Representations of quivers and Lie algebras

Day 2: Proving Gabriel's Theorem (part 1)

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1. Last time. . .
2. Finite type ⇒ Dynkin
3. Reflection functors

Last time...

- A **quiver** Q is a directed graph, with vertices Q_0 and edges Q_1.
- A **representation** V of a quiver Q consists of:
  - for every vertex $x \in Q_0$, a vector space $V(x)$;
  - for every edge $\alpha \in Q_1$, a linear map $V(\alpha)$ between the spaces at its endpoints.
- A representation which is not isomorphic to a nontrivial direct sum is **indecomposable**.

Question

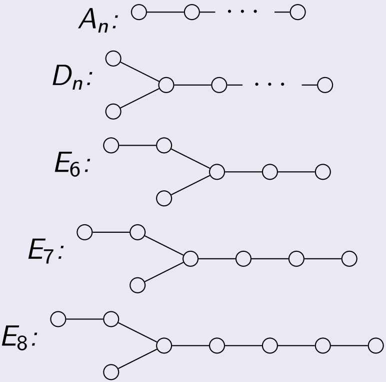
What are the indecomposable representations of a quiver up to isomorphism?

- Say a quiver is **finite type** if it has finitely many indecomposable representations.

Last time...

Theorem (Gabriel)

A quiver is finite type if and only if its underlying undirected graph is one of these:



In this case, the indecomposable representations correspond bijectively to positive roots.

Last time...

Definition

A **(finite, crystallographic) root system** in V is a finite collection of nonzero vectors Φ (called **roots**) such that:

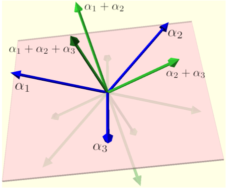
(1) For each $\alpha \in \Phi$, Φ contains $-\alpha$, but no other multiple of α.

(2) For $\alpha, \beta \in \Phi$, $s_\alpha(\beta) \in \Phi$.

(3) For $\alpha, \beta \in \Phi$, $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$.

- We divide a root system into **positive** and **negative** roots.
- The **simple roots** are a special basis of positive roots such that every other positive root is a nonnegative linear combination.
- We assemble a **Cartan matrix** and **Dynkin diagram** using the inner products of the simple roots with each other.

Last time. . .



$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\circ \to \circ \to \circ$$

indecomposables	positive roots
$k \xrightarrow{0} 0 \xrightarrow{0} 0$	α_1
$0 \xrightarrow{0} k \xrightarrow{0} 0$	α_2
$0 \xrightarrow{0} 0 \xrightarrow{0} k$	α_3
$k \xrightarrow{1} k \xrightarrow{0} 0$	$\alpha_1+\alpha_2$
$0 \xrightarrow{0} k \xrightarrow{1} k$	$\alpha_2+\alpha_3$
$k \xrightarrow{1} k \xrightarrow{1} k$	$\alpha_1+\alpha_2+\alpha_3$

Quiver representations as points of a space

- Representations of a quiver are easy to parametrize.
- Let Q be a quiver and $\alpha = (\alpha(x))_{x \in Q_0}$ a dimension vector.
- Let

$$\mathrm{Rep}(Q, \alpha) := \{\text{representations } V \text{ of } Q \mid V(x) = k^{\alpha(x)}\}$$

- Then we identify

$$\begin{aligned} \mathrm{Rep}(Q, \alpha) &\cong \bigoplus_{e \in Q_1} \mathrm{Hom}(k^{\alpha(\mathrm{tail}(e))}, k^{\alpha(\mathrm{head}(e))}) \\ &\cong \prod_{e \in Q_1} \mathbb{A}^{\alpha(\mathrm{tail}(e)) \times \alpha(\mathrm{head}(e))} \end{aligned}$$

- For example:

$$\mathrm{Rep}(1 \to 2 \to 3, (3, 2, 4)) : k^3 \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} x_7 & x_8 \\ x_9 & x_{10} \\ x_{11} & x_{12} \\ x_{13} & x_{14} \end{pmatrix}} k^4$$

Quiver representations as points of a space

- When are two representations isomorphic in this context?

$$\begin{array}{ccc} k^{\alpha(x)} & \xrightarrow{V(e)} & k^{\alpha(y)} \\ \downarrow g_x & & \downarrow g_y \\ k^{\alpha(x)} & \xrightarrow{g_y V(e) g_x^{-1}} & k^{\alpha(y)} \end{array}$$

- Define

$$\mathrm{GL}_\alpha = \prod_{x \in Q_0} \mathrm{GL}_\alpha(k);$$

then this acts on $\mathrm{Rep}(Q, \alpha)$ by

$$(g_x)_{x \in Q_0} \cdot (V(e))_{e \in Q_1} = (g_{\mathrm{tail}(e)} V(e) g_{\mathrm{head}(e)}^{-1})_{e \in Q_1}$$

- Orbits of $\mathrm{GL}_\alpha \circlearrowright \mathrm{Rep}(Q, \alpha)$ are precisely isomorphism classes.

Dimension counting

- **Assume Q is finite type.**
- Because there are finitely many indecomposable representations of Q, for any dimension vector α, there are finitely many representations of dimension α, up to isomorphism.
- Thus $\mathrm{Rep}(Q, \alpha)$ has finitely many GL_α-orbits for each α. In particular, GL_α must be "big enough" to cover $\mathrm{Rep}(Q, \alpha)$ this way — which is not always possible!
- We must at least have $\dim \mathrm{GL}_\alpha \geq \dim \mathrm{Rep}(Q, \alpha)$.
- Moreover, GL_α has a nontrivial subgroup acting trivially on $\mathrm{Rep}(Q, \alpha)$: $\boxed{\{(\lambda \cdot \mathrm{Id})_{x \in Q_0} \mid \lambda \in k^*\}}$
- Thus

$$\dim GL_\alpha - 1 \geq \dim \mathrm{Rep}(Q, \alpha).$$

Dimension counting

- What is $\dim \mathrm{GL}_\alpha$? $\boxed{\sum_{x\in Q_0} \alpha(x)^2}$
- What is $\dim \mathrm{Rep}(Q,\alpha)$? $\boxed{\sum_{e\in Q_1} \alpha(\mathrm{tail}(e))\alpha(\mathrm{head}(e))}$
- Then, for there to be finitely many orbits, we must have

$$\dim \mathrm{GL}_\alpha - 1 \geq \dim \mathrm{Rep}(Q,\alpha)$$
$$\sum_{x\in Q_0} \alpha(x)^2 - \sum_{e\in Q_1} \alpha(\mathrm{tail}(e))\alpha(\mathrm{head}(e)) \geq 1$$

- Define

$$B_Q(\alpha,\beta) = \sum_{x\in Q_0} \alpha(x)\beta(x) - \sum_{e\in Q_1} \alpha(\mathrm{tail}(e))\beta(\mathrm{head}(e))$$

and

$$\langle \alpha,\beta\rangle_Q = B_Q(\alpha,\beta) + B_Q(\beta,\alpha)$$

Dimension counting

- Consolidating the above,

$$\langle \alpha, \beta \rangle_Q = 2 \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{x \to y} (\alpha(x)\beta(y) + \beta(x)\alpha(y))$$

Proposition

$$\langle \alpha, \beta \rangle_Q = \alpha^T C \beta$$

where C is the Cartan matrix associated to the undirected graph underlying Q.

Corollary

If $\langle \alpha, \alpha \rangle_Q = 2B_Q(\alpha, \alpha) > 0$ for all α, the matrix C is positive definite.

- This forces Q to be a Dynkin diagram!

Reflection functors

- The next step: any Dynkin quiver has indecomposable representations corresponding to positive roots.
- We'll want some notion of reflection for representations.
- What should this do on the level of dimension vectors?

Proposition

Let Φ be a root system with Dynkin diagram G and simple roots $\alpha_1, \ldots, \alpha_n$. Then applying s_i to $\sum_j c_j \alpha_j$ replaces the coefficient c_i with

$$\left(\sum_{j - i} c_j \right) - c_i$$

and leaves the other coefficients unchanged.

Reflection functors

- Consider a quiver Q and representation V.
- Let x be a **sink** of the quiver Q: no arrows point out.
- Let $s_x(Q)$ be the quiver Q with all arrows into x reversed.
- Consider the map

$$\varphi_{\partial x} : \bigoplus_{y \to x} V(y) \xrightarrow{\sum_{y \to x} V(y \to x)} V(x)$$

- Then we define a representation $\Phi_x^+(V)$ of $s_x(Q)$ on vertices by

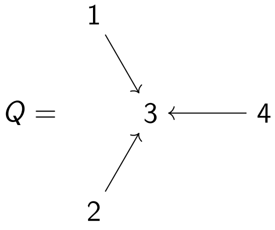
$$\Phi_x^+(V)(y) = \begin{cases} \ker(\varphi_{\partial x}) & y = x \\ V(y) & \text{otherwise} \end{cases}$$

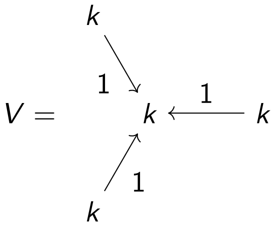
and on edges by

$$\Phi_x^+(V)(y \to z) = \begin{cases} \pi_z|_{\ker(\varphi_{\partial x})} & y = x \\ V(y \to z) & \text{otherwise} \end{cases}$$

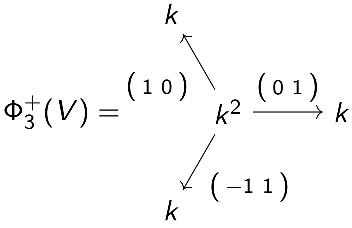
where $\pi_z : \bigoplus_{y \to x} V(y) \to V(z)$ is projection.

Reflection functors: example





$\varphi_{\partial x} : k^3 \xrightarrow{(1\ 1\ 1)} k \qquad \ker(\varphi_{\partial x}) = \operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) \cong k^2$



Reflection functors

- Consider a quiver Q and representation V.
- Let x be a **source** of the quiver Q: no arrows point in.
- Consider the map

$$\varphi_{\partial x} : V(x) \xrightarrow{(V(x \to y))_{x \to y}} \bigoplus_{y \to x} V(y)$$

- Let $s_x(Q)$ be the quiver Q with all arrows out of x reversed.
- Then we define a representation $\Phi_x^-(V)$ of $s_x(Q)$ on vertices by

$$\Phi_x^-(V)(y) = \begin{cases} \operatorname{coker}(\varphi_{\partial x}) & y = x \\ V(y) & \text{otherwise} \end{cases}$$

and on edges by

$$\Phi_x^-(V)(z \to y) = \begin{cases} \iota_z & y = x \\ V(z \to y) & \text{otherwise} \end{cases}$$

where $\iota_z : V(z) \to \bigoplus_{x \to y} V(y) \to \operatorname{coker}(\varphi_{\partial x})$ is inclusion followed by projection.

Properties of reflection functors

- We continue considering a quiver Q with sink x.
- The construction $\ker\left(\bigoplus_{y\to x} V(y) \to V(x)\right)$ looks like what we want: adding together the data of all the neighbors of x and then taking x away.
- But for this to reflect the dimension vector like we want, we need the map to be surjective.
- Let S_x be the representation which is k at x and 0 everywhere else.

Proposition

The map $\bigoplus_{y\to x} V(y) \to V(x)$ fails to be surjective if and only if V has S_x as a direct summand.

Properties of reflection functors

Proposition

The map $\bigoplus_{y \to x} V(y) \to V(x)$ fails to be surjective if and only if V has S_x as a direct summand.

Proof (sketch).

$\Leftarrow$: If V has S_x as a direct summand, any nonzero vector in $S_x(x)$ will not be hit by the maps into x (in S_x, those maps are all 0). $\Rightarrow$: If there is some $v \in V(x)$ which isn't hit by any map into x, we can break $\text{span}(v)$ off as a summand of $V(x)$ which doesn't interact with any other part of V. □

- Let $\text{Rep}_x(Q)$ be the collection of representations which don't have S_x as a summand.

Properties of reflection functors: dimension vectors

- If the map $\varphi_{\partial x} : \bigoplus_{y \to x} V(y) \to V(x)$ is surjective, we have

$$\dim(\ker(\varphi_{\partial x})) = \left(\sum_{y \,—\, x} \dim(V(y)) \right) - \dim(V(x))$$

Lemma

For $V \in \mathrm{Rep}_x(Q)$, $\dim(\Phi_x^+(V)) = s_x(\dim(V))$, where $\dim(V)$ is viewed as a combination of simple roots and s_x is the reflection by the simple root at x.

- Success! (Kind of.)

Properties of reflection functors: back and forth

- If the map $\varphi_{\partial x} : \bigoplus_{y \to x} V(y) \to V(x)$ is surjective, then we can recover $V(x)$ as the cokernel of the map

$$\ker(\varphi_{\partial x}) \to \bigoplus_{y \to x} V(y)$$

- Chasing some more arrows gives the more precise:

Lemma

The functors $\Phi_x^+ : \mathrm{Rep}_x(Q) \to \mathrm{Rep}_x(s_x(Q))$ *and* $\Phi_x^- : \mathrm{Rep}_x(s_x(Q)) \to \mathrm{Rep}_x(Q)$ *are inverse equivalences of categories.*

- On the other hand, what's $\Phi_x^+(S_x)$? $\boxed{0}$.
- So the $\Phi_x^\pm$ show that the representation theories of Q and $s_x(Q)$ are *almost* the same.

Properties of reflection functors: direct sum

- Each step we took in defining the reflection functor preserves the direct sum operation, thus:

Lemma

$$\Phi_x^+(V \oplus W) \cong \Phi_x^+(V) \oplus \Phi_x^+(W)$$

Corollary

If V is an indecomposable representation other than S_x, $\Phi_x^+(V)$ is an indecomposable representation of $s_x(Q)$.

- There's an important parallel in the theory of root systems:

Proposition

If α is a positive root other than α_x, $s_x(\alpha)$ is a positive root.

- However, we have $s_x(\alpha_x) = -\alpha_x$, $\Phi_x^+(S_x) = 0$.

Exercises

Exercise

Write down an example of a quiver representation and perform the appropriate reflection functor at a sink or source.

Exercise

Check the proofs of reflection functors stated here to your satisfaction.

Next time. . .

- A whirlpool of reflection functors!
- Lie algebras appear at last!