Representation Theory of Finite-Dimensional Algebras

Day 2: Projectives, Injectives, and Duality

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Last time...

- We introduced the example of path algebras of quivers.
- We looked at properties of the radical of an Artinian ring Λ.

Definition

The radical of Λ consists of the elements which annihilate all (semi)simple left Λ-modules.

Theorem

The radical $\mathfrak{r}$ is

- *the biggest nilpotent ideal,*
- *the smallest ideal with $\Lambda/\mathfrak{r}$ semisimple, and thus*
- *the unique ideal with both these properties.*

- We showed that the radical of a path algebra is generated by all the arrows.

Digression: everything* comes from path algebras

*over an algebraically closed field up to Morita equivalence

Theorem

Let Λ be any finite-dimensional algebra over an algebraically closed field. Then there exists a (potentially cyclic) quiver Q and an ideal $\mathfrak{a} \subset kQ$ such that $(kQ/\mathfrak{a})$-mod $\cong \Lambda$-mod.

More precisely, given $\mathfrak{r} := \text{rad}(\Lambda)$:

- Vertices of Q correspond to summands of $\Lambda/\mathfrak{r}$ (simple modules).
- Arrows of Q correspond to summands of $\mathfrak{r}/\mathfrak{r}^2$.

The radical of a module

Just as with a ring, we can define the radical of a module:

Definition

The **radical** of a module is the intersection of its maximal submodules.

However, in the Artinian case this isn't anything new:

Proposition

$$\mathrm{rad}(A) = \mathfrak{r}A$$

Proposition

$$\mathrm{rad}(A) = \mathfrak{r}A$$

Proof.

$\supset$: Suppose instead there is some maximal submodule M not containing $\mathfrak{r}A$. Then

$$\mathfrak{r}A + M = A$$

and by repeatedly multiplying by $\mathfrak{r}$ and adding M, we see

$$\mathfrak{r}^i A + M = A, \quad \forall i$$

which, since $\mathfrak{r}$ is nilpotent, is a contradiction.

$\subset$: We just showed $\mathrm{rad}(A) \supset \mathfrak{r}A$, so $\mathrm{rad}(A/\mathfrak{r}A) = \mathrm{rad}(A)/\mathfrak{r}A$. But also $A/\mathfrak{r}A$ is semisimple, so $\mathrm{rad}(A/\mathfrak{r}A) = 0$.

□

Composition series and length

Definition

Let A be a module for a ring Λ. A filtration

$$0 =: A_0 \subset A_1 \subset \cdots \subset A_n := A$$

is a **composition series** if all the quotients A_{i+1}/A_i are simple.

Theorem (Jordan-Hölder Theorem)

The simple quotients A_{i+1}/A_i are unique up to rearrangement and isomorphism.

Definition

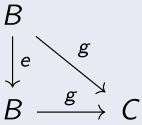
The integer n above is the **length** of A, denoted $\ell(A)$.

Every Artinian ring, and (f.g.) module over it, has a finite length.

Minimal morphisms

Definition

A map $g : B \to C$ is **right minimal** if any map $e : B \to B$ making



commute is an isomorphism.

- Intuition: No extraneous stuff in B we can kill off.
- Similarly, define a **left minimal** morphism by reversing all the arrows.

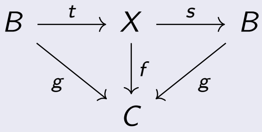
Pulling off a minimal piece

Theorem

Let $f : B \to C$ be any morphism. Then there is a direct sum decomposition $B \cong B_0 \oplus M$ such that $f|_{B_0} : B_0 \to C$ is right minimal and $f|_M = 0$.

Proof.

Consider the collection of all nonzero morphisms $g : X \to C$ such that there exist maps $s : X \to B$ and $t : B \to X$ making this diagram commute:



Choose $g : X \to C$ such that X is of minimal length. First, we claim that g is right minimal. □

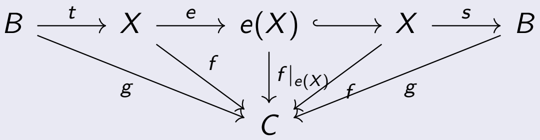
Pulling off a minimal piece

Theorem

Let $f : B \to C$ be any morphism. Then there is a direct sum decomposition $B \cong B_0 \oplus M$ such that $f|_{B_0} : B_0 \to C$ is right minimal and $f|_M = 0$.

Proof.

If $e : X \to X$ is a nonisomorphism, then $e(X) \subsetneq X$ is even shorter than X. The following diagram commutes:



which contradicts the minimality of $\ell(X)$. □

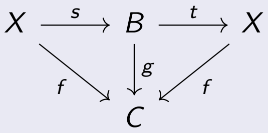
Pulling off a minimal piece

Theorem

Let $f : B \to C$ be any morphism. Then there is a direct sum decomposition $B \cong B_0 \oplus M$ such that $f|_{B_0} : B_0 \to C$ is right minimal and $f|_M = 0$.

Proof.

Then let's rearrange the diagram:



Since f is right minimal, ts is an isomorphism. So t is a split epimorphism, $B \cong X \oplus \ker(t)$, and chasing through the diagram shows the result. □

Projective covers

Theorem

Let $\pi : P \to A$ be a surjection with P projective. The following are equivalent:

(a) *π is right minimal.*

(b) *For any $X \subsetneq P$ a proper submodule, $\pi|_X : X \to A$ is not surjective.*

(c) $\ker(\pi) \subset \mathfrak{r}P$.

(d) *The induced map $\overline{\pi} : P/\mathfrak{r}P \to A/\mathfrak{r}A$ is an isomorphism.*

All different ways of saying "P is no larger than it needs to be."

Lemma

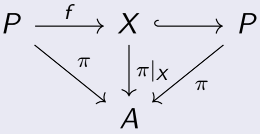
Let $\pi : P \to A$ be a surjection with P projective.

If *π is right minimal*

then *for any $X \subsetneq P$ a proper submodule, $\pi|_X : X \to A$ is not surjective.*

Proof.

Let $X \subset P$ be such that $\pi|_X : X \to A$ is surjective. Then we can lift $\pi : P \to A$ through $\pi|_X : X \to A$, and get this commutative diagram:



But since π is right minimal, the composition $P \xrightarrow{f} X \hookrightarrow P$ is an isomorphism. So $X \hookrightarrow P$ is surjective and $X = P$. □

Lemma

Let $\pi : P \to A$ be a surjection with P projective.

If *for any $X \subsetneq P$ a proper submodule, $\pi|_X : X \to A$ is not surjective*

then $\ker(\pi) \subset \mathfrak{r}P$.

Proof.

We show $\ker(\pi) \subset \mathrm{rad}(P)$.
Suppose instead that there is some maximal submodule $M \subset P$ not containing $\ker(\pi)$. Then

$$\ker(\pi) + M = P.$$

But this implies

$$\pi(M) = \pi(P),$$

a contradiction. □

Lemma

Let $\pi : P \to A$ be a surjection with P projective.

If $\ker(\pi) \subset \mathfrak{r}P$

then π *is right minimal.*

Proof.

We know we can write $P \cong P_0 \oplus Q$ such that $\pi|_{P_0} : P_0 \to A$ is minimal and $\pi(Q) = 0$. This implies that

$$Q \subset \ker(\pi) \subset \mathrm{rad}(P).$$

But if $Q' \subset Q$ is a maximal submodule, then

$$\mathrm{rad}(P) \subset P_0 \oplus Q',$$

so $\mathrm{rad}(P)$ cannot contain Q. The only way to avoid this is if $Q = 0$, so π is right minimal. □

Projective covers

Theorem

Let $\pi : P \to A$ be a surjection with P projective. The following are equivalent:

(a) *π is right minimal.*

(b) *For any $X \subsetneq P$ a proper submodule, $\pi|_X : X \to A$ is not surjective.*

(c) $\ker(\pi) \subset \mathfrak{r}P$.

(d) *The induced map $\overline{\pi} : P/\mathfrak{r}P \to A/\mathfrak{r}A$ is an isomorphism.*

Definition

A right minimal epimorphism $\pi : P \to A$ with P projective is a **projective cover**.

A few basic things about projective covers

Proposition

Any module has a projective cover.

Proof.

Write it as a quotient of a free module, and then split a right minimal morphism off from that. □

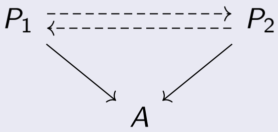
A few basic things about projective covers

Proposition

Projective covers are unique up to isomorphism.

Proof.

Suppose $P_1 \to A$ and $P_2 \to A$ are two projective covers.



Lift the two maps along each other, to get maps $f : P_1 \to P_2$ and $g : P_2 \to P_1$ which commute with the covers.

By right minimality, the compositions $P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_1$ and $P_2 \xrightarrow{g} P_1 \xrightarrow{f} P_2$ are isomorphisms. Thus f is both injective and surjective, and an isomorphism. □

A few basic things about projective covers.

Proposition

If $P_1 \to A_1$ *and* $P_2 \to A_2$ *are projective covers, so is* $P_1 \oplus P_2 \to A_1 \oplus A_2$.

Proposition

For P *projective,* $P \to P/\mathfrak{r}P$ *is a projective cover.*

Proof.

Both statements follow from the "$P/\mathfrak{r}P \to A/\mathfrak{r}A$ is an isomorphism" criterion. □

Corollary

For projective modules P, Q,

$$P \cong Q \Leftrightarrow P/\mathfrak{r}P \cong Q/\mathfrak{r}Q.$$

Simple downstairs $\leftrightarrow$ indecomposable projective upstairs

Proposition

A projective module P is indecomposable if and only if $P/\mathfrak{r}P$ is simple.

Proof.

If $P \cong P_1 \oplus P_2$, $P/\mathfrak{r}P \cong P_1/\mathfrak{r}P_1 \oplus P_2/\mathfrak{r}P_2$. If $P/\mathfrak{r}P \cong S_1 \oplus S_2$, and we have projective covers $P_1 \to S_1$ and $P_2 \to S_2$, then $P \cong P_1 \oplus P_2$. □

Corollary

The operations of projective cover and semisimple quotient give a bijection between simple modules and indecomposable projective ones.

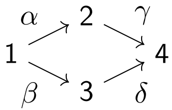
In particular, there are only finitely many indecomposable projective modules.

- kQe_x corresponds to "paths starting from x".
- kQ decomposes as a direct sum

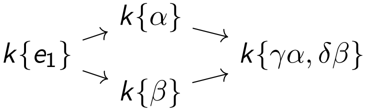
$$kQ \cong \bigoplus_{\text{vertex } x} kQe_x$$

so these are all projective.
- In $kQe_x/\mathfrak{r}kQe_x$, only e_x remains. This is the simple supported at x.
- So the kQe_x are exactly the indecomposable projectives!

Q



kQe_1



Duality

Definition

A duality between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of contravariant functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ such that FG and GF are naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{C}}$ respectively.

- The contravariant version of equivalence.
- Turns every categorical construction in $\mathcal{C}$ into its "co-" version in $\mathcal{D}$.

Duality #1

- A convention: we identify left Λ^{op}-modules with right Λ-modules.
- Define a contravariant functor $(-)^* : \Lambda\text{-mod} \to \Lambda^{\mathrm{op}}\text{-mod}$ by

$$A^* := \mathrm{Hom}_\Lambda(A, \Lambda)$$

  with action

$$(a^* \lambda)(-) = a^*(-)\lambda$$

- On a morphism $f : A \to B$:

$$f^*(b^*)(-) = b^*(f(-))$$

Duality #1

Proposition

Let $\mathcal{P}(\Lambda)$ be the full subcategory of projective Λ-modules. Then $(-)^*$ gives a categorical duality

$$\mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda^{\text{op}})$$

Proof (sketch).

The map

$$\begin{aligned} A &\to A^{**} := \text{Hom}_\Lambda(\text{Hom}_\Lambda(A, \Lambda), \Lambda) \\ a &\mapsto (a^* \mapsto a^*(a)) \end{aligned}$$

isn't always an isomorphism, but it is for $A = \Lambda$.
Because Hom commutes with direct sums, this map is also an isomorphism for free modules Λ^n, and also for direct summands of free modules, i.e.: projectives. □

Duality #1 for path algebras

- For a quiver Q, Q^{op}, the **opposite quiver**, is obtained by reversing all arrows of Q.

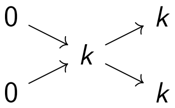
Proposition

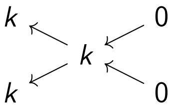
$$(kQ)^{\text{op}} \cong k(Q^{\text{op}})$$

- An element of $\text{Hom}_{kQ}(kQe_x, kQ)$ is determined by where we send e_x.
- e_x can be sent to any combination of paths ending at x:

$$e_x a^*(e_x) = a^*(e_x e_x) = a^*(e_x)$$

- This identifies $(kQe_x)^*$ with $e_x kQ$, which we identify with $k(Q^{\text{op}})e_x$.





Duality #2

- We still haven't used "finite-dimensional over a field". But we're about to! From here on, assume Λ is a finite-dimensional k-algebra.
- Define a contravariant functor $D : \Lambda\text{-mod} \to \Lambda^{\mathrm{op}}\text{-mod}$ by

$$DA := \mathrm{Hom}_k(A, k)$$

  with action

$$(f\lambda)(-) = f(\lambda \cdot -)$$

- On a morphism $\varphi : A \to B$:

$$\varphi^*(f)(-) = f(\varphi(-))$$

Duality #2

Proposition

$$D : \Lambda\text{-}mod \to \Lambda^{\mathrm{op}}\text{-}mod$$

is a duality.

Proof.

This time, the map

$$A \to D(DA) := \mathrm{Hom}_k(\mathrm{Hom}_\Lambda(A, k), k)$$
$$a \mapsto (f \mapsto f(a))$$

is always an isomorphism. □

Duality #2 for path algebras

- Suppose A is a representation of Q. What does DA look like as a representation of Q^{op}?
- The space at x is given by $(DA)e_x$:

$$(DA)e_x = \text{Hom}_k(A, k)e_x = \{f(e_x \cdot -) \mid f : A \to k\}$$

which amounts to restricting f to $e_x A$; thus

$$\text{Hom}_k(A, k)e_x \cong \text{Hom}_k(e_x A, k).$$

- Given an arrow $\alpha : x \to y$ and $f \in \text{Hom}_k(e_y A, k)$, we have

$$(f\alpha)(-) = f(\alpha \cdot -) \in \text{Hom}_k(e_x A, k)$$

- Altogether:
  - $DA(x)$ is the dual space of $A(x)$
  - $DA(\alpha^*)$ is the dual map to $A(\alpha)$ (where α^* is the reversed arrow)

Injective modules

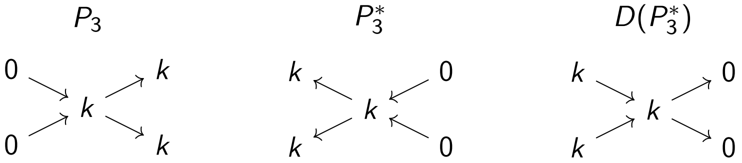
Because D is a duality, it sends projectives to injectives and vice versa.

Proposition

The maps

$$P \mapsto D(P^*)$$
$$I \mapsto (DI)^*$$

define a bijection between projective and injective Λ-modules.



- Indecomposable projective at x: paths starting at x
- Indecomposable injective at x: paths ending at x

Next time. . .

- The radical meets its evil twin!
- Our two duality operations team up again!
- A secret cache of unlimited indecomposable modules is unearthed!

All this and more. . .