Representation Theory of Finite-Dimensional Algebras

Day 5: Almost Split Sequences and the Brauer-Thrall Conjecture

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Last time...

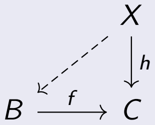
- We defined the **Auslander-Reiten transform** $D\,\mathrm{Tr}$, which turns nonprojective indecomposable modules into noninjective indecomposable ones.
- We used this to construct an infinite sequence of indecomposable modules for $k[x, y]/(x, y)^2$
- We introduced the concept of **almost split sequences**.

Almost split morphisms

Definition

A morphism $f : B \to C$ is **right almost split** if:

- It is not a split surjection.
- If $h : X \to C$ is not a split surjection, it factors through f:

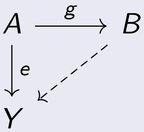


Almost split morphisms

Definition

A morphism $g : A \to B$ is **left almost split** if:

- It is not a split injection.
- If $e : A \to Y$ is not a split injection, it factors through g:



Almost split sequences

Definition

An exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an **almost split sequence** if f is left almost split and g is right almost split.

The key theorem

Theorem

(1) *Let C be an indecomposable, non-projective module. Then there exists an almost split sequence*

$$0 \to D\,\mathrm{Tr}\,C \to B \to C \to 0$$

and any almost split sequence ending at C is isomorphic to this one.

(2) *Let A be an indecomposable, non-injective module. Then there exists an almost split sequence*

$$0 \to A \to B \to \mathrm{Tr}\,DA \to 0$$

and any almost split sequence starting from A is isomorphic to this one.

Another reference for today

Idun Reiten, *The use of almost split sequences in the representation theory of artin algebras*.

Almost split sequences: other perspectives

Definition

A morphism is **(right/left) minimal almost split** if it is both (right/left) minimal and (right/left) almost split.

Proposition

For an exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

the following are equivalent:

(1) *The sequence is almost split.*

(2) *g is minimal right almost split.*

(3) *g is right almost split and $A = D\,\mathrm{Tr}\,C$*

(4) *The dual versions of these things.*

Minimal almost split morphisms to/from (pro/in)jectives

Even though projective modules can't be the right ends of almost split sequences, they still admit minimal right almost split morphisms:

Proposition

Let P be an indecomposable projective module. Then the inclusion $\mathfrak{r}P \hookrightarrow P$ is minimal right almost split.

Proof.

Any map $X \to P$ which isn't a split surjection isn't a surjection, so it factors through the unique maximal submodule $\mathfrak{r}P \subset P$. □

Even though injective modules can't be the left ends of almost split sequences, they still admit minimal left almost split morphisms:

Proposition

Let I be an indecomposable injective module. Then the projection $I \twoheadrightarrow I/\operatorname{soc} I$ is minimal left almost split.

Irreducible morphisms

Definition

A morphism $f : A \to B$ is **irreducible** if:

- It is not a split injection or a split surjection.
- For any maps $g : A \to X$, $h : X \to B$ such that $hg = f$, either g is a split injection or h is a split surjection.

Proposition

For a nonzero morphism $f : A \to B$ with B indecomposable, the following are equivalent:

- *f is irreducible.*
- *There exists a map $f' : A' \to B$ such that $(f, f') : A \oplus A' \to B$ is minimal right almost split.*
- *There exists a map $f' : A \to B'$ such that $(f, f') : A \to B \oplus B'$ is minimal left almost split.*

Almost split sequences and finiteness

- The mere existence of almost split sequences has an interesting consequence:

Proposition

For a fixed indecomposable module B, there are only finitely many indecomposable modules admitting nonzero irreducible morphisms to or from B.

- Importantly, this is true even if there are infinitely many indecomposables.

The Brauer-Thrall conjecture

Definition

An algebra Λ is **finite type** if it has finitely many indecomposable representations.

Theorem (Brauer-Thrall "Conjecture")

An algebra Λ is finite type if and only if there is a bound on the length of the indecomposables.

Proof of the Brauer-Thrall conjecture

- In what follows, assume that there is a bound on the lengths of indecomposable Λ-modules.

Lemma

If $f_i : A_i \to A_{i+1}$ are nonisomorphisms between indecomposable modules A_i for $1 \leq i \leq 2^n - 1$, and $\ell(A_i) \leq n$ for all i, then $f_{2^n-1} \cdots f_1 = 0$.

- In particular, there is a bound on the length of chains

$$A_0 \to A_1 \to \cdots \to A_n$$

such that the modules are indecomposable, the morphisms are not isomorphisms, and the composition is nonzero.

Proof of the Brauer-Thrall conjecture

Lemma

Let $f : X \to C$ be a nonzero nonisomorphism between indecomposable modules. Then there is a finite chain of irreducible morphisms between indecomposable modules $X \to Y_1 \to \cdots \to Y_n \to C$ with nonzero composition. (I'll call this an "irreducible chain".)

Proof.

Take a minimal right almost split morphism $g : B \to C$. Because $f : X \to C$ is not an isomorphism, we can find $h : X \to B$ such that $f = gh$. Now break B down into indecomposables $B_1 \oplus \cdots \oplus B_m$. There must be some i such that the composition of the restricted maps $X \to B_i \to C$ is nonzero. Note that $B_i \to C$ is irreducible. If $X \to B_i$ is an isomorphism, we're done. Otherwise, repeat this process with the map $X \to B_i$. By the previous slide's observation, it must eventually stop. □

Proof of the Brauer-Thrall conjecture

Corollary

For any nonsimple indecomposable module X, there is an irreducible chain starting at X and ending at a simple module.

Proof.

Let S be a summand of $X/\mathfrak{r}X$, which is semisimple. Then we have a nonzero nonisomorphism $X \to X/\mathfrak{r}X \to S$. By the previous result, there is also an irreducible chain from X to S. □

Proposition

For a fixed indecomposable module C, there are only finitely many indecomposable modules occurring in irreducible chains ending at C.

Proof.

There are only finitely many indecomposables admitting irreducible maps to C. Repeat, and use the boundedness of irreducible chains. □

Proof of the Brauer-Thrall conjecture

- There are finitely many simples, and every indecomposable module falls into one of finitely many chains ending at those simples, which have bounded length. So Λ is finite type!
- Brauer-Thrall in action: recall the parametrized family of indecomposable quiver representations

$$\mathbb{C} \underset{t}{\overset{1}{\rightrightarrows}} \mathbb{C}$$

- These all have length 2, which tells us there must be more indecomposables lurking out there. . . .

More on irreducible morphisms

- We previously showed that, if Λ is finite type and $X \to C$ is a nonisomorphism between indecomposables, we can construct an irreducible chain

$$X \to Y_1 \to \cdots Y_n \to C$$

- We constructed this chain by lifting our map through right almost split morphisms and pulling off irreducible morphisms from those.
- If we keep track of all the chains we build this way and add them together, we get the original morphism back!

Proposition

For a finite type algebra Λ, any nonzero nonisomorphism between indecomposable modules is a sum of compositions of irreducible maps between indecomposables.

- This gives us a sense in which irreducible morphisms are building blocks.

The Auslander-Reiten quiver

Definition

The **Auslander-Reiten quiver** of an algebra Λ has

- vertices given by isomorphism classes of indecomposable modules
- arrows given by nonzero irreducible maps

Also, if our field isn't algebraically closed, we need to put some labels on the arrows, but let's not worry about that.

- The connection between almost split sequences and irreducible maps gives us some nice constraints on what the Auslander-Reiten quiver can look like.

A simple example

- Consider the quiver $1 \to 2 \to 3$. We denote its indecomposables by their dimension vectors: for example, 110 denotes $k \to k \to 0$.
- Through various tricks, we can find 3 almost split sequences:

$$0 \longrightarrow 001 \longrightarrow 011 \longrightarrow 010 \longrightarrow 0$$

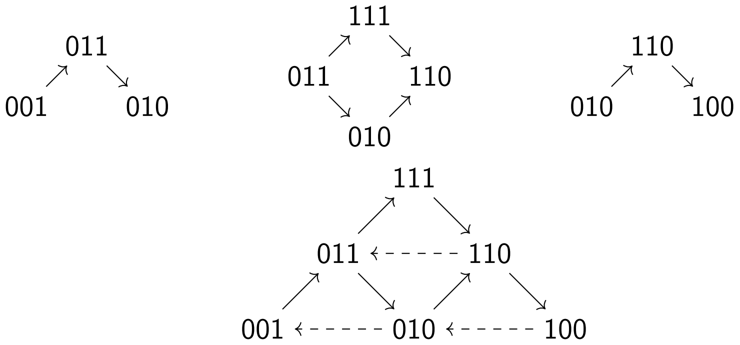
$$0 \longrightarrow 011 \longrightarrow \begin{array}{c} 111 \\ \oplus \\ 010 \end{array} \longrightarrow 110 \longrightarrow 0$$

$$0 \longrightarrow 010 \longrightarrow 110 \longrightarrow 100 \longrightarrow 0$$

- Each one of these sequences gives a piece of the Auslander-Reiten quiver:

$$\begin{array}{ccc} & 011 & \\ \nearrow & & \searrow \\ 001 & & 010 \end{array} \qquad \begin{array}{ccccc} & & 111 & & \\ & \nearrow & & \searrow & \\ 011 & & & & 110 \\ & \searrow & & \nearrow & \\ & & 010 & & \end{array} \qquad \begin{array}{ccc} & 110 & \\ \nearrow & & \searrow \\ 010 & & 100 \end{array}$$

A simple example



- Here we also include a dashed arrow indicating the Auslander-Reiten transform.
- Note that the projectives are those which don't start an arrow, and the injectives are those which don't end one.

What does the Auslander-Reiten quiver tell us?

- Note that for C indecomposable nonprojective, any irreducible $\alpha : B \to C$ is matched by an irreducible $\sigma(\alpha) : D\,\mathrm{Tr}\, C \to B$.
- Furthermore, since this matching comes from exact sequences, we have for any fixed C the **mesh relation**

$$\sum_{\alpha : B \to C} \alpha \sigma(\alpha) = 0$$

Definition

The **mesh category** has

- As objects, vertices in the Auslander-Reiten quiver.
- For two objects A, B, $\mathrm{Hom}(A, B)$ is the space of formal linear combinations of paths from A to B in the Auslander-Reiten quiver, modulo the mesh relation.

What does the Auslander-Reiten quiver tell us?

Theorem (Bautista-Gabriel-Roiter-Salmerón 1985)

Let k be an algebraically closed field, char $k \neq 2$. *Let Λ be a finite-dimensional k-algebra of finite type. Then the full subcategory of indecomposable modules is equivalent to the mesh category.*

Thank you!