Representation Theory of Finite-Dimensional Algebras

Day 1: Motivation and The Basics

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References

Reference

Auslander, Reiten, and Smalø. *Representation Theory of Artin Algebras.*

General conventions

- k is a field.
- All vector spaces are assumed to be finite-dimensional k-vector spaces.
- All modules are assumed to be finitely-generated, unless otherwise stated.

A motivating simple example

A fundamental theorem of linear algebra:

Theorem

Let $f : V \to W$ be a linear map between vector spaces. Then we can choose bases of V and W with respect to which f is given by a matrix of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Once more with diagrams

Theorem

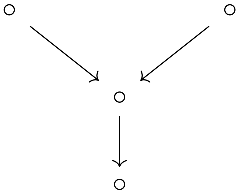
Let $f : V \to W$ be a linear map between vector spaces. Then there exist isomorphisms $\varphi : V \to k^n$, $\psi : W \to k^m$, and a map $k^n \to k^m$ given by a matrix M of the special form above, such that the diagram

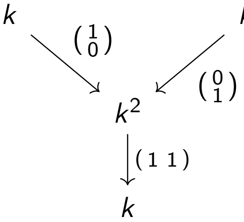
$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ k^n & \xrightarrow{M} & k^m \end{array}$$

commutes.

Quiver representations

- A **quiver** Q is a directed graph, with vertices Q_0 and edges Q_1.
- A **representation** V of a quiver Q assigns to every vertex x a vector space $V(x)$ and to every edge α a linear map $V(\alpha)$ between the spaces at its endpoints.





- A **morphism of representations** $h : V_1 \to V_2$ consists of maps $h_x : V_1(x) \to V_2(x)$ for each vertex x which commute with the maps associated to the arrows.

Once more with jargon

Theorem

Let $f : V \to W$ be a linear map between vector spaces. Then there exist isomorphisms $\varphi : V \to k^n$, $\psi : W \to k^m$, and a map $k^n \to k^m$ given by a matrix M of the special form above, such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} \\ k^n & \xrightarrow{M} & k^m \end{array}$$

commutes.

Theorem

Every representation of the quiver $\circ \longrightarrow \circ$ is isomorphic to one of the form $k^n \xrightarrow{M} k^m$ where M has the special form from above.

Once more with reductionism

- The **direct sum** of two representations of a quiver is defined by just taking the direct sum of the spaces at each vertex and the direct sum of the maps at each edge.
- In terms of matrices, this amounts to, at each edge, putting together the matrices into a block diagonal matrix.
- Our "special form from above" is a block diagonal matrix, and pulling it apart, we get

Theorem

Every representation of the quiver $\circ \longrightarrow \circ$ *is isomorphic to a direct sum of the indecomposable representations*

$$k \to 0$$

$$0 \to k$$

$$k \xrightarrow{1} k$$

Path algebras

- Given a quiver Q, a **path** is a word $\alpha_n \cdots \alpha_2 \alpha_1$ in the edges, where the end vertex of α_i is the start vertex of α_{i+1}.
  - We also have, for each vertex x, a stationary path e_x.
- The **path algebra** kQ consists of formal k-linear combinations of paths in Q.
  - The product pq of two paths is their concatenation if the end of q matches up with the start of p, and 0 otherwise.

$$1 \xrightarrow{\alpha} 2$$

kQ has basis e_1, e_2, α.

$$\begin{aligned} e_1\alpha &= 0 & \alpha e_1 &= \alpha \\ e_2\alpha &= \alpha & \alpha e_2 &= 0 \\ e_1 e_1 &= e_1 & e_1 e_2 &= 0 \\ e_2 e_2 &= e_2 & e_2 e_1 &= 0 \\ \alpha\alpha &= 0 \end{aligned}$$

Interlude: another general convention

- kQ is finite-dimensional $\Leftrightarrow$ Q has no oriented cycles.
- Since we want things to be finite-dimensional, we'll assume Q **has no oriented cycles** unless stated otherwise.
- This is not to say that cycles aren't interesting, but they're outside the scope of this course.

Modules of path algebras are quiver representations

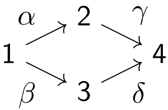
Theorem

The category of left kQ-modules is equivalent to the category of representations of Q.

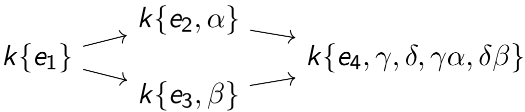
How do we turn a kQ-module M into a quiver representation?

- At vertex x, put the space $e_x M$.
- For an edge α, the map $e_{\text{start}(\alpha)}M \to e_{\text{end}(\alpha)}M$ is just left multiplication by α.

Q



kQ



How do we turn a quiver representation V into a kQ-module?

- Our module is $\bigoplus_{\text{vertices } x} V(x)$.
- Multiplication by a path α sends the $V(\text{start}(\alpha))$ summand into the $V(\text{end}(\alpha))$ summand, and is 0 on all the other summands.

Once more with algebras

Theorem

Let Q be the quiver $1 \xrightarrow{\alpha} 2$ Then the path algebra kQ has exactly 3 indecomposable representations, which we can describe explicitly.

Specifically,

$$(0 \to k) \leftrightarrow (kQ)e_2$$
$$(k \to k) \leftrightarrow (kQ)e_1$$
$$(k \to 0) \leftrightarrow \frac{(kQ)e_1}{(kQ)\alpha}$$

Key examples of finite-dimensional algebras

- Path algebras (and their quotients)
  - In the words of Auslander-Reiten-Smalø, "describ[ing] how a finite number of linear transformations can act simultaneously on a finite dimensional vector space"
- Matrix algebras
- Finite group algebras
  - We're particularly interested in the messy positive characteristic case.

Our nemesis: failure of semisimplicity

- In several popular representation-theoretic contexts, we have **semisimplicity**: every representation is a direct sum of simple ones.
- That is emphatically not the case here.

Theorem

Let Q be an acyclic quiver. The category kQ-mod is semisimple if and only if Q has no edges.

Some problems we have to deal with

- What are the projective and injective representations, if not everything?

$$
\begin{aligned}
0 \to k &: \text{ projective} \\
k \to 0 &: \text{ injective} \\
k \to k &: \text{ both}
\end{aligned}
$$

- What are the indecomposables, if not the same things as the simples?
- What structure can exact sequences have, if they don't necessarily split?

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & k & \longrightarrow & k & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & k & \longrightarrow & k & \longrightarrow & 0
\end{array}
$$

What control does finite-dimensionality give us?

- We have multiple **duality** operations which link together projectives and injectives and create nice symmetry.
- Fully classifying indecomposable representations may be a hopeless task in general, but an operation called the **Auslander-Reiten transform** generates new indecomposables from old ones, and gives some understanding of when classification is out of reach.
- We can construct **almost split sequences**, which give the module category some unexpected structure.

Basic structure: Artinian rings

- Why are finite-dimensional algebras useful to us? They are Artinian:

$$I_0 \supset I_1 \supset \cdots \supset I_n = I_{n+1} = \cdots$$

  - In fact, we could do everything with finitely generated Artinian algebras over commutative Artinian rings.
- So we'll need some general facts about Artinian rings.
- In what follows, Λ is assumed to be Artinian.

The obstruction to semisimplicity: the radical

Definition

The (Jacobson) **radical** of a ring Λ, $\text{rad}(\Lambda)$, is the intersection of all maximal left ideals of Λ.

Definition

The **radical** of Λ consists of the elements which annihilate every simple left Λ-module.

- If Λ is implied, write $\mathfrak{r} := \text{rad}(\Lambda)$.
- This is the part of the ring which is invisible from the perspective of semisimple modules.
- In particular,

$$\Lambda \text{ semisimple} \Rightarrow \mathfrak{r} = 0$$

The obstruction to semisimplicity: the radical

Proposition

(1) $\mathfrak{r} = 0 \Leftrightarrow \Lambda$ *is semisimple.*

(2) $\Lambda/\mathfrak{r}$ *is semisimple.*

Proof.

(1) [sketch] We already know $\Leftarrow$. To show $\Rightarrow$, let $\mathfrak{a} \subset \Lambda$ be a simple submodule. Since $\mathfrak{r} = 0$, there is a maximal left ideal $\mathfrak{m}$ not containing $\mathfrak{a}$. Then:

- $\mathfrak{m} \cap \mathfrak{a} = 0$, because $\mathfrak{a}$ is minimal.
- $\mathfrak{m} + \mathfrak{a} = \Lambda$, because $\mathfrak{m}$ is maximal.

and so $\Lambda = \mathfrak{a} \oplus \mathfrak{m}$. In this way, we can pull off simple submodules as summands until (by Artinianity) we run out.

(2) $\text{rad}(\Lambda/\mathfrak{r}) = 0$.

□

The radical and simple modules

Already, we have a nice bit of finiteness!

Theorem

An Artinian ring Λ has finitely many simple modules up to isomorphism.

Proof.

Any simple module is annihilated by $\mathfrak{r}$, so it is also a simple $\Lambda/\mathfrak{r}$-module. But these are just the summands of $\Lambda/\mathfrak{r}$, of which there are finitely many. □

The radical and nilpotence

Lemma (Nakayama's Lemma)

Let M be a nonzero finitely generated module over Λ. Then $\mathfrak{r}M \subsetneq M$.

Proposition

The radical of an Artinian ring is nilpotent.

Proof.

Consider the chain

$$\Lambda \supset \mathfrak{r} \supset \mathfrak{r}^2 \supset \mathfrak{r}^3 \supset \cdots$$

By Nakayama's Llama, all these inclusions are proper as long as $\mathfrak{r}^i \neq 0$. But because Λ is Artinian, the chain must stabilize, so it eventually hits 0. □

Semisimplicity + nilpotence = radical

Lemma

Any nilpotent ideal $\mathfrak{a}$ is contained in the radical.

Proof.

Suppose instead that some maximal left ideal $\mathfrak{m}$ does not contain $\mathfrak{a}$. Then

$$\mathfrak{a} + \mathfrak{m} = \Lambda.$$

Multiplying by $\mathfrak{a}$ and adding $\mathfrak{m}$ to both sides gives

$$\mathfrak{a}^2 + \mathfrak{a}\mathfrak{m} + \mathfrak{m} = \mathfrak{a}^2 + \mathfrak{m} = \mathfrak{a} + \mathfrak{m} = \Lambda.$$

Repeating this process, we get

$$\mathfrak{a}^i + \mathfrak{m} = \Lambda$$

for all i, but this contradicts the nilpotence of $\mathfrak{a}$. □

Semisimplicity + nilpotence = radical

Lemma

Any nilpotent ideal $\mathfrak{a}$ is contained in the radical.

Lemma

The radical is contained in any ideal $\mathfrak{a}$ with $\Lambda/\mathfrak{a}$ semisimple.

Proof.

$\Lambda/\mathfrak{a}$, being semisimple, is annihilated by $\mathfrak{r}$. □

Theorem

The radical is the unique ideal that is nilpotent and induces a semisimple quotient.

Radicals of path algebras

This is typically an easy criterion to check.

Proposition

Let Q be a quiver and kQ its path algebra. Let $\mathfrak{r} \subset kQ$ be the ideal generated by all the arrows. Then $\mathfrak{r}$ is the radical of kQ.

Proof.

- **Nilpotence:** $\mathfrak{r}^i$ is spanned by all paths of length $\geq i$. Since Q is acyclic, these will eventually run out.
- **Semisimplicity:** $kQ/\mathfrak{r}$ is spanned by the orthogonal idempotents e_x, so

$$kQ/\mathfrak{r} \cong \prod_{\text{vertex } x} k,$$

□

Simple modules of path algebras

Proposition

Let Q be a quiver and kQ its path algebra. Let $\mathfrak{r} \subset kQ$ be the ideal generated by all the arrows. Then $\mathfrak{r}$ is the radical of kQ.

- This illustrates why semisimplicity is anathema to path algebras.
- It also tells us what the simple representations are:

Corollary

For a quiver Q, each simple representation is given by k at one vertex and 0 elsewhere.

Digression: everything* comes from path algebras

*over an algebraically closed field up to Morita equivalence

Theorem

Let Λ be any finite-dimensional algebra over an algebraically closed field. Then there exists a (potentially cyclic) quiver Q and an ideal $\mathfrak{a} \subset kQ$ such that $(kQ/\mathfrak{a})$-mod $\cong \Lambda$-mod.

More precisely, given $\mathfrak{r} := \text{rad}(\Lambda)$:

- Vertices of Q correspond to summands of $\Lambda/\mathfrak{r}$ (simple modules).
- Arrows of Q correspond to summands of $\mathfrak{r}/\mathfrak{r}^2$.

Next time...

- Powerful production of projective modules!
- The mystique of minimal morphisms!
- Daring deeds of duality!

You won't want to miss it!