } G$$
$$[x_i, [x_i, x_j]] = 0 \text{ if } i \text{ and } j \text{ are adjacent in } G$$

Corollary

$U(\mathfrak{n}_+)$ is the associative algebra with generators $x_1, \ldots, x_n$ and relations

$$x_i x_j - x_j x_i = 0 \text{ if } i \text{ and } j \text{ are not adjacent in } G$$
$$x_i^2 x_j - 2x_i x_j x_i + x_j x_i^2 = 0 \text{ if } i \text{ and } j \text{ are adjacent in } G$$

Serre relations in the Hall algebra

- We just need to show that the Hall algebra satisfies

$$[S_i][S_j] - [S_j][S_i] = 0 \text{ if } i \text{ and } j \text{ are not adjacent}$$
$$[S_i]^2[S_j] - 2[S_i][S_j][S_i] + [S_j][S_i]^2 = 0 \text{ if } i \text{ and } j \text{ are adjacent}$$

- If i is not adjacent to j, any extension of S_i by S_j is just two unrelated 1-dimensional spaces — that is, $S_i \oplus S_j$.
- This has only one S_i or S_j subrepresentation, so

$$[S_i][S_j] = [S_j][S_i] = [S_i \oplus S_j].$$

which is what we want.

Serre relations in the Hall algebra

- To show the relations when i and j are adjacent, it suffices to consider the quiver $1 \to 2$.
- Since we're computing $[S_1]^2[S_2]$, $[S_1][S_2][S_1]$, and $[S_2][S_1]^2$, we're interested in representations of the form $k^2 \to k$. Up to isomorphism, there are only 2:

$$N_1 := k^2 \xrightarrow{(1\ 0)} k$$
$$N_0 := k^2 \xrightarrow{0} k$$

- Then you can directly compute in $H(Q, \mathbb{F}_q)$:

$$[S_1]^2[S_2] = (q+1)[S_1^2][S_2] = (q+1)[N_0] + (q+1)[N_1]$$
$$[S_1][S_2][S_1] = [S_1][S_1 \oplus S_2] = (q+1)[N_0] + [N_1]$$
$$[S_2][S_1]^2 = (q+1)[S_2][S_1^2] = (q+1)[N_0]$$

Serre relations in the Hall algebra

$$[S_1]^2[S_2] = (q+1)[N_0] + (q+1)[N_1]$$
$$[S_1][S_2][S_1] = (q+1)[N_0] + [N_1]$$
$$[S_2][S_1]^2 = (q+1)[N_0]$$

- It's a quick check from here that

$$[S_1]^2[S_2] - (q+1)[S_1][S_2][S_1] + q[S_2][S_1]^2 = 0$$

  and specializing $q \to 1$ gives the identity we want!
- Thus the map φ is well-defined.

Generators of $H(Q)$

- We define φ by sending $x_i \mapsto [S_i]$. So we need to show that

Lemma

The elements $[S_i]$ generate $H(Q)$.

- This proceeds in two steps. First, we show the Hall algebra is generated by indecomposables.
- Recall the construction of a list of indecomposables from before:

Lemma

There is a list of vertices $v_1, \dots, v_\ell$ such that the sequence

$$I_j := \Phi^-_{v_1} \cdots \Phi^-_{v_{j-1}}(S_{v_j}), 1 \leq j \leq \ell$$

contains every indecomposable representation exactly once.

Generation by indecomposables

Lemma

The representations I_j, in the given order, satisfy

$$\mathrm{Hom}(I_a, I_b) = 0, a > b$$
$$\mathrm{Ext}^1(I_a, I_b) = 0, a \leq b$$

Proof (sketch).

We can unwind the reflection functors defining I_a and I_b:

$$\mathrm{Hom}(\Phi^-_{v_1} \cdots \Phi^-_{v_{a-1}}(S_{v_a}), \Phi^-_{v_1} \cdots \Phi^-_{v_{b-1}}(S_{v_b})) \cong \mathrm{Hom}(\Phi^-_{v_b} \cdots \Phi^-_{v_{a-1}}(S_{v_a}), S_{v_b})$$

If v_b is a sink of Q, any map $V \to S_{v_b}$ splits. If this Hom is nonzero, $\Phi^-_{v_b} \cdots \Phi^-_{v_{a-1}}(S_{v_a})$ has S_{v_b} as a summand — but this can't happen for any representation of form $\Phi^-_{v_b}(V)$.
The Ext^1 case is similar. □

Generation by indecomposables

Lemma

For any representation $V \cong \bigoplus_{j=1}^{\ell} I_j^{c_j}$,

$$[V] = \frac{[I_1]}{c_1!}\frac{[I_2]}{c_2!}\cdots\frac{[I_\ell]}{c_\ell!}$$

Proof (sketch).

In general, if $\mathrm{Hom}(W, V) = 0$ and $\mathrm{Ext}^1(V, W) = 0$, $[V] \cdot [W] = [V \oplus W]$:

- $\mathrm{Ext}^1(V, W) = 0$ means $[V \oplus W]$ is the only term that shows up.
- $\mathrm{Hom}(W, V) = 0$ means its coefficient is 1.

Thus $[V] = [I_1^{c_1}] \cdots [I_\ell^{c_\ell}]$. Further, $[I_j^{c_j}] = \frac{[I_j]}{c_j!}$ for the same reason as for S_y; this uses the properties that $\mathrm{Hom}(I_j, I_j) = k$ and $\mathrm{Ext}^1(I_j, I_j) = 0$, which it inherits from S_y through reflection functors. □

Generation by simples

- The second step is to show that all the $[I_j]$ are in the algebra generated by the $[S_v]$.

Lemma

(1) *The simple representations of an acyclic quiver are precisely the S_v.*

(2) *Any representation admits a filtration whose quotients are representations S_v.*

- Given a filtration for I_j, the product of its simple factors will have the form $c[I_j] + \sum_r c_r[V_r]$, where I_j and all V_r have the same dimension vector.
- The other $[V_r]$ will be decomposable, so we can break them down and proceed by induction on dimension vectors.
- Thus the simples are all we need, and φ is surjective!

The Poincaré-Birkhoff-Witt basis

- Once we know φ is surjective, we can show it's also injective by counting dimensions.
- $U(\mathfrak{n}_+)$ and $H(Q)$ are both infinite dimensional, but we can break them into pieces and show that the dimensions are the same on each side.
- This is possible because $U(\mathfrak{n}_+)$ admits a convenient basis:

Theorem

Let $\mathfrak{g}$ be any Lie algebra with basis $g_1, \ldots, g_n$. Then $U(\mathfrak{g})$ has a basis consisting of elements

$$g_{i_1} g_{i_2} \cdots g_{i_\ell}, \quad i_1 \leq i_2 \leq \cdots \leq i_\ell$$

(including the empty product 1).

- So $\mathfrak{g}$ has a basis like that of a polynomial ring — but, of course, the multiplication is different.

The PBW basis in our case

- $\mathfrak{n}_+$ has a basis indexed by positive roots, so $U(\mathfrak{n}_+)$ has a basis indexed by unordered tuples of positive roots.
- To each basis element, assign a weight:

$$x_{\alpha^{(1)}} x_{\alpha^{(2)}} \cdots x_{\alpha^{(\ell)}} \mapsto \alpha^{(1)} + \ldots + \alpha^{(\ell)}$$

- Let $U(\mathfrak{n}_+)_{\mathsf{d}}$ be the subspace spanned by basis elements of weight d.
- Now let $H(Q)_{\mathsf{d}}$ be the subspace spanned by representations of dimension d.
- Working through the definition of φ shows that it maps $U(\mathfrak{n}_+)_{\mathsf{d}}$ to $H(Q)_{\mathsf{d}}$.
- Both spaces have the same dimension: the number of unordered tuples of positive roots summing to d.
  - For $U(\mathfrak{n}_+)_{\mathsf{d}}$, this follows from the PBW basis.
  - For $H(Q)_{\mathsf{d}}$, it follows from the decomposition of representations into indecomposables.

In summary. . .

- By interpreting a quiver as a directed Dynkin diagram, we unlock a connection between indecomposable representations and positive roots.
- The first clue to this connection comes from reflection functors, which "categorify" reflections of roots.
- But we can also ask how the related Lie algebra is manifested in these representations. The Ringel-Hall algebra shows that it captures the behavior of extensions.

Where to go from here?

- Rather than specializing q to 1, leave it ambiguous, in order to work with the *quantized* universal enveloping algebra.
- Find a *canonical* basis of the universal enveloping algebra, constructed geometrically, with nice properties (Lusztig, 1990)
- Move to infinite dimensions — general quivers and Kac-Moody Lie algebras (see Kirillov's book!)
- Try to capture the entire universal enveloping algebra, using derived categories to introduce "shifted" representations corresponding to negative roots (Bridgeland 2013)

Thanks for joining me!