Morning Session

1. If $M$ is a manifold with boundary, then the double of $M$ is defined by identifying two copies of $M$ along their boundaries by the identity map. Let $M = D^2 - \bigcup_i D_\epsilon(x_i)$ where $\{D_\epsilon(x_i)\}$ are $n$ mutually disjoint open discs of radius $\epsilon$ in the interior of $D^2$ centered at $\{x_i\}$. Let $W$ be the double of $M$. Determine the fundamental group and Euler characteristic of $W$.

Solution. In general, given two compact manifolds $X,Y$ identified on a closed connected submanifold $A$, the Euler characteristic is $\chi(X) + \chi(Y) - \chi(A)$. The Euler characteristic of a closed disc is 1, while the Euler characteristic of a circle is 0. Considering $D^2$ as the union of $M$ and $n$ closed discs, which intersect on $n$ circles,

$$\chi(D^2) = \chi(M) + n - 0$$

implies

$$\chi(W) = \chi(M) + \chi(M) - 0 = 2 - 2n.$$ 

□

2. Let $U_1, U_2, \ldots$ be a countable open covering of a metric space $X$. A refinement is an open covering $V_1, V_2, \ldots$ of $X$ such that for each $i$, $V_i \subset U_j$ for some $j$. Show that there exists a refinement $V_1, V_2, \ldots$ which is star-finite i.e., for each $i$, $V_i \cap V_j \neq \emptyset$ for at most finitely many values of $j$.

Solution. $X$ is a metric space so in particular is paracompact Hausdorff. Thus there exists a partition of unity $\{\phi_n\}_n$ on $X$ dominated by $\{U_n\}_n$. By definition, the family $\{\text{Supp } \phi_n\}$ is locally finite. Therefore for each $n$ let $V_n$ denote the interior of $\{\text{Supp } \phi_n\} \subset U_n$. □

3. Let $M$ be a compact smooth manifold of dimension $n$, and let $f : M \to \mathbb{R}^n$ be a smooth map. Show that $f$ has a singular point.

Solution. Assume $f$ has no singular points. In other words, $df_x : T_x M \to \mathbb{R}^n$ is an isomorphism for all $x \in M$. So $f$ is a local diffeomorphism. Then since every $x \in M$ has a neighborhood mapping diffeomorphically to $\mathbb{R}^n$, $f(M)$ must be open. Note $M$ is compact implies $f(M)$ is compact, so $f(M) \neq \mathbb{R}^n$. But $f(M)$ is also closed, hence clopen (and non-empty), a contradiction. Therefore $f$ must have a singular point. □
4. Let \( \mathbb{R}P^2 \) and \( T \) denote, in this order, the real projective plane and the torus \( S^1 \times S^1 \). Prove that any map \( f : \mathbb{R}P^2 \to T \) is homotopic to a constant map.

**Solution.** A map \( f : \mathbb{R}P^2 \to T \) induces a homomorphism on the fundamental groups \( f_* : \pi_1(\mathbb{R}P^2) \to \pi_1(T) \), or, \( f_* : \mathbb{Z}_2 \to \mathbb{Z} \times \mathbb{Z} \). Since \( \mathbb{Z}_2 \) cannot inject into \( \mathbb{Z} \times \mathbb{Z} \), \( f_* \) must be the zero map. Particularly, \( f_* \circ \pi_1(\mathbb{R}P^2) = 0 \) implies \( f \) factors through the universal cover of \( T \), which is \( \mathbb{R} \times \mathbb{R} \).

\[
\begin{array}{ccc}
\mathbb{R} \times \mathbb{R} & \xrightarrow{\sim} & \mathbb{R}P^2 \\
& \searrow & \downarrow f \\
& & T
\end{array}
\]

The plane deformation retracts to the origin, which then maps to a single point in \( T \). Therefore composing with the lift gives a homotopy to a constant map. \( \square \)

5. Consider the covering map \( f : S^2 \to \mathbb{R}P^2 \).

Let \( X \) be the homotopy pushout of the diagram

\[
\begin{array}{ccc}
S^2 & \xrightarrow{f} & \mathbb{R}P^2 \\
\downarrow f & & \downarrow \\
\mathbb{R}P^2 & & \\
\end{array}
\]

Calculate the homology groups of \( X \). (Recall that a homotopy pushout of a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow f & & \downarrow \\
Z & & \\
\end{array}
\]

is \( (X \times [0, 1]) \amalg Y \amalg Z/ \sim \) with the quotient topology, where \( \sim \) is the smallest equivalence relation satisfying \((x, 0) \sim f(x), (x, 1) \sim g(x) \) for every \( x \in X \).)

**Solution.** The resulting space is a thickened sphere, where the boundaries are both projective planes. Thus removing any intermediate sphere \( S^2 \times \{c\} \), with \( c \in (0, 1) \), results in two disjoint projective planes, since the thickened part can now retract to the boundary. Let \( R_1 \) denote the outer projective plane when \( S^2 \times \{1\} \) is removed, \( R_2 \) denote the inner projective plane when \( S^2 \times \{\frac{3}{4}\} \) is removed, and \( S \) denote the open set \( S^2 \times \left(\frac{1}{4}, \frac{3}{4}\right) \). To find the homology groups of \( X \), use a Mayer-Vietoris sequence:

\[
\cdots \to 0 \to H_2(S) \to H_2(R_1) \oplus H_2(R_2) \to H_2(X) \to H_1(S) \to H_1(R_1) \oplus H_1(R_2) \to H_1(X) \to 0
\]

becomes

\[
0 \to \mathbb{Z} \to 0 \oplus 0 \to H_2(X) \to 0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to H_1(X) \to 0.
\]

Therefore \( H_2(X) \cong 0 \) and \( H_1(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). The higher homology groups vanish. Finally, \( X \) is path connected so \( H_0(X) \cong 0 \). \( \square \)
Afternoon Session

1. Let $X$ be obtained by gluing two solid tori $D^2 \times S^1$ along their boundary via the map $f : \partial D^2 \times S^1 \to \partial D^2 \times S^1$ given by $f(x, y) = (y^p x, y)$ where $p$ is a fixed positive integer.
(a) For which values of $p$ can $X$ be given the structure of a topological manifold?
(b) Compute $\pi_1(X)$.

Solution.
(a) Define $\partial D^2 \times S^1 \to \partial D^2 \times S^1$ by $(u, v) \mapsto (v^{-p}u, v)$. Then $g \circ f$ and $f \circ g$ are both the identity, so $f$ is a homeomorphism. Therefore, for all values of $p$, $X$ can be given the structure of a topological manifold.
□

(b) Use VanKampen’s Theorem. Each solid torus has fundamental group $\mathbb{Z}$; let $a, b$ be their respective generators. Then for $(x, y)$ on the boundary the inclusion of the first coordinate in each of the tori is trivial, while for the second is the identity. Therefore
$$\pi_1(X) = a\mathbb{Z} * b\mathbb{Z} \cong \mathbb{Z}.$$  

□

2. Consider the space $O_{n+1,2} = \{(x_1, x_2) | <x_1, x_2> \geq 0\} \subset S^n \times S^n$ where $S^n$ is the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$ with standard inner product. Denote by $p : O_{n+1,2} \to S^n$ the projection on the first factor. Prove that there is a section $s : S^n \to O_{n+1,2}$ (i.e. a continuous map $s$ such that $ps = Id$) if and only if $n$ is odd.

Solution. Suppose a section $s : S^n \to O_{n+1,2}$ exists. Since $(x, x) \neq 0$ for any $x \in S^n$, $s$ followed by projection to the second factor gives a continuous map $S^n \to S^n$ with no fixed points. When $n$ is even, this is not possible since the map would induce a non-vanishing vector field on $S^n$, which is not possible.

Conversely, if $n$ is odd, define
$$s : (x_1, x_2, \ldots, x_{n+1}) \mapsto (x_2, -x_1, x_3, -x_4, \ldots, x_{n+1}, -x_n)$$
where coordinates are given in the ambient space $\mathbb{R}^{n+1}$. □

3. Let $X, Y$ be topological spaces with $Y$ compact. Let $p : X \times Y \to X$ be the projection to the first factor. Show that $p$ maps each closed subset of $X \times Y$ onto a closed subset of $X$.

Solution. Let $W$ be a closed set in $X \times Y$, and assume $u \in X$ is a limit point for $p(W)$, with $u \notin p(W)$. Then any neighborhood of $u$ meets $p(W)$ and in fact, while $\{u\} \times Y$ is disjoint from $W$, $(U \times Y) \cap W = \emptyset$, for any neighborhood $U$ of $u$. For each $(u, y) \in \{u\} \times Y$, there is a basic neighborhood $U \times V$ which is disjoint from $W$; otherwise $(u, y)$ is a limit point of $W$, hence contained in $W$, a contradiction. A cover of $\{u\} \times Y$ with such neighborhoods has a finite subcover $\bigcup_{i=1}^{n} (U_i \times V_i)$ because $\{u\} \times Y$ is compact.
But
\[
\left(\bigcap_{i=1}^n U_i\right) \times Y \subset \bigcup_{i=1}^n (U_i \times V_i)
\]
is a neighborhood of \(\{u\} \times Y\) which does not intersect \(W\), and that is a contradiction. Conclude if \(u \notin p(W)\), then \(u\) cannot be a limit point for \(p(W)\), so \(p(W)\) is closed. \(\square\)

4. Let \(X\) be the union of the three coordinate axes in \(\mathbb{R}^3\). Calculate the homology of \(\mathbb{R}^3 - X\).

**Solution.** Write \(Y = \mathbb{R}^3 - X\). Since the origin is removed, \(Y\) retracts to a sphere with six discs removed, or, a disc with five discs removed. Consider \(D^2\) as the union of \(Y\) with five disjoint discs; the intersection is on five disjoint circles. Use a Mayer-Vietoris sequence to calculate homology:

\[
\cdots \to 0 \to H_2(Y) \oplus 0 \to 0 \to \mathbb{Z}^5 \to H_1(Y) \oplus 0 \to 0
\]

implies \(H_i(Y) = 0\) for \(i \geq 2\), \(H_1(Y) \cong \mathbb{Z}^5\). Finally, \(Y\) is path connected, so \(H_0(Y) \cong \mathbb{Z}\). \(\square\)

5. Let \(S^2 \subset \mathbb{R}^3\) be the standard unit sphere and

\[
X = \{(x, y, z) \in S^2 : y^2z = x^3 - xz^2\}
\]

Is \(X\) a smooth submanifold of \(\mathbb{R}^3\)?

**Solution.** Define \(f : \mathbb{R}^3 \to \mathbb{R}^2\) by

\[
(x, y, z) \mapsto (x^2 + y^2 + z^2, y^2z - x^3 + xz^2).
\]

Then \(X\) is a smooth submanifold of \(\mathbb{R}^3\) if \((1, 0)\) is a regular value. The Jacobian is

\[
df_{(x,y,z)} = \begin{pmatrix}
2x & 2y & 2z \\
-3x^2 + z^2 & 2yz & y^2 + 2xz
\end{pmatrix}
\]

which has rank < 2 when the \(2 \times 2\) minors vanish, i.e.,

\[
4xyz + 6x^2y - 2yz^2 = 0
\]

\[
2xy^2 + 10xz^2 - 2z^3 = 0
\]

\[
2y^3 + 4xyz - 4yz^2 = 0.
\]

By factoring equations in the system, deduce from the third equation if \(y = 0\), then \(x = \pm \sqrt{\frac{1}{6}}, z = \pm \sqrt{\frac{5}{6}}\). If \(x = -z \neq 0\) then \(z = \pm \sqrt{\frac{1}{6}}, y = \pm \sqrt{\frac{2}{3}}\) and if \(x = -z = 0\) then \(y = \pm 1\). Finally, from the third equation \(3x = z\) implies \(y = 0\), which was already considered. In the first equation, if \(4xz + 6x - 2z^2 = 0\) then \(x = 0\) implies \(y = \pm \sqrt{\frac{2}{3}}, z = \pm \sqrt{\frac{1}{3}}\), or \(x = 1\) implies \(y = z = 0\). And, if the second equation vanishes then using the defining map for \(X\), the system

\[
z(z^2 - 5x^2) = x(1 - x^2 - z^2)
\]

\[
x(x^2 - z^2) = z(1 - x^2 - z^2)
\]
implies $z = \pm \sqrt{3}x, y = \pm \sqrt{2\sqrt{3}x}$. The condition that the point lie on $S^2$ then gives $x = \pm \sqrt{\frac{2\pm \sqrt{3}}{8}}, y = \pm \sqrt{\frac{2\sqrt{3}+3}{4}}, z = \pm \sqrt{\frac{6\pm \sqrt{3}}{8}}$.

Now, each of the above points must satisfy the entire system
\[
\begin{align*}
x^2 + y^2 + z^2 &= 1 \\
y^2z &= x^3 - xz^2 \\
4xyz + 6x^2y - 2yz^2 &= 0 \\
2xy^2 + 10x^2z - 2z^3 &= 0 \\
2y^3 + 4xyz - 4yz^2 &= 0.
\end{align*}
\]

But they don’t. Conclude the Jacobian of $f$ always has maximum rank, in particular at points in $f^{-1}(1,0)$. Therefore $(1,0)$ is a regular value, which implies $X$ is a submanifold of $\mathbb{R}^3$. $\square$