Problem 1. Say that a metric space \( X \) has property (A) if the image of every continuous function \( f: X \to \mathbb{R} \) is an interval, which may be open, closed, or half-open. Prove that \( X \) has property (A) if and only if it is connected.

Solution. Assert rays are also intervals, since they are homeomorphic to them. Suppose \( X \) is not connected, so has a nontrivial separation \( X = A \sqcup B \). Let \([a, b]\) be a closed interval in \( \mathbb{R} \) such that \( a \neq b \). Urysohn’s lemma gives a continuous map \( X \to [a, b] \) such that \( f(x) = a \) for all \( x \in A \) and \( f(x) = b \) for all \( x \in B \). Then \( f(X) = \{a, b\} \) and so \( X \) cannot have property (A).

Conversely, suppose \( X \) is connected and assume \( f(X) \) is not an interval, for some continuous \( f: X \to \mathbb{R} \). In particular, \( f(X) \) is not connected, so has a nontrivial (relatively) clopen set \( Z \). Then \( f^{-1}(Z) \) is clopen in \( X \) by continuity of \( f \), and is nonempty by choice of \( Z \). Furthermore, \( f^{-1}(Z) \) is strictly contained in \( X \) or else \( f(X) = Z \), contradicting the choice of \( Z \). Conclude \( f^{-1}(Z) \) is a nontrivial clopen set in \( X \). But then that contradicts \( X \) connected. So property (A) must hold. \( \square \)

Problem 2. Consider \( \text{SL}_n \mathbb{R} \) as a group and as a topological space with the topology induced from \( \mathbb{R}^{n^2} \). Show that if \( H \subset \text{SL}_n \mathbb{R} \) is an abelian subgroup, then the closure \( \overline{H} \) of \( \text{SL}_n \mathbb{R} \) is also an abelian subgroup.

Solution. Matrix multiplication and addition are continuous functions from \( \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2} \). Suppose \( A \) is a limit point for \( H \). So any neighborhood of \( A \) contains a point in \( H \). In particular, for any \( \epsilon \) there exists a \( \delta \)-neighborhood of \( A \) containing \( A' \in H \) such that for \( B \in H \),

\[
\|AB - A'B\| < \frac{\epsilon}{2} \\
\|BA' - BA\| < \frac{\epsilon}{2}
\]

By the Triangle Inequality,

\[
\|AB - BA\| \leq \|AB - A'B\| + \|BA' - BA\| < \epsilon.
\]

If \( B \) is a limit point of \( H \) as well, then the same argument applies since \( A \) and \( B \) each commute with everything in \( H \). \( \square \)

Problem 3. Recall that the complex projective space \( \mathbb{C}P^d \) is the quotient space of \( \mathbb{C}^{d+1} \setminus \{0\} \) under the equivalence relation \( x \sim y \) if and only if there is \( \lambda \in \mathbb{C} \) with \( x = \lambda y \). Prove that \( \mathbb{C}P^d \) is a compact, connected, orientable manifold of dimension \( 2d \).
Solution. $\mathbb{C}^{d+1} \setminus \{0\}$ is homotopy equivalent to $S^{2d+1} \subset \mathbb{R}^{2(d+1)}$, so $\mathbb{C}P^d$ is equivalently constructed as a quotient of $S^{2d+1}$. $S^{2d+1}$ is compact and connected then implies $\mathbb{C}P^d$ is compact and connected.

In general, $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a real $2n$-cell via the quotient map $S^{2n-1} \to \mathbb{C}P^{n-1}$. This means $\mathbb{C}P^d$ is a manifold with CW-structure consisting of exactly one cell in each even real dimension, up to $2d$. In particular, its first homology group is trivial. As the abelianization of the fundamental group, this implies $\pi_1(\mathbb{C}P^d)$ is trivial. This implies $\mathbb{C}P^d$ is its own universal cover, so must be orientable, and it is simply connected. □

**Problem 4.** Consider the 2-dimensional torus $\mathbb{T}^2$ and the topological space

$$X = \mathbb{T}^2 \times [-1, 1] / \sim$$

where $(x, t) \sim (x', t')$ if either $(x, t) = (x', t')$, or $t = t' \in \{-1, 1\}$. Compute $H_*(X; \mathbb{Z})$.

**Solution.** Write $X = (\mathbb{T}^2 \times [-1,0] / \sim) \cup (\mathbb{T}^2 \times [0,1] / \sim)$ and let $U$ and $V$ denoted the respective components. Each of $U$ and $V$ is homotopy equivalent to a point because of the equivalence relation in defining $X$. Their intersection is a torus, $T$. Use a Mayer-Vietoris sequence to compute the homology:

$$\cdots \to 0 \to H_3(X) \to H_2(T) \to H_2(U) \oplus H_2(V) \to H_2(X) \to H_1(T) \to \cdots \to H_1(U) \oplus H_1(V) \to H_1(X) \to H_0(T) \to H_0(U) \oplus H_0(V) \to H_0(X) \to 0.$$  

Then the homology groups are

$$H_3(X) \cong H_2(T) \cong \mathbb{Z},$$

$$H_2(X) \cong H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z},$$

$$H_1(X) \cong 0, \text{ and}$$

$$H_0(X) \cong \mathbb{Z},$$

because $X$ is connected. The higher homology groups vanish. □

**Problem 5.** Let $S^4$ be the 4-dimensional sphere and suppose that $\pi : S^4 \to M$ is a local homeomorphism onto an orientable manifold. Prove that $\pi$ is a homeomorphism and give an example showing that the orientability condition is necessary.

**Solution.** First, $S^4$ is compact Hausdorff, so $\pi$ must be a cover. Since $S^4$ is simply connected, $S^4$ must then be the universal cover of $M$. $S^4$ is path connected and locally path connected, and the universal cover is regular, so $M \cong S^4/G$ where $G$ acts freely and properly discontinuously on $S^4$. The only nontrivial free and properly discontinuous group action on $S^4$ is $\mathbb{Z}_2$ because 4 is even. But $S^4/\mathbb{Z}_2 \cong \mathbb{R}P^4$ is nonorientable. Since $M$ is orientable, it must be the quotient of the trivial action on $S^4$, hence $M \cong S^4$. $\mathbb{R}P^4$ gives the desired counterexample when $M$ is not orientable. □
Afternoon Session

Problem 1. Let $G$ be a cyclic group, $G \acts S^1$ an effective action by rotations and endow the quotient $S^1/G$ with the quotient topology. Prove that $S^1/G$ is $T_0$ if and only if $G$ is finite.

Solution. Suppose $G$ is infinite. Rotation must be by $2\pi q$, where $q$ is irrational, because $G$ is cyclic and must act effectively on $S^1$. Assume $S^1/G$ is $T_0$. Let $x \neq y \in S^1/G$ and $U \subset S^1/G$ open with $x \in U$, $y \notin U$. The preimage of $U$ covers $S^1$, hence intersects the preimage of $y$. But then that implies $y \in U$, a contradiction. So $S^1/G$ cannot be $T_0$.

Conversely, let $n < \infty$ denote the order of $G$. Choose $x, y \in S^1/G$. The preimage $K$ of $\{x, y\}$ consists of $2n$ points. $S^1$ is a metric space with the induced metric $d$ from $\mathbb{R}^2$. Put $\delta = \min_{x' \neq y'} d(x', y')$ and $\epsilon = \frac{2}{3} \delta$. The $\epsilon$-balls centered at each point of $K$ with $S^1$ are disjoint, as are their images in $S^1/G$. This means $S^1/G$ is in fact Hausdorff, hence $T_0$. □

Problem 2. Consider the standard sphere $S^2 = \{x \in \mathbb{R}^3, \|x\| = 1\}$ and

$$T^1S^2 = \{(x, y) \in S^2 \times S^2, x \perp y\}$$

with the induced topology. Prove that $T^1S^2$ is homeomorphic to $SO_3$.

Solution. Let $e_1, e_2$ and $e_3$ denote the standard unit vectors in $\mathbb{R}^3$. Suppose $(x, y) \in T^1S^2$. Regarding $x$ and $y$ as vectors, there is a unique rotation $T_{x,y} \in SO_3$ such that $T_{x,y}(e_1) = x, T_{x,y}(e_2) = y$. Define $h : T^1S^2 \to SO_3$ by $h : (x, y) \mapsto T_{x,y}$. Rotation is continuous, so a neighborhood of $T \in SO_3$ maps $e_1, e_2$ to a neighborhood of $x, y$. Hence $h$ is continuous. Define the inverse $k : SO_3 \to T^1S^2$ by $k : T \mapsto (T(e_1), T(e_2))$. By the same argument $k$ is continuous, and by construction $h$ and $k$ are inverses to eachother. Therefore $T^1S^2 \cong SO_3$. □

Problem 3. Suppose that $M^d \subset \mathbb{R}^n$ is a $d$-dimensional smooth submanifold of $n$-dimensional Euclidean space. Prove that $\mathbb{R}^n \setminus M^d$ is simply connected if $n - d \geq 3$.

Solution. A loop $\gamma \subset \mathbb{R}^n$ is homotopy equivalent to $S^1 \subset P$, where $P$ is a plane. If $\gamma$ is not contractible in $\mathbb{R}^n \setminus M^d$ then $P$ must intersect $M^d$ in $\mathbb{R}^n$. If $n - d \geq 3$ then $d + 2 < n$. Therefore, if $P \cap M^d \neq \emptyset$ then at no point can the intersection be transversal. A perturbation of $P$ makes the intersection empty. Then $\gamma$ is contractible on $P$, so $\mathbb{R}^n \setminus M^d$ is simply connected. □

Problem 4. Let $K$ be the image of an embedding of $S^1 \times D^2$ into $S^3$. Compute $H_1(S^3 \setminus K; \mathbb{Z})$.

Solution. Write $S^3 = K \cup S^3 \setminus K$; since $K$ is the solid torus, the intersection is the boundary $\partial K$, which is a torus, $T$. Use a Mayer-Vietoris equence to find $H_1(S^3 \setminus K; \mathbb{Z})$. Note $K$ is homotopy equivalent to $S^1$, hence has the same homology groups.

$$\cdots \to 0 \to H_2(S^3) \to H_1(T) \to H_1(K) \oplus H_1(S^3 \setminus K) \to H_1(S^3) \to \cdots$$
becomes
\[ 0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus H_1(S^3 \setminus K) \to 0 \]
and so \( H_1(S^3 \setminus K) \cong \mathbb{Z} \). \( \square \)

**Problem 5.** Suppose \( X \) is a connected compact CW-complex. Prove that \( H_1(X; \mathbb{Z}) \) is finite if and only if every map \( X \to S^1 \) is homotopic to a constant map.

**Solution.** Any map \( X \to S^1 \) induces a map on homology, \( H_i(X) \to H_i(S^1) \). Consider the double complex
\[
\begin{array}{cccc}
0 & \to & H_2(S^2) & \to & H_1(S^1) & \to & H_0(S^1) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \to & H_3(X) & \to & H_2(X) & \to & H_1(X) & \to & H_0(X) & \to & 0
\end{array}
\]
where the rows are exact and the squares commute (since \( H_i \) is a functor). If \( H_1(X) \) is finite then it cannot map non-trivially onto \( \mathbb{Z} \), so \( H_1(X) \to H_1(S^1) \) is the zero map. Since \( H_0(S^1) \cong H_0(X) \cong \mathbb{Z} \), by commutativity \( H_1(X) \to H_0(X) \) must be the zero map. Then every 1-ball in \( X \) has a boundary; since \( X \) is connected, it is thus contractible. So \( X \to S^1 \) factors through a point, which maps to a constant.

Conversely, if \( f : X \to S^1 \) is homotopic to a constant map, then the induced map on fundamental groups, and hence first homology groups, is trivial. Since this works for any such \( f \), \( H_1(X) \) must be either finite or trivial. \( \square \)