Representation Theory of Finite-Dimensional Algebras

Day 3: Indecomposables and the Auslander-Reiten Transform

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Last time...

- We looked at the **radical of a module** A, and saw that it's just $\mathfrak{r}A$.
- We defined right minimal morphisms, in particular **projective covers**: surjections from a projective which are as small as possible.
- We showed that, through projective covers, indecomposable projective modules correspond to simple ones.
- In particular, the indecomposable projective modules of a path algebra kQ are the ideals kQe_x—"paths starting at x".

Duality

Definition

A duality between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of contravariant functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ such that FG and GF are naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{C}}$ respectively.

- The contravariant version of equivalence.
- Turns every categorical construction in $\mathcal{C}$ into its "co-" version in $\mathcal{D}$.

Duality #1

- We denote the category of left Λ-modules by Λ-mod, and the category of right Λ-modules by mod-Λ.
  - We can also identify mod-Λ with Λ^{op}-mod, where Λ^{op} is the opposite ring
- Define a contravariant functor $(-)^* : \Lambda\text{-mod} \to \text{mod-}\Lambda$ by

$$A^* := \mathrm{Hom}_\Lambda(A, \Lambda)$$

  with action

$$(a^*\lambda)(-) = a^*(-)\lambda$$

- On a morphism $f : A \to B$:

$$f^*(b^*)(-) = b^*(f(-))$$

Duality #1

Proposition

Let $\mathcal{P}(\Lambda\text{-mod})$ be the full subcategory of projective Λ-modules. Then $(-)^*$ gives a categorical duality

$$\mathcal{P}(\Lambda\text{-}mod) \to \mathcal{P}(mod\text{-}\Lambda)$$

Proof (sketch).

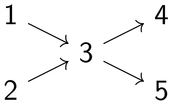
The map

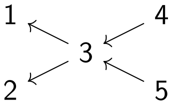
$$A \to A^{**} := \mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(A, \Lambda), \Lambda)$$
$$a \mapsto (a^* \mapsto a^*(a))$$

isn't always an isomorphism, but it is for $A = \Lambda$.
Because Hom commutes with direct sums, this map is also an isomorphism for free modules Λ^n, and also for direct summands of free modules, i.e.: projectives. □

Duality #1 for path algebras

- For a quiver Q, Q^{op}, the **opposite quiver**, is obtained by reversing all arrows of Q.





Proposition

$$\textit{mod-}kQ \cong k(Q^{\text{op}})\textit{-mod}$$

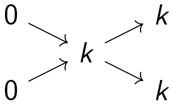
- To view a right kQ-module A as a representation of Q^{op}, put Ae_x on vertex x.
- If $\alpha : x \to y$ is an arrow, let $\alpha^* : y \to x$ be the reverse arrow. The map $A(\alpha^*) : Ae_y \to Ae_x$ is right multiplication by α.

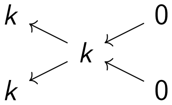
Duality #1 for path algebras

- An element of $(kQe_x)^* := \mathrm{Hom}_{kQ}(kQe_x, kQ)$ is determined by where we send e_x.
- e_x can be sent to any combination of paths ending at x:

$$e_x a^*(e_x) = a^*(e_x e_x) = a^*(e_x)$$

- This identifies $(kQe_x)^*$ with $e_x kQ$, and in turn with $k(Q^{\mathrm{op}})e_x$.





Duality #2

- We still haven't used "finite-dimensional over a field". But we're about to! From here on, assume Λ is a finite-dimensional k-algebra.
- Define a contravariant functor $D : \Lambda\text{-mod} \to \text{mod-}\Lambda$ by

$$DA := \mathrm{Hom}_k(A, k)$$

  with action

$$(f\lambda)(-) = f(\lambda \cdot -)$$

- On a morphism $\varphi : A \to B$:

$$\varphi^*(f)(-) = f(\varphi(-))$$

Duality #2

Proposition

$$D : \Lambda\text{-}mod \to mod\text{-}\Lambda$$

is a duality.

Proof.

This time, the map

$$\begin{aligned} A &\to D(DA) := \mathrm{Hom}_k(\mathrm{Hom}_\Lambda(A, k), k) \\ a &\mapsto (f \mapsto f(a)) \end{aligned}$$

is always an isomorphism. Need only check it is a Λ-morphism. □

Duality #2 for path algebras

- Suppose A is a representation of Q. What does DA look like as a representation of Q^{op}?
- The space at x is given by $(DA)e_x$:

$$(DA)e_x = \mathrm{Hom}_k(A, k)e_x = \{f(e_x \cdot -) \mid f : A \to k\}$$

$f(e_x \cdot -)$ is determined by its value on $e_x A$. Then

$$\mathrm{Hom}_k(A, k)e_x \cong \mathrm{Hom}_k(e_x A, k).$$

- Given an arrow $\alpha : x \to y$ and $f \in \mathrm{Hom}_k(e_y A, k)$, we have

$$(f\alpha)(-) = f(\alpha \cdot -) \in \mathrm{Hom}_k(e_x A, k)$$

- In summary:
  - $DA(x)$ is the dual space of $A(x)$
  - $DA(\alpha^*)$ is the dual map of $A(\alpha)$

Injective modules

Because D is a duality, it sends projectives to injectives and vice versa.

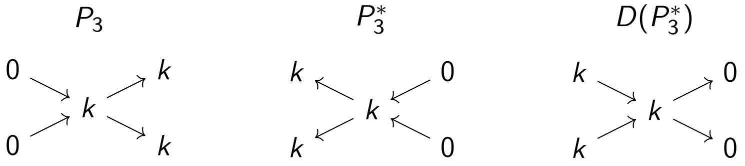
Proposition

The maps

$$P \mapsto D(P^*)$$
$$I \mapsto (DI)^*$$

define a bijection between projective and injective Λ-modules.

Injective modules for path algebras



- Indecomposable projective at x: paths starting at x
- Indecomposable injective at x: paths ending at x

The radical and the socle

- We know the radical is important. How does it interact with duality?

Definition

The **radical** of a module A, $\mathrm{rad}(A)$ is the intersection of all maximal submodules.

Definition

The **socle** of a module A, $\mathrm{soc}(A)$ is the sum of all simple submodules.

- Note any two distinct simple submodules have 0 intersection. Thus $\mathrm{soc}(A)$ is the *direct* sum of all simple submodules.

Proposition

$\mathrm{soc}(A)$ *is the largest semisimple submodule of* A.

Proposition

$\mathrm{soc}(A)$ *consists of all elements annihilated by* $\mathfrak{r}$.

The radical and the socle

Proposition

$$D(A/\mathfrak{r}A) \cong \operatorname{soc}(DA)$$
$$D(\mathfrak{r}A) \cong DA/\operatorname{soc}(DA)$$

Proof.

An exercise in thinking categorically. (You shouldn't need to use the definition of D). □

Minimal projective presentations

- The interplay between our two duality operations, $(-)^*$ and D, gave a nontrivial connection between projective and injective modules.
- Can we do a similar thing with arbitrary modules?
- To open up arbitrary modules to $(-)^*$, use projective presentations.

Definition

A **minimal projective presentation** of a module A is an exact sequence

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

such that:

- P_0 and P_1 are projective.
- P_0 is a projective cover of A.
- P_1 is a projective cover of $\ker(f_0)$.

- Note this is unique, up to isomorphism.

The transpose

Definition

Let A be a left Λ-module, and let $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$ be its minimal projective presentation. Then the **transpose** of A is the right Λ-module that makes this sequence exact:

$$P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{\pi} \mathrm{Tr}(A) \to 0$$

that is,

$$\mathrm{Tr}(A) := \mathrm{coker}(f_1^*)$$

Digression: this is not quite Ext

- This construction may look kind of familiar.
- If

$$\cdots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

is a projective resolution of A, then the cohomology of

$$P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{f_2^*} P_2^* \to \cdots$$

at index 1 is $\ker(f_2^*)/\operatorname{im}(f_1^*) = \operatorname{Ext}^1_\Lambda(A, \Lambda)$.
- If $P_2 = 0$ (so A has projective dimension 1), then $\operatorname{Tr}(A) \cong \operatorname{Ext}^1_\Lambda(A, \Lambda)$. In general, this isn't true.

Digression: this is not quite a functor

- Annoyingly, since our construction relies on a *minimal* projective presentation, Tr is not functorial. But there is a remedy.
- Say a morphism $f : A \to B$ **factors through a projective** if there exists a projective module P and morphisms $g : A \to P$, $h : P \to B$ such that $f = hg$.
- For Λ-modules A, B, define

$$\underline{\mathrm{Hom}}_\Lambda(A, B) := \frac{\mathrm{Hom}_\Lambda(A, B)}{\text{maps factoring through a projective module}}$$

- Then define a category $\underline{\Lambda\text{-mod}}$ whose objects are Λ-modules, but whose morphisms are given by these quotient spaces. This is called the **stable module category**.
  - In a sense, we are killing off the projective modules.
- Then $\mathrm{Tr} : \underline{\Lambda\text{-mod}} \to \underline{\text{mod-}\Lambda}$ is a functor.

Example with quiver representations

Consider the quiver

$$1 \to 2 \begin{smallmatrix} \nearrow 3 \\ \searrow 4 \end{smallmatrix}$$

We will calculate the transpose of S_2, the simple supported at 2:

$$0 \to k \begin{smallmatrix} \nearrow 0 \\ \searrow 0 \end{smallmatrix}$$

First note that its projective cover is $P_2 := (kQ)e_2$, the projective spanned by paths starting at 2:

$$0 \to k \begin{smallmatrix} \nearrow k \\ \searrow k \end{smallmatrix} \longrightarrow 0 \to k \begin{smallmatrix} \nearrow 0 \\ \searrow 0 \end{smallmatrix}$$

Example with quiver representations

The kernel of the projective cover then breaks down as $P_3 \oplus P_4$:

$$0 \to 0 \begin{array}{l} \nearrow k \\ \searrow k \end{array} \longrightarrow 0 \to k \begin{array}{l} \nearrow k \\ \searrow k \end{array} \longrightarrow 0 \to k \begin{array}{l} \nearrow 0 \\ \searrow 0 \end{array}$$

$$P_3 \oplus P_4 \to P_2 \to S_2$$

Our task is then to find the cokernel of $P_2^* \to P_3^* \oplus P_4^*$:

$$k \leftarrow k \begin{array}{l} \swarrow 0 \\ \nwarrow 0 \end{array} \longrightarrow k^2 \leftarrow k^2 \begin{array}{l} \swarrow k \\ \nwarrow k \end{array}$$

This turns out to be

$$k \leftarrow k \begin{array}{l} \swarrow k \\ \nwarrow k \end{array}$$

with all maps the identity.

A key property of the transpose

Proposition

Suppose A is indecomposable and not projective, and that

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} A \to 0$$

is a minimal projective presentation of A. Then

$$P_0^* \xrightarrow{f_1^*} P_1^* \xrightarrow{\pi} \mathrm{Tr}(A) \to 0$$

is a minimal projective presentation of $\mathrm{Tr}(A)$.

First, what happens if A *is* projective?

Proposition

If A is projective, $\mathrm{Tr}(A) = 0$.

This fits with the claim that "killing off projectives" plays nicely with Tr.

A key property of the transpose

Proof.

Let $E_0 \xrightarrow{g_1} E_1 \xrightarrow{\widetilde{\pi}} \mathrm{Tr}(A) \to 0$ be a minimal projective presentation of $\mathrm{Tr}(A)$.
First, write $P_1^* \cong E_1 \oplus K_1$, where $\pi|_{E_1} = \widetilde{\pi}$ and $\pi|_{K_1} = 0$.
Then $\ker(\pi) \cong \ker(\widetilde{\pi}) \oplus K_1$, so it has projective cover $E_0 \oplus K_1$. Thus we can split $P_0^* \cong E_0 \oplus K_1 \oplus K_0$, where f_1^* maps E_0 to E_1 via g^*, $K_1 \to K_1$ via the identity, and K_0 to 0.
Altogether, this means we can write
$f_1^* : P_0^* \cong E_0 \oplus K_1 \oplus K_0 \to P_1^* \cong E_1 \oplus K_1$ with the matrix

$$\begin{pmatrix} g_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

□

A key property of the transpose

Proof.

We can write $f_1^* : P_0^* \cong E_0 \oplus K_1 \oplus K_0 \to P_1^* \cong E_1 \oplus K_1$ with the matrix

$$\begin{pmatrix} g_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

But now hit everything with $(-)^*$ again, to get back $f_1 : P_1 \to P_0$. This tells us that

$$E_1^* \oplus K_1^* \xrightarrow{\begin{pmatrix} g_1^* & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} E_0^* \oplus K_1^* \oplus K_0^* \xrightarrow{f_0} A \to 0$$

is the minimal projective presentation we started with.
We can see that $K_1^* \subset \ker(f_0)$; then $K_1^* = 0$, since f_0 is right minimal.
Looking at the rest of the sequence, we get $A \cong \operatorname{coker}(g_1^*) \oplus K_0^*$. Since A is indecomposable, one of these summands is 0. □

A key property of the transpose

Proof.

We have $A \cong \operatorname{coker}(g_1^* : E_1^* \to E_0^*) \oplus K_0^*$. Since A is indecomposable, one of these summands is 0.

- If $K_0^* = 0$, both K_0 and K_1 are 0, implying $P_0^* \to P_1^* \to \operatorname{Tr}(A)$ was actually a minimal projective presentation, and we are done.
- If $\operatorname{coker}(g_1^*) = 0$, $A \cong K_0^*$, which is projective. But we assumed A isn't projective.

□

Some nice consequences regarding the transpose

Proposition

(1) $\mathrm{Tr}(A \oplus B) \cong \mathrm{Tr}(A) \oplus \mathrm{Tr}(B)$

Suppose A is indecomposable and not projective. Then

(2) $\mathrm{Tr}(\mathrm{Tr}(A)) \cong A$.

(3) $\mathrm{Tr}(A)$ *is indecomposable.*

Proof.

(1) Projective covers and $(-)^*$ commute with direct sums.

(2) We can reuse the minimal projective presentation

$$P_0^* \to P_1^* \to \mathrm{Tr}(A) \to 0$$

to compute $\mathrm{Tr}(\mathrm{Tr}(A))$, in the process just getting the original presentation of A back.

□

Some nice consequences regarding the transpose

Proposition

(1) $\mathrm{Tr}(A \oplus B) \cong \mathrm{Tr}(A) \oplus \mathrm{Tr}(B)$

Suppose A is indecomposable and not projective. Then

(2) $\mathrm{Tr}(\mathrm{Tr}(A)) \cong A$.

(3) $\mathrm{Tr}(A)$ *is indecomposable.*

Proof.

(3) Suppose instead $\mathrm{Tr}(A)$ is decomposable. We can assume $\mathrm{Tr}(A)$ has a nonprojective indecomposable summand B_1: if not, it would be projective, and $\mathrm{Tr}(\mathrm{Tr}(A)) = 0$, contradicting (2).
Then write $\mathrm{Tr}(A) \cong B_1 \oplus B_2$. We have
$A \cong \mathrm{Tr}(\mathrm{Tr}(A)) \cong \mathrm{Tr}(B_1) \oplus \mathrm{Tr}(B_2)$.
Since A is indecomposable, $\mathrm{Tr}(B_2) = 0$. But then
$\mathrm{Tr}(\mathrm{Tr}(B_1)) = \mathrm{Tr}(A) = B_1 \oplus B_2$, contradicting (2).

□

The Auslander-Reiten transform!

Proposition

The transpose gives a bijection between indecomposable nonprojective left Λ-modules and indecomposable nonprojective right Λ-modules.

Now we use the duality D to move things back into the realm of left modules!

Definition

The **Auslander-Reiten transform** is the operation $D\,\mathrm{Tr}$.

Proposition

The Auslander-Reiten transform $D\,\mathrm{Tr}$ gives a bijection between indecomposable nonprojective left modules and indecomposable noninjective left modules.

Back to quivers

Earlier, we looked at the quiver

$$1 \to 2 \begin{array}{l} \nearrow 3 \\ \searrow 4 \end{array}$$

and found the transpose of the simple S_2:

$$\mathrm{Tr} \quad 0 \to k \begin{array}{l} \nearrow 0 \\ \searrow 0 \end{array} \quad = \quad k \leftarrow k \begin{array}{l} \swarrow k \\ \nwarrow k \end{array}$$

Now when we apply the duality D, this flips all the arrows back:

$$D\,\mathrm{Tr} \quad 0 \to k \begin{array}{l} \nearrow 0 \\ \searrow 0 \end{array} \quad = \quad k \to k \begin{array}{l} \nearrow k \\ \searrow k \end{array}$$

In short: $D\,\mathrm{Tr}(S_2) = P_1$.

The significance of this

- The Auslander-Reiten transform generates new indecomposable modules from old ones—a nontrivial feature.
- It is also, in many cases, reasonable to compute.
- It's tangled up with the structure of the module category in ways we'll see more of in the next two days.

Next time...

- A deep dive on an example that doesn't use quivers!
- Too many indecomposable modules!
- What does "almost split" mean anyway?!

See it all, tomorrow!