1. Normalization of the Governing Equation

Weak statement: $$\int\left[\frac{\underline{J}\bullet\delta\underline{I}}{M}+\frac{j\delta i}{m}\right]dA=-\delta G \quad (1)$$

After expansion: $$\int\left[\frac{\underline{J}\bullet\delta\underline{I}}{M}+\frac{v_n\delta r_n}{m}+\frac{v_n\nabla\bullet(\underline{I})}{m}+\frac{\nabla\bullet\underline{J}\delta r_n}{m}+\frac{\nabla\bullet\underline{J}\nabla\bullet(\underline{I})}{m}\right]dA=-\delta G \quad (2)$$

Matrix form: $$H\dot{q}=f \quad (3)$$

$$A\begin{bmatrix} H_{11} & \cdots & & \\ \cdots & \cdots & & \\ & & H_{44} & \cdots \\ & & \cdots & \cdots \end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \cdots \\ J_1 \\ \cdots\end{bmatrix}=\gamma_s\begin{bmatrix}C_1\\ \cdots\\ \cdots\\ \cdots\end{bmatrix}+\left(w_e+g\right)l_0\begin{bmatrix}C_2\\ \cdots\\ \cdots\\ \cdots\end{bmatrix}$$

where $A \sim l_0^2/2$ is the area of the each triangle element, and l_0 is the characteristic side length of the triangle; m is the mobility of interface migration; γ_s, w_e and g are surface energy, elastic energy and phase free energy difference; C_1, C_2 and the other terms in the two brackets on RHS are some dimensionless coefficients.

After calculation, we know H_{11} has the form of $\alpha^2/12/m$, where α is the first component of the normalized normal direction of the element, and m is the evaporation-condensation rate; the other similar terms related to $v_n\delta r_n/m, v_n\nabla\bullet(\underline{I})/m, \nabla\bullet\underline{J}\delta r_n/m$ and $\nabla\bullet\underline{J}\nabla\bullet(\underline{I})/m$ have the same form. H_{44} has the form of $1/M/24+1/[2m(\mathrm{a}^2+\mathrm{b}^2\text{-}2\mathrm{ab})]$, where a,b are the two lengths of the triangle, and M is the mobility of atoms on surface; the other similar terms related to $\underline{J}\bullet\delta\underline{I}/M$ have the same form.

Define the following lengths L, b and time scales τ_M, τ_m:

$$\begin{cases} L=\dfrac{\gamma_s E}{\sigma^2} \\ b=\dfrac{\gamma_s}{g} \\ Q=\dfrac{L}{b}=\dfrac{\gamma_s E}{\sigma^2 b}\end{cases} \quad \text{and} \quad \begin{cases} \tau_M=\dfrac{L^4}{M\gamma_s} \\ \tau_m=\dfrac{L^2}{m\gamma_s} \\ P=\dfrac{\tau_M}{\tau_m}=\dfrac{L^2 m}{M}=\left(\dfrac{\gamma_s E}{\sigma^2}\right)^2\dfrac{m}{M}\end{cases}$$

where E is the Young's module and the σ is the applied biaxial stress on the film.

Normalize the coordinates x, y, z by L, and the time t by τ_m, we can get the following normalized governing equation, namely,

$$\begin{bmatrix} H'_{11} & \cdots & & \\ \cdots & \cdots & & \\ & & H'_{44} & \cdots \\ & & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \dot{x}_1^n \\ \cdots \\ J_1^n \\ \cdots \end{bmatrix} = \begin{bmatrix} C_1 \\ \cdots \\ \cdots \\ \cdots \end{bmatrix} + (1+Q) l_0^n \begin{bmatrix} C'_2 \\ \cdots \\ \cdots \\ \cdots \end{bmatrix}$$

where $H'_{11} = \alpha^2 / 12$ and H'_{44} has the form of $C_3 + C_4 P$, which are both dimensionless.

All of the physical parameters are now integrated into the two dimensionless parameters P and Q. The competition between elastic energy and phase difference is characterized by P; and Q determines the domination of diffusion process or evaporation/condensation process. The normalization of the governing equation is very useful. We can tune some dimensionless parameters like P and Q to observe the effects upon different material systems, without surveying the whole parameter picture.

Following shows the units of all the physical quantities used in the above equations:

$$\dot{x} : \frac{m}{s}, \qquad J : \frac{m^3}{s} / m = \frac{m^2}{s}, \qquad \gamma_s : \frac{Nm}{m^2} = \frac{N}{m}, \qquad w_e, g : \frac{Nm}{m^3} = \frac{N}{m^2}$$

$$\text{Mobility M} : \frac{Nm}{m^3}, \qquad \text{Mobility m} : \frac{Nm}{m^3}$$