Representations of quivers and Lie algebras

Day 4: Lie algebras! Ringel-Hall algebras! Universal enveloping algebras!

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Last time...

- We finished the proof of Gabriel's theorem!
- The key step: starting with an indecomposable representation of a Dynkin quiver, apply reflection functors until its dimension vector goes negative.
- This signifies that we've hit some S_y.
- Then we unwind all the reflections we performed, showing the dimension vector of our representation is a root.

Last time...

- Moreover, there's a nice way of spitting out all the indecomposables, one for each root:

Lemma

There exists a sequence of vertices $x_1, \dots, x_n$ such that

(1) *x_i is a sink of $s_{x_{i-1}} \cdots s_{x_1}(Q)$ for each i (the sequence is* adapted to *the orientation of Q)*

(2) *The sequence $s_{x_1} \cdots s_{x_{j-1}}(\alpha_{x_j})$, $1 \leq j \leq \ell$, hits every positive root exactly once.*

(3) *The sequence $\Phi^-_{x_1} \cdots \Phi^-_{x_{j-1}}(S_{x_j})$, $1 \leq j \leq \ell$, hits every indecomposable representation exactly once.*

- More about this later.

Last time...

- A **Lie algebra** is a vector space $\mathfrak{g}$ with a bilinear, antisymmetric map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

- A Lie algebra is **simple** if it has no proper ideals (for example, $\mathfrak{sl}_n$)
- A simple Lie algebra has a **Cartan subalgebra** $\mathfrak{h}$, analogous to the diagonal matrices in $\mathfrak{sl}_n$.
- For any $\alpha \in \mathfrak{h}^* := \mathrm{Hom}_k(\mathfrak{h}, k)$, we have a **root space**

$$V_\alpha := \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g \ \forall h \in \mathfrak{h}\}$$

  and together, these give a **root space decomposition**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} V_\alpha$$

Last time. . .

- There's a reason we call them root spaces. If $V_\alpha \neq 0$, say α is a **root**.

Lemma

The collection of roots of a semisimple Lie algebra forms a root system.

- In the case of $\mathfrak{sl}_n$, we get the A_{n-1} root system, and:
  - the root spaces correspond to off-diagonal entries;
  - the positive roots correspond to entries below the diagonal;
  - the simple roots correspond to the entries immediately below the diagonal.

The Lie bracket and the root space decomposition

Lemma

$$[V_\alpha, V_\beta] \subset V_{\alpha+\beta}$$

Proof.

For $v \in V_\alpha$, $w \in V_\beta$, $h \in \mathfrak{h}$:

$$\begin{aligned}[h,[v,w]] &= -[v,[w,h]] - [w,[h,v]] = [v,[h,w]] + [[h,v],w] \\ &= [v,\beta(h)w] + [\alpha(h)v,w] = (\alpha+\beta)(h)[v,w]\end{aligned}$$

□

- In particular, the positive and negative roots give a coarser breakdown into subalgebras:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Reconstructing the algebra from the root system

- Gabriel's theorem connects indecomposable representations to positive roots. Can we bring in the rest of the Lie algebra?
- Since the roots are what we have to work with, it's helpful to reconstruct the Lie algebra from its roots.
- This construction is not too complicated, but it's still complicated enough that the statement requires its own slide.

The Serre relations

Theorem

Suppose $\mathfrak{g}$ is a simple Lie algebra with root system Φ and simple roots $\alpha_1, \ldots, \alpha_n$. Then $\mathfrak{g}$ is isomorphic to the Lie algebra with generators

$$x_1, \ldots, x_n, y_1, \ldots, y_n, h_1, \ldots, h_n$$

and the following relations (where $R(\alpha_j, \alpha_i) = 2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}$, in the simply laced case just $\langle \alpha_j, \alpha_i \rangle$):

(1) $[h_i, h_j] = 0$ *for all* i, j.

(2) $[x_i, y_i] = h_i$, $[x_i, y_j] = 0$ *for all* $i \neq j$.

(3) $[h_i, x_j] = R(\alpha_j, \alpha_i)x_j$, $[h_i, y_j] = -R(\alpha_j, \alpha_i)y_j$ *for all* i, j

(4) $\mathrm{ad}(x_i)^{-R(\alpha_j, \alpha_i)+1}(x_j) = 0$, $\mathrm{ad}(y_i)^{-R(\alpha_j, \alpha_i)+1}(y_j) = 0$ *for all* i, j.

Where are the Serre relations coming from?

- The generators $x_1, \ldots, x_n$ span the root spaces for the positive roots.
- The generators $y_1, \ldots, y_n$ span the root spaces for the negative roots
- The generators $h_1, \ldots, h_n$ are a basis for the Cartan subalgebra.

(1) $\boxed{[h_i, h_j] = 0}$

- The Cartan subalgebra satisfies $[\mathfrak{h}, \mathfrak{h}] = 0$.

(2) $\boxed{[x_i, y_i] = h_i;\ [x_i, y_j] = 0 \text{ for } i \neq j}$

- We have $[x_i, y_i] \in [V_{\alpha_i}, V_{-\alpha_i}] \subset V_0 = \mathfrak{h}$.
- On the other hand, $[x_i, y_j] \in V_{\alpha_i - \alpha_j}$. Since every root is either a positive or negative combination of simple roots, this must be 0.

(3) $\boxed{[h_i, x_j] = 2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} x_j,\ [h_i, y_j] = -2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} y_j}$

- x_j spans the root space for $\langle \alpha_j, - \rangle$, and y_j spans the root space for $\langle -\alpha_j, - \rangle$.
- That h_i corresponds to $2\frac{\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ follows from careful manipulation of the Killing form.

Where are the Serre relations coming from?

(4) $\mathrm{ad}(x_i)^{-R(\alpha_j,\alpha_i)+1}(x_j) = 0$, $\mathrm{ad}(y_i)^{-R(\alpha_j,\alpha_i)+1}(y_j) = 0$ for all i, j.

- Since $x_i \in V_{\alpha_i}$ and $x_j \in V_{\alpha_j}$, $\mathrm{ad}(x_i)^{-R(\alpha_j,\alpha_i)}(x_j) \in V_{\alpha_j - R(\alpha_j,\alpha_i)\alpha_i} = V_{s_i(\alpha_j)}$.
- An intuition: applying $\mathrm{ad}(x_i)$ one more time "overshoots".
- In the simply-laced case:
    - If i and j are not adjacent in the Dynkin diagram, $[x_i, x_j] = 0$.
    - If i and j are adjacent in the Dynkin diagram, $[x_i, [x_i, x_j]] = 0$

The Ringel-Hall algebra

- Fix a quiver Q and a **finite field** $\mathbb{F}_q$.
- For representations V, M_1, M_2 of Q over $\mathbb{F}_q$, define

$$F^V_{M_1 M_2} = \#\{\text{subrepresentations } W \subset V \mid W \cong M_2, V/W \cong M_1\}.$$

Definition

The **Ringel-Hall algebra** $H(Q, \mathbb{F}_q)$ is a $\mathbb{C}$-algebra with:

- a basis indexed by isomorphism classes of representations of Q over $\mathbb{F}_q$.
- multiplication defined by

$$[M_1] \cdot [M_2] := \sum_{[L]} F^L_{M_1 M_2} [L]$$

- Essentially, $[M_1] \cdot [M_2]$ combines all the extensions of M_1 by M_2 — but we have to be careful how we count them.

Basic properties

- $H(Q, \mathbb{F}_q)$ is associative, because there's a natural way to define a product of any number of elements:

Theorem

Define

$$F^V_{M_1 M_2 \cdots M_\ell} := \#\{\text{filtrations } V = U_0 \supset U_1 \supset \cdots \supset U_\ell = 0 \mid U_{i-1}/U_i \cong M_i \forall i\}$$

Then

$$[M_1] \cdot [M_2] \cdot \ldots \cdot [M_\ell] = \sum_{[L]} F^L_{M_1 \cdots M_\ell} [L]$$

no matter how the terms on the left are grouped.

An example

- $H(Q, \mathbb{F}_q)$ is not commutative. An example: let Q be $1 \to 2$.
  - We compute $[S_1] \cdot [S_2]$.
  - Any extension $0 \to S_2 \to L \to S_1 \to 0$ must have dimension vector $\boxed{1 \to 1}$.
  - There are two representations of this dimension vector:

$$\mathbb{F}_q \xrightarrow{1} \mathbb{F}_q$$

$$\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q (\cong S_1 \oplus S_2)$$

  - How many subrepresentations of $\mathbb{F}_q \xrightarrow{1} \mathbb{F}_q$ are isomorphic to S_2? $\boxed{1.}$
  - How many subrepresentations of $\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q$ are isomorphic to S_2? $\boxed{1.}$
  - Thus

$$[S_1] \cdot [S_2] = [\mathbb{F}_q \xrightarrow{1} \mathbb{F}_q] + [\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q]$$

An example

- $H(Q, \mathbb{F}_q)$ is not commutative. An example: let Q be $1 \to 2$.
  - We compute $[S_2] \cdot [S_1]$.
  - Any extension $0 \to S_1 \to L \to S_2 \to 0$ must have dimension vector $\boxed{1 \to 1}$.
  - There are two representations of this dimension vector:

$$\mathbb{F}_q \xrightarrow{1} \mathbb{F}_q$$

$$\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q (\cong S_1 \oplus S_2)$$

  - How many subrepresentations of $\mathbb{F}_q \xrightarrow{1} \mathbb{F}_q$ are isomorphic to S_1? $\boxed{\mathbf{0}.}$
  - How many subrepresentations of $\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q$ are isomorphic to S_1? $\boxed{1.}$
  - Thus

$$[S_2] \cdot [S_1] = [\mathbb{F}_q \xrightarrow{0} \mathbb{F}_q] = [S_1 \oplus S_2].$$

Another example

- Now let Q be just $\circ$ (so representations are just vector spaces)
- We compute $[k^m] \cdot [k]$.
- How many subrepresentations of k^{m+1} are isomorphic to k?
- These are parametrized by $\mathbb{P}^m$, which over $\mathbb{F}_q$ has order

$$\frac{q^{m+1} - 1}{q - 1} = 1 + q + \ldots + q^m = [m]_q \text{ (the ``q-analog'' of } m)$$

- Thus $[k^m] \cdot [k] = [n]_q [k^{m+1}]$.
- In general, we have an isomorphism with $\mathbb{C}[x]$ given by

$$[k^m] \mapsto \frac{x^m}{[m]_q [m-1]_q \cdots [1]_q}$$

Structure constants are polynomials

Theorem (Ringel)

For any Dynkin quiver Q and representations V, M_1, M_2 of Q, the structure constant $F^V_{M_1 M_2}$ is a polynomial in q.

- Knowing this, we define a **universal Hall algebra** $H(Q, \mathbb{C}[t])$ with multiplication defined using these polynomials.
- The algebra we want comes from specializing $t = 1$. We denote it by just $H(Q)$.
- Essentially, we are undoing the deformation clued by the q-analogs above.

The universal enveloping algebra

- Lie algebras are nice, but (especially for representation theory) we're more familiar with associative algebras.
- Fortunately, there's a canonical way of turning any Lie algebra into an associative algebra:

Definition

Let $\mathfrak{g}$ be a Lie algebra. The **tensor algebra**, $T(\mathfrak{g})$, is the algebra

$$\bigoplus_{i=0}^{\infty} \mathfrak{g}^{\otimes i}$$

with multiplication given by tensor product. The **universal enveloping algebra**, $U(\mathfrak{g})$, is the quotient of $T(\mathfrak{g})$ by the relations

$$g \otimes h - h \otimes g = [g, h]$$

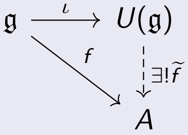
for all $g, h \in \mathfrak{g}$.

The universal enveloping algebra

- If we turn A into a Lie algebra with $[a, b] = ab - ba$, taking the universal enveloping algebra **does not** recover A — it will typically be much larger.
- However, $U(\mathfrak{g})$ has nice categorical properties. It's universal in the following sense:

Theorem

Let $\mathfrak{g}$ be a Lie algebra, A an associative algebra (viewable as a Lie algebra with $[a, b] = ab - ba$), and $f : \mathfrak{g} \to A$ a homomorphism of Lie algebras. Then there exists a unique morphism of associative algebras $\widetilde{f} : U(\mathfrak{g}) \to A$ making this diagram commute:



The universal enveloping algebra

- A **representation** of a Lie algebra $\mathfrak{g}$ is a vector space V and bilinear map $- \cdot - : \mathfrak{g} \times V \to V$ such that

$$[g, h] \cdot v = g \cdot (h \cdot v) - h \cdot (g \cdot v)$$

Theorem

The categories of representations of $\mathfrak{g}$ and modules over $U(\mathfrak{g})$ are equivalent.

- This essentially follows from the above universality statement with $A = \mathrm{GL}(V)$.

Ringel's theorem

- We're now equipped to state the first key result strengthening the connection between Gabriel's theorem and Lie algebras:

Theorem (Ringel)

There is an isomorphism $U(\mathfrak{n}_+) \to H(Q)$ *sending* $e_x \mapsto [S_x]$.

Exercises

- Two facts we'll be interested in using tomorrow:

Exercise

Under what circumstances do we have

$$[M] \cdot [N] = [M \oplus N]$$

in the Hall algebra?

Exercise

Compute the following products in the Hall algebra of the quiver $1 \to 2$:

$$[S_1]^2 \cdot [S_2]$$
$$[S_2] \cdot [S_1]^2$$
$$[S_1] \cdot [S_2] \cdot [S_1]$$

Next time...

- As much of the proof of Ringel's theorem as I can manage!