Representation Theory of Finite-Dimensional Algebras

Day 4: Indecomposables and Almost Split Sequences

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1 Recap

2 The Auslander-Reiten transform

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Last time...

- We introduced two notions of duality between left and right modules:
  - $A^* := \mathrm{Hom}_\Lambda(A, \Lambda)$, which is only a duality for projectives
  - $DA := \mathrm{Hom}_k(A, k)$, which works on anything
- We gave a bijection between projective modules and injective ones:

$$P \mapsto D(P^*).$$

- To use $(-)^*$ with modules which aren't projective, we defined the transpose.

Last time...

Definition

An exact sequence

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

is a **minimal projective presentation** if P_0 is a projective cover of A, and P_1 is a projective cover of $\ker(f_0)$.

Definition

Let

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

be a minimal projective presentation of the left module A. Then the tranpose $\mathrm{Tr}(A)$ is the right module which makes the following sequence exact:

$$P_0^* \xrightarrow{f_1^*} P_1^* \to \mathrm{Tr}(A) \to 0$$

That is, $\mathrm{Tr}(A) = \mathrm{coker}(f_1^*)$.

Key properties of the transpose

Proposition

If P is projective, then $\mathrm{Tr}(P) = 0$.

Proposition

Suppose A is indecomposable and not projective, and that

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

is a minimal projective presentation of A. Then

$$P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{\pi} \mathrm{Tr}(A) \to 0$$

is a minimal projective presentation of $\mathrm{Tr}(A)$.

Some nice consequences regarding the transpose

Proposition

(1) $\mathrm{Tr}(A \oplus B) \cong \mathrm{Tr}(A) \oplus \mathrm{Tr}(B)$

Suppose A is indecomposable and not projective. Then

(2) $\mathrm{Tr}(\mathrm{Tr}(A)) \cong A$.

(3) $\mathrm{Tr}(A)$ *is indecomposable.*

Proof.

(1) Projective covers and $(-)^*$ commute with direct sums.

(2) We can reuse the minimal projective presentation

$$P_0^* \to P_1^* \to \mathrm{Tr}(A) \to 0$$

to compute $\mathrm{Tr}(\mathrm{Tr}(A))$, in the process just getting the original presentation of A back.

□

Some nice consequences regarding the transpose

Proposition

(1) $\mathrm{Tr}(A \oplus B) \cong \mathrm{Tr}(A) \oplus \mathrm{Tr}(B)$

Suppose A is indecomposable and not projective. Then

(2) $\mathrm{Tr}(\mathrm{Tr}(A)) \cong A$.

(3) $\mathrm{Tr}(A)$ *is indecomposable.*

Proof.

(3) Suppose instead $\mathrm{Tr}(A)$ is decomposable. We can assume $\mathrm{Tr}(A)$ has a nonprojective indecomposable summand B_1: if not, it would be projective, and $\mathrm{Tr}(\mathrm{Tr}(A)) = 0$, contradicting (2).
Then write $\mathrm{Tr}(A) \cong B_1 \oplus B_2$. We have
$A \cong \mathrm{Tr}(\mathrm{Tr}(A)) \cong \mathrm{Tr}(B_1) \oplus \mathrm{Tr}(B_2)$.
Since A is indecomposable, $\mathrm{Tr}(B_2) = 0$. But then
$\mathrm{Tr}(\mathrm{Tr}(B_1)) = \mathrm{Tr}(A) = B_1 \oplus B_2$, contradicting (2).

□

The Auslander-Reiten transform!

Proposition

The transpose gives a bijection between indecomposable nonprojective left Λ-modules and indecomposable nonprojective right Λ-modules.

Now we use the duality D to move things back into the realm of left modules!

Definition

The **Auslander-Reiten transform** is the operation $D\,\mathrm{Tr}$.

Proposition

The Auslander-Reiten transform $D\,\mathrm{Tr}$ *gives a bijection between indecomposable nonprojective left modules and indecomposable noninjective left modules.*

Back to quivers

Earlier, we looked at the quiver

$$1 \to 2 \begin{array}{l} \nearrow 3 \\ \searrow 4 \end{array}$$

and found the transpose of the simple S_2:

$$\mathrm{Tr} \quad 0 \to k \begin{array}{l} \nearrow 0 \\ \searrow 0 \end{array} \quad = \quad k \leftarrow k \begin{array}{l} \swarrow k \\ \nwarrow k \end{array}$$

Now when we apply the duality D, this flips all the arrows back:

$$D\,\mathrm{Tr} \quad 0 \to k \begin{array}{l} \nearrow 0 \\ \searrow 0 \end{array} \quad = \quad k \to k \begin{array}{l} \nearrow k \\ \searrow k \end{array}$$

In short: $D\,\mathrm{Tr}(S_2) = P_1$.

Working with indecomposables

- The complexity of our module category boils down to how many indecomposables there are, and how long they are.
- This can vary wildly:
  - $\circ \to \circ$ has three:
    $$\mathbb{C} \to 0, \quad 0 \to \mathbb{C}, \quad \mathbb{C} \to \mathbb{C}$$
  - $\circ \rightrightarrows \circ$ has a parametrized family (and more besides!):
    $$\mathbb{C} \overset{1}{\underset{t}{\rightrightarrows}} \mathbb{C}$$
- A powerful aspect of the Auslander-Reiten transform is that it generates new indecomposables from old ones.

Case study: $k[x, y]/(x, y)^2$

- This is a very small algebra, only 3-dimensional. Nonetheless:

Theorem

$\Lambda := k[x, y]/(x, y)^2$ *has infinitely many nonisomorphic indecomposable modules.*

- Proving this will bring together all the tools we've developed so far.
- The key idea: starting with a simple module, iterate the inverse Auslander-Reiten transform Tr D.

Step 1: first observations

Let $\Lambda := k[x, y]/(x, y)^2$

- This algebra has several things going for it:
- It's commutative, so Λ-mod $\cong$ mod-Λ in a nice way.
- It's a local ring, with maximal ideal (x, y). Thus $\mathfrak{r} = (x, y)$.
- Then $S := \Lambda/\mathfrak{r}$ is the unique simple module, and the only indecomposable projective module is Λ itself.
- $\mathfrak{r}^2 = 0$. So for any module C, $\mathfrak{r}C$ is annihilated by $\mathfrak{r}$, and $\mathfrak{r}C$ is semisimple. In particular, $\mathfrak{r} \cong S^2$.

In fact:

Lemma

For C indecomposable but not simple, $\mathfrak{r}C = \operatorname{soc} C$.

Step 2: radicals and socles

Lemma

For C indecomposable but not simple, $\mathfrak{r}C = \operatorname{soc} C$.

Proof.

Suppose not. Then we can write $\operatorname{soc} C \cong \mathfrak{r}C \oplus K$ for some other semisimple K. Then the composition

$$K \hookrightarrow C \to C/\mathfrak{r}C$$

is injective.
Since $C/\mathfrak{r}C$ is semisimple, this composition is a split injection, so we can get a map $C/\mathfrak{r}C \to K$ making this composition the identity:

$$K \hookrightarrow C \to C/\mathfrak{r}C \to K$$

But this shows $K \hookrightarrow C$ is also a split injection. Since C is indecomposable, $K \cong C$. This implies C is simple. □

Step 3: minimal projective presentations

Now let C be any indecomposable.

- Suppose $\ell(\mathfrak{r}C) = s$ and $\ell(C/\mathfrak{r}C) = t$.
- Since $\mathfrak{r}C$ and $C/\mathfrak{r}C$ are both semisimple, and we only have one simple module,

$$\mathfrak{r}C \cong S^s, \quad C/\mathfrak{r}C \cong S^t$$

- If $P \to C$ is a projective cover, $P/\mathfrak{r}P \to C/\mathfrak{r}C \cong S^t$ is an isomorphism. Then $P \cong \Lambda^t$.

Step 3: minimal projective presentations

$\ell(\mathfrak{r}C) = s$, $\ell(C/\mathfrak{r}C) = t$

- To compute the kernel of the cover $\Lambda^t \to C$, we use the snake lemma:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \Lambda^t & = & \Lambda^t & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{r}C \cong S^s & \longrightarrow & C & \longrightarrow & C/\mathfrak{r}C \cong S^t & \longrightarrow & 0 \end{array}$$

$$0 \longrightarrow \ker(\Lambda^t \to C) \longrightarrow \ker(\Lambda^t \to S^t) \longrightarrow S^s \longrightarrow 0$$

- We have $\ker(\Lambda^t \to S^t) \cong \mathfrak{r}^t \cong S^{2t}$, which forces $\ker(\Lambda^t \to C) \cong S^{2t-s}$. This, in turn, has projective cover Λ^{2t-s}.
- Thus the minimal projective presentation looks like

$$\Lambda^{2t-s} \to \Lambda^t \to C \to 0$$

Step 3b: minimal projective presentations

$\ell(\mathfrak{r}C) = s$, $\ell(C/\mathfrak{r}C) = t$

From duality, as long as $C \not\cong S$, we know that

$$\mathfrak{r}DC = D(C/\operatorname{soc} C) = D(C/\mathfrak{r}C)$$
$$DC/\mathfrak{r}DC = D(\operatorname{soc} C) = D(\mathfrak{r}C)$$

so $\ell(\mathfrak{r}DC) = t, \ell(DC/\mathfrak{r}DC) = s$, and DC has the minimal projective presentation

$$\Lambda^{2s-t} \to \Lambda^{s} \to DC \to 0$$

In turn, we have a minimal projective presentation

$$\Lambda^{s} \to \Lambda^{2s-t} \to \operatorname{Tr} DC \to 0$$

Step 4: Iterating Tr D

First, DS is also simple, with projective presentation

$$\Lambda^2 \to S^2 \cong \mathfrak{r} \hookrightarrow \Lambda \to DS \to 0$$

For C indecomposable, not simple:

$$\Lambda^{2t-s} \to \Lambda^t \to C \to 0$$
$$\Lambda^{2s-t} \to \Lambda^s \to DC \to 0$$

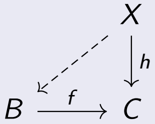
$$\Lambda^2 \to \Lambda \to DS \to 0$$
$$\Lambda \to \Lambda^2 \to \operatorname{Tr} DS \to 0 \qquad s = 3, t = 2$$
$$\Lambda^4 \to \Lambda^3 \to D \operatorname{Tr} DS \to 0$$
$$\Lambda^3 \to \Lambda^4 \to (\operatorname{Tr} D)^2 S \to 0 \qquad s = 5, t = 4$$
$$\Lambda^6 \to \Lambda^5 \to D(\operatorname{Tr} D)^2 S \to 0$$
$$\Lambda^5 \to \Lambda^6 \to (\operatorname{Tr} D)^3 S \to 0 \qquad s = 7, t = 6$$

Almost split morphisms

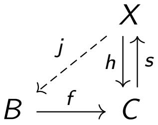
Definition

A morphism $f : B \to C$ is **right almost split** if:

- It is not a split surjection.
- If $h : X \to C$ is not a split surjection, it factors through f:



Note that if h *is* a split surjection and factors through f, f is also a split surjection:



$$f(js) = hs = \mathrm{id}_C$$

Almost split morphisms

Definition

A morphism $g : A \to B$ is **left almost split** if:

- It is not a split injection.
- If $e : A \to Y$ is not a split injection, it factors through g:

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow{\scriptstyle e} & \swarrow & \\ Y & & \end{array}$$

Definition

An exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an **almost split sequence** if f is left almost split and g is right almost split.

An example of an almost split sequence

Let $\Lambda = k[x]/(x^n)$.

- The indecomposable modules are $k[x]/(x^i)$ for $1 \leq i \leq n$.
- For $i \neq n$, consider the exact sequence

$$0 \to k[x]/(x^i) \xrightarrow{g} k[x]/(x^{i-1}) \oplus k[x]/(x^{i+1}) \xrightarrow{f} k[x]/(x^i) \to 0$$

  where $g(p) = (\overline{p}, xp)$, $f(p, q) := xp - \overline{q}$.

- To see why this is almost split, consider the case of an indecomposable $k[x]/(x^j)$ mapping to $k[x]/(x^i)$. Either:
  - $k[x]/(x^j) \to k[x]/(x^i)$ is not surjective, and factors

$$k[x]/(x^j) \to k[x]/(x^{i-1}) \to k[x]/(x^i)$$

  - $k[x]/(x^j) \to k[x]/(x^i)$ is surjective but not injective, and factors

$$k[x]/(x^j) \to k[x]/(x^{i+1}) \to k[x]/(x^i)$$

Which modules can appear in almost split sequences?

Suppose $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is an almost split sequence.

- Note that C cannot be projective: every surjection to a projective module splits.
- Dually, A cannot be injective.

Proposition

A and C are indecomposable.

Proof.

Suppose C breaks into indecomposables $C_1 \oplus \cdots \oplus C_n$. Each inclusion of a summand $C_i \hookrightarrow C$ factors through $f : B \to C$; but summing these maps together gives a factorization of $\mathrm{id}_C : C \to C$ through $f : B \to CB$. But this means the sequence splits. The case of A is dual. □

The key theorem

Theorem

(1) *Let C be an indecomposable, non-projective module. Then there exists an almost split sequence*

$$0 \to D\,\mathrm{Tr}\,C \to B \to C \to 0$$

and any almost split sequence ending at C is isomorphic to this one.

(2) *Let A be an indecomposable, non-injective module. Then there exists an almost split sequence*

$$0 \to A \to B \to \mathrm{Tr}\,DA \to 0$$

and any almost split sequence starting from A is isomorphic to this one.

Next time...

You decide!