1 Introduction

2 Artinian rings: preliminaries
General conventions

- $k$ is a field.
- All vector spaces are assumed to be finite-dimensional $k$-vector spaces.
- All modules are assumed to be finitely-generated, unless otherwise stated.
A motivating simple example

A fundamental theorem of linear algebra:

Theorem

Let \( f : V \rightarrow W \) be a linear map between vector spaces. Then we can choose bases of \( V \) and \( W \) with respect to which \( f \) is given by a matrix of the form

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
Theorem

Let $f : V \to W$ be a linear map between vector spaces. Then there exist isomorphisms $\varphi : V \to k^n$, $\psi : W \to k^m$, and a map $k^n \to k^m$ given by a matrix $M$ of the special form above, such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow{} & & \downarrow{} \\
k^n & \xrightarrow{M} & k^m
\end{array}
\]

commutes.
Quiver representations

- A **quiver** $Q$ is a directed graph, with vertices $Q_0$ and edges $Q_1$.

- A **representation** $V$ of a quiver $Q$ assigns to every vertex $x$ a vector space $V(x)$ and to every edge $\alpha$ a linear map $V(\alpha)$ between the spaces at its endpoints.

- A **morphism of representations** $h : V_1 \to V_2$ consists of maps $h_x : V_1(x) \to V_2(x)$ for each vertex $x$ which commute with the maps associated to the arrows.
Once more with jargon

**Theorem**

Let $f : V \to W$ be a linear map between vector spaces. Then there exist isomorphisms $\varphi : V \to k^n$, $\psi : W \to k^m$, and a map $k^n \to k^m$ given by a matrix $M$ of the special form above, such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
\downarrow{\varphi} & & \downarrow{\psi} \\
k^n & \xrightarrow{M} & k^m
\end{array}
\]

commutes.

**Theorem**

Every representation of the quiver $\circ \xrightarrow{\quad} \circ$ is isomorphic to one of the form $k^n \xrightarrow{M} k^m$ where $M$ has the special form from above.
Once more with reductionism

- The **direct sum** of two representations of a quiver is defined by just taking the direct sum of the spaces at each vertex and the direct sum of the maps at each edge.
- In terms of matrices, this amounts to, at each edge, putting together the matrices into a block diagonal matrix.
- Our “special form from above” is a block diagonal matrix, and pulling it apart, we get

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**Theorem**

*Every representation of the quiver* \( \circ \rightarrow \circ \) *is isomorphic to a direct sum of the indecomposable representations*

\[
\begin{align*}
k & \rightarrow 0 \\
0 & \rightarrow k \\
k & \xrightarrow{1} k
\end{align*}
\]
Path algebras

- Given a quiver $Q$, a **path** is a word $\alpha_n \cdots \alpha_2 \alpha_1$ in the edges, where the end vertex of $\alpha_i$ is the start vertex of $\alpha_{i+1}$.
  - We also have, for each vertex $x$, a stationary path $e_x$.
- The **path algebra** $kQ$ consists of formal $k$-linear combinations of paths in $Q$.
  - The product $pq$ of two paths is their concatenation if the end of $q$ matches up with the start of $p$, and 0 otherwise.

\[
\begin{align*}
1 \xrightarrow{\alpha} 2 \\
kQ \text{ has basis } e_1, e_2, \alpha.
\end{align*}
\]

\[
\begin{align*}
e_1 \alpha &= 0 & \alpha e_1 &= \alpha \\
e_2 \alpha &= \alpha & \alpha e_2 &= 0 \\
e_1 e_1 &= e_1 & e_1 e_2 &= 0 \\
e_2 e_2 &= e_2 & e_2 e_1 &= 0 \\
\alpha \alpha &= 0
\end{align*}
\]
Interlude: another general convention

- $kQ$ is finite-dimensional $\iff Q$ has no oriented cycles.
- Since we want things to be finite-dimensional, we’ll assume $Q$ has no **oriented cycles** unless stated otherwise.
- This is not to say that cycles aren’t interesting, but they’re outside the scope of this course.
Modules of path algebras are quiver representations

**Theorem**

The category of left $kQ$-modules is equivalent to the category of representations of $Q$.

How do we turn a $kQ$-module $M$ into a quiver representation?

- At vertex $x$, put the space $e_x M$.
- For an edge $\alpha$, the map $e_{\text{start}(\alpha)} M \to e_{\text{end}(\alpha)} M$ is just left multiplication by $\alpha$.

How do we turn a quiver representation $V$ into a $kQ$-module?

- Our module is $\bigoplus_{\text{vertices } x} V(x)$.
- Multiplication by a path $\alpha$ sends the $V(\text{start}(\alpha))$ summand into the $V(\text{end}(\alpha))$ summand, and is 0 on all the other summands.
Once more with algebras

**Theorem**

Let $Q$ be the quiver $1 \xrightarrow{\alpha} 2$. Then the path algebra $kQ$ has exactly 3 indecomposable representations, which we can describe explicitly.

Specifically,

$(0 \to k) \leftrightarrow (kQ)e_2$

$(k \to k) \leftrightarrow (kQ)e_1$

$(k \to 0) \leftrightarrow \frac{(kQ)e_1}{(kQ)\alpha}$
Key examples of finite-dimensional algebras

- Path algebras (and their quotients)
  - In the words of Auslander-Reiten-Smalø, “describ[ing] how a finite number of linear transformations can act simultaneously on a finite dimensional vector space”
- Matrix algebras
- Finite group algebras
  - We’re particularly interested in the messy positive characteristic case.
In several popular representation-theoretic contexts, we have \textbf{semisimplicity}: every representation is a direct sum of simple ones. That is emphatically not the case here.

\begin{itemize}
  \item Theorem
  \end{itemize}

\begin{align*}
\text{Let } Q \text{ be an acyclic quiver. The category } kQ\text{-mod is semisimple if and only if } Q \text{ has no edges.}
\end{align*}
Some problems we have to deal with

- What are the projective and injective representations, if not everything?
  
  \[
  0 \rightarrow k : \text{projective}
  \]
  
  \[
  k \rightarrow 0 : \text{injective}
  \]
  
  \[
  k \rightarrow k : \text{both}
  \]

- What are the indecomposables, if not the same things as the simples?

- What structure can exact sequences have, if they don’t necessarily split?
  
  \[
  \begin{array}{c}
  0 \rightarrow k \rightarrow k \rightarrow 0 \rightarrow 0 \\
  \downarrow \downarrow \downarrow \\
  0 \rightarrow 0 \rightarrow k \rightarrow k \rightarrow 0
  \end{array}
  \]
What control does finite-dimensionality give us?

- We have multiple **duality** operations which link together projectives and injectives and create nice symmetry.
- Fully classifying indecomposable representations may be a hopeless task in general, but an operation called the **Auslander-Reiten transform** generates new indecomposables from old ones, and gives some understanding of when classification is out of reach.
- We can construct **almost split sequences**, which give the module category some unexpected structure.
Why are finite-dimensional algebras useful to us? They are Artinian:

\[ I_0 \supset I_1 \supset \cdots \supset I_n = I_{n+1} = \cdots \]

In fact, we could do everything with finitely generated Artinian algebras over commutative Artinian rings.

So we’ll need some general facts about Artinian rings.

In what follows, \( \Lambda \) is assumed to be Artinian.
The (Jacobson) **radical** of a ring $\Lambda$, $\text{rad}(\Lambda)$, is the intersection of all maximal left ideals of $\Lambda$.

The **radical** of $\Lambda$ consists of the elements which annihilate every simple left $\Lambda$-module.

- If $\Lambda$ is implied, write $\tau := \text{rad}(\Lambda)$.

- This is the part of the ring which is invisible from the perspective of semisimple modules.

- In particular,

  $$\Lambda \text{ semisimple} \Rightarrow \tau = 0$$
The obstruction to semisimplicity: the radical

Proposition

1. \( r = 0 \iff \Lambda \text{ is semisimple.} \)
2. \( \Lambda / r \text{ is semisimple.} \)

Proof.

(1) [sketch] We already know \( \iff \). To show \( \Rightarrow \), let \( a \subset \Lambda \) be a simple submodule. Since \( r = 0 \), there is a maximal left ideal \( m \) not containing \( a \). Then:
   - \( m \cap a = 0 \), because \( a \) is minimal.
   - \( m + a = \Lambda \), because \( m \) is maximal.

and so \( \Lambda = a \oplus m \). In this way, we can pull off simple submodules as summands until (by Artinianity) we run out.

(2) \( \text{rad}(\Lambda / r) = 0. \)
The radical and simple modules

Already, we have a nice bit of finiteness!

**Theorem**

An Artinian ring $\Lambda$ has finitely many simple modules up to isomorphism.

**Proof.**

Any simple module is annihilated by $\tau$, so it is also a simple $\Lambda/\tau$-module. But these are just the summands of $\Lambda/\tau$, of which there are finitely many.
The radical and nilpotence

**Lemma (Nakayama’s Lemma)**

Let $M$ be a nonzero finitely generated module over $\Lambda$. Then $\mathfrak{r}M \subsetneq M$.

**Proposition**

The radical of an Artinian ring is nilpotent.

**Proof.**

Consider the chain

$$\Lambda \supset \mathfrak{r} \supset \mathfrak{r}^2 \supset \mathfrak{r}^3 \supset \cdots$$

By Nakayama’s Llama, all these inclusions are proper as long as $\mathfrak{r}^i \neq 0$. But because $\Lambda$ is Artinian, the chain must stabilize, so it eventually hits 0.
Lemma

Any nilpotent ideal $\alpha$ is contained in the radical.

Proof.

Suppose instead that some maximal left ideal $m$ does not contain $\alpha$. Then

$$\alpha + m = \Lambda.$$ 

Multiplying by $\alpha$ and adding $m$ to both sides gives

$$\alpha^2 + \alpha m + m = \alpha^2 + m = \alpha + m = \Lambda.$$ 

Repeating this process, we get

$$\alpha^i + m = \Lambda$$

for all $i$, but this contradicts the nilpotence of $\alpha$. \qed
Semisimplicity + nilpotence = radical

Lemma
Any nilpotent ideal $\alpha$ is contained in the radical.

Lemma
The radical is contained in any ideal $\alpha$ with $\Lambda/\alpha$ semisimple.

Proof.
$\Lambda/\alpha$, being semisimple, is annihilated by $\tau$.

Theorem
The radical is the unique ideal that is nilpotent and induces a semisimple quotient.
This is typically an easy criterion to check.

**Proposition**

Let $Q$ be a quiver and $kQ$ its path algebra. Let $	au \subset kQ$ be the ideal generated by all the arrows. Then $\tau$ is the radical of $kQ$.

**Proof.**

- **Nilpotence:** $\tau^i$ is spanned by all paths of length $\geq i$. Since $Q$ is acyclic, these will eventually run out.
- **Semisimplicity:** $kQ/\tau$ is spanned by the orthogonal idempotents $e_x$, so

$$kQ/\tau \cong \prod_{\text{vertex } x} k,$$
Proposition

Let $Q$ be a quiver and $kQ$ its path algebra. Let $\tau \subset kQ$ be the ideal generated by all the arrows. Then $\tau$ is the radical of $kQ$.

- This illustrates why semisimplicity is anathema to path algebras.
- It also tells us what the simple representations are:

Corollary

For a quiver $Q$, each simple representation is given by $k$ at one vertex and $0$ elsewhere.
Digression: everything* comes from path algebras

*over an algebraically closed field up to Morita equivalence

Theorem

Let $\Lambda$ be any finite-dimensional algebra over an algebraically closed field. Then there exists a (potentially cyclic) quiver $Q$ and an ideal $\alpha \subset kQ$ such that $(kQ/\alpha)$-mod $\cong \Lambda$-mod.

More precisely, given $\tau := \text{rad}(\Lambda)$:

- Vertices of $Q$ correspond to summands of $\Lambda/\tau$ (simple modules).
- Arrows of $Q$ correspond to summands of $\tau/\tau^2$. 
Next time... 

- Powerful production of projective modules!
- The mystique of minimal morphisms!
- Daring deeds of duality!
You won’t want to miss it!