Schubert Polynomials and Schur Polynomials
Schubert Polynomials were introduced by Lascoux and Schützenberger to study the cohomology of the complete flag variety [LS82].

They form a $\mathbb{Z}$-linear basis of $\mathbb{Z}[x_1, x_2, \ldots]$
The Definition of $\mathcal{S}_w$

- Start with the **longest** permutation in $S_n$

  
  \[ w_0 = n \ldots n-1 \ldots 1 \quad \mathcal{S}_{w_0} := x_1^{n-1}x_2^{n-2} \ldots x_{n-1} \]

- The rest are defined recursively by **divided difference** operators:

  \[
  \partial_i f := \frac{f - s_i \cdot f}{x_i - x_{i+1}} \quad \text{and} \quad \mathcal{S}_{ws_i} := \partial_i \mathcal{S}_w \text{ if } w(i) > w(i+1)
  \]

Anna Weigandt  
Prism Tableaux and ASMs
Schubert Polynomials for $S_3$

Schubert polynomials are well defined and stable under the inclusion

$$S_n \hookrightarrow S_\infty.$$
**Goal**: We want to understand the coefficients of the monomials of $\mathfrak{S}_w$ using a combinatorial model.
**Goal:** We want to understand the coefficients of the monomials of $\mathcal{S}_w$ using a combinatorial model.

**Many earlier models:** A. Kohnert, S. Billey-C. Jockusch-R. Stanley, S. Fomin-A. Kirillov, S. Billey-N. Bergeron, ...
A permutation $u \in S_n$ is **Grassmannian** if it has a unique descent, i.e. a position $i$ so that $u(i) > u(i + 1)$.
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$u = 1 2 3 5 7 10 11 \vert 4 6 8 9$
A permutation $u \in S_n$ is Grassmannian if it has a unique descent, i.e. a position $i$ so that $u(i) > u(i + 1)$.

$u = 1 \ 2 \ 3 \ 5 \ 7 \ 10 \ 11 \mid 4 \ 6 \ 8 \ 9$

If $u \in S_n$ is Grassmannian, then $\Psi_u$ is a Schur polynomial. These are typically indexed using partitions, weakly decreasing sequences of nonnegative integers.

$$\lambda = (\lambda_1, \ldots, \lambda_k).$$
A permutation $u \in S_n$ is **Grassmannian** if it has a unique **descent**, i.e. a position $i$ so that $u(i) > u(i + 1)$.

$$u = 1 \ 2 \ 3 \ 5 \ 7 \ 10 \ 11 \ \mid \ 4 \ 6 \ 8 \ 9$$

If $u \in S_n$ is Grassmannian, then $\mathcal{S}_u$ is a **Schur polynomial**. These are typically indexed using **partitions**, weakly decreasing sequences of nonnegative integers.

$$\lambda = (\lambda_1, \ldots, \lambda_k).$$

Schur polynomials form a basis the ring of symmetric polynomials.
Let $u = 1 \ 2 \ 3 \ 5 \ 7 \ 10 \ 11 \ | \ 4 \ 6 \ 8 \ 9$. 

![Diagram of a permutation as a Young diagram]
Let $u = 1 2 3 5 7 10 11 \mid 4 6 8 9$. 
Let $u = 1 2 3 5 7 10 11 \mid 4 6 8 9$. 
Let $u = 1 \ 2 \ 3 \ 5 \ 7 \ 10 \ 11 \ | \ 4 \ 6 \ 8 \ 9$.

$\lambda = (4, 4, 2, 1)$ and $d = 7$. 
Write $[\lambda, d]_g$ for the inverse map.

$$s_\lambda(x_1, x_2, \ldots, x_d) = \mathcal{S}_{[\lambda, d]_g}(x_1, x_2, \ldots)$$
Reverse Semistandard Tableaux

\[ \lambda = (4, 4, 2, 1) \]
Reverse Semistandard Tableaux

\[ \lambda = (4, 4, 2, 1) \quad T = \begin{array}{cccc}
2 &   &   & \\
3 & 1 &   & \\
4 & 3 & 2 & 2 \\
6 & 6 & 4 & 4 \\
\end{array} \]
Reverse Semistandard Tableaux

\[ \lambda = (4, 4, 2, 1) \]

\[ T = \begin{array}{cccc}
2 \\
3 & 1 \\
4 & 3 & 2 & 2 \\
6 & 6 & 4 & 4 \\
\end{array} \]

\[ \text{wt}(T) = x_1 x_2^3 x_3^2 x_4^3 x_6^2 \]
Write $\text{RSSYT}(\lambda, d)$ for the set of reverse semistandard tableaux of shape $\lambda$ with labels from the set $[d] = \{1, \ldots, d\}$.
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The **Schur polynomial** $s_\lambda(x_1, \ldots, x_d)$ can be expressed as a sum over $RSSYT(\lambda, d)$

$$s_\lambda(x_1, \ldots, x_d) = \sum_{T \in RSSYT(\lambda, d)} \text{wt}(T).$$
Schur Polynomials

Write $\text{RSSYT}(\lambda, d)$ for the set of reverse semistandard tableaux of shape $\lambda$ with labels from the set $[d] = \{1, \ldots, d\}$.

The Schur polynomial $s_\lambda(x_1, \ldots, x_d)$ can be expressed as a sum over $\text{RSSYT}(\lambda, d)$

$$s_\lambda(x_1, \ldots, x_d) = \sum_{T \in \text{RSSYT}(\lambda, d)} \text{wt}(T).$$

If $\lambda = (3, 1)$ then $s_\lambda(x_1, x_2) = x_1x_2^2 + x_1^2x_2^2 + x_1^3x_2$. 

\[
\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
2 & 2 & 1 \\
\hline
\end{array}
\hspace{1cm}
\begin{array}{|c|c|c|}
\hline
1 & 2 & 2 \\
\hline
2 & 2 & 2 \\
\hline
\end{array}
\hspace{1cm}
\begin{array}{|c|c|c|}
\hline
1 & 2 & 2 \\
\hline
2 & 2 & 2 \\
\hline
\end{array}
\]
Prism Tableaux
Prism tableaux were introduced in joint work with A. Yong to give a tableau based formula for Schubert polynomials [WY15].
Fix $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ and $d = (d_1, \ldots, d_k)$ with $d_i \geq \ell(\lambda^{(i)}).

A **prism tableau** for $(\lambda, d)$ is an element of

$$\text{AllPrism}(\lambda, d) = \text{RSSYT}(\lambda^{(1)}, d_1) \times \ldots \times \text{RSSYT}(\lambda^{(k)}, d_k).$$
Fix $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ and $d = (d_1, \ldots, d_k)$ with $d_i \geq \ell(\lambda^{(i)})$.

A prism tableau for $(\lambda, d)$ is an element of

$$\text{AllPrism}(\lambda, d) = \text{RSSYT}(\lambda^{(1)}, d_1) \times \ldots \times \text{RSSYT}(\lambda^{(k)}, d_k).$$

Let $\lambda = ((1), (3, 2), (2, 1, 1))$ and $d = (2, 5, 6)$.

$$\left(\begin{array}{c}
1 \\
2 2 \\
3 3 2 \\
\hline
1 \\
2 \\
6 3
\end{array}\right)$$
Let $\lambda = ((1), (3, 2), (2, 1, 1))$ and $d = (2, 5, 6)$.

\[
\begin{pmatrix}
1, & \begin{array}{ccc}
2 & 2 & \\
3 & 3 & 2
\end{array}, & \begin{array}{c}
1 \\
2 \\
6 \\
3
\end{array}
\end{pmatrix}
\]
Let $\lambda = ((1), (3, 2), (2, 1, 1))$ and $d = (2, 5, 6)$.
The weight monomial of $\mathcal{T}$ is

$$\text{wt}(\mathcal{T}) = \prod_{i} x_i^{n_i}$$

where $n_i$ is the number of antidiagonals which have a label (in any color) of value $i$. 
The **weight monomial** of $\mathcal{T}$ is

$$\text{wt}(\mathcal{T}) = \prod_{i} x^{n_i}_i$$

where $n_i$ is the number of **antidiagonals** which have a label (in any color) of value $i$. 
The weight monomial of $T$ is

$$\text{wt}(T) = \prod_i x_i^{n_i}$$

where $n_i$ is the number of antidiagonals which have a label (in any color) of value $i$. 

$$\text{wt}(T) = x_1^2 x_2^3 x_3^3 x_6$$
\( \mathcal{T} \) is **minimal** if the degree of \( \text{wt}(\mathcal{T}) \) is the minimum possible degree among all prism tableaux for \((\lambda, d)\).
\( \mathcal{T} \) is \textbf{minimal} if the degree of \( \text{wt}(\mathcal{T}) \) is the minimum possible degree among all prism tableaux for \((\lambda, d)\).

\[
\mathcal{T} = \begin{array}{c}
1 & 1 \\
3 & \end{array} \\
\mathcal{T}' = \begin{array}{c}
2 & 1 & 1 \\
3 & \end{array}
\]

\[
\text{wt}(\mathcal{T}) = x_1^2 x_3 \\
\text{wt}(\mathcal{T}') = x_1^2 x_2 x_3
\]
Unstable Triples

Labels \((\ell_c, \ell_d, \ell'_e)\) in the same antidiagonal \((\nearrow)\) of \(T\) form an unstable triple if \(\ell < \ell'\) and the tableau \(T'\) obtained by replacing \(\ell_c\) with \(\ell'_c\) is itself a prism tableau.

\[
\begin{array}{cccc}
1 \\
21 & 2 \\
32 & 3 & 2 \\
6 & 3
\end{array}
\]
Unstable Triples

Labels $(\ell_c, \ell_d, \ell'_e)$ in the same antidiagonal (↗) of $\mathcal{T}$ form an **unstable triple** if $\ell < \ell'$ and the tableau $\mathcal{T}'$ obtained by replacing $\ell_c$ with $\ell'_c$ is itself a prism tableau.

\[
\begin{array}{cccc}
1 \\
212 \\
3232 \\
63 \\
\end{array}
\quad \rightarrow \quad 
\begin{array}{cccc}
1 \\
212 \\
3332 \\
63 \\
\end{array}
\]
Unstable Triples

Labels \((\ell_c, \ell_d, \ell'_e)\) in the same antidiagonal \(\leftrightarrow\) of \(T\) form an unstable triple if \(\ell < \ell'\) and the tableau \(T'\) obtained by replacing \(\ell_c\) with \(\ell'_c\) is itself a prism tableau.

\[
\begin{array}{c}
1 \\
212 \\
3232 \\
63 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
1 \\
212 \\
3332 \\
63 \\
\end{array}
\]

\[\text{wt}(T) = x_1^2x_2^3x_3^3x_6\]

\[\text{wt}(T') = x_1^2x_2^3x_3^3x_6\]
Let $\text{Prism}(\lambda, d)$ be the set of minimal prism tableaux for $(\lambda, d)$ which have no unstable triples.

$$A_{\lambda,d} = \sum_{\mathcal{T} \in \text{Prism}(\lambda,d)} wt(\mathcal{T}).$$
Let $\text{Prism}(\lambda, d)$ be the set of minimal prism tableaux for $(\lambda, d)$ which have no unstable triples.

$$\mathcal{A}_{\lambda, d} = \sum_{\mathcal{T} \in \text{Prism}(\lambda, d)} \text{wt}(\mathcal{T}).$$

\[
\lambda = ((1), (1)) \quad d = (1, 2) \quad \begin{array}{c} 1 \\ 2 \end{array} \quad \begin{array}{c} 1 \\ 1 \end{array}
\]

$$\mathcal{A}_{\lambda, d} = x_1 x_2 + x_1^2$$
Some Special Cases

When $\lambda = (\lambda)$ and $d = (d)$,

$$A_{\lambda,d} = s_{\lambda}(x_1, \ldots, x_d).$$
Some Special Cases

When $\lambda = (\lambda)$ and $d = (d)$,

$$A_{\lambda,d} = s_\lambda(x_1, \ldots, x_d).$$

In [WY15], we showed how to associate to each permutation $w \in S_n$ a tuple $(\lambda_w, d_w)$ so that

$$A_{\lambda_w, d_w} = S_w.$$
Some Special Cases

When \( \lambda = (\lambda) \) and \( d = (d) \),

\[ \mathcal{A}_{\lambda,d} = s_\lambda(x_1, \ldots, x_d). \]

In \([WY15]\), we showed how to associate to each permutation \( w \in S_n \) a tuple \((\lambda_w, d_w)\) so that

\[ \mathcal{A}_{\lambda_w,d_w} = S_w. \]

In general, \( \mathcal{A}_{\lambda,d} \) is a \textbf{multiplicity free} sum of Schubert polynomials, determined by an \textbf{alternating sign matrix}. 
Alternating Sign Matrices
A matrix $A$ is an alternating sign matrix (ASM) if:

- $A$ has entries in $\{-1, 0, 1\}$
- Rows and columns sum to 1
- Non-zero entries alternate in sign along rows and columns

$$
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & -1 & 1 \\
1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
$$

Let $\text{ASM}(n)$ be the set of $n \times n$ ASMs.
Define the \textbf{corner sum function} $r_A(a, b) = \sum_{i=1}^{a} \sum_{j=1}^{b} a_{ij}$. 
Define the **corner sum function** $r_A(a, b) = \sum_{i=1}^{a} \sum_{j=1}^{b} a_{ij}$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad r_A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$
Order the set of corner sum matrices by entrywise comparison.

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\quad r_A = \begin{pmatrix}
0 & 1 & 1 \\
0 & 1 & 2 \\
1 & 2 & 3
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & 0
\end{pmatrix}
\quad r_B = \begin{pmatrix}
0 & 1 & 1 \\
1 & 1 & 2 \\
1 & 2 & 3
\end{pmatrix}
\]

\[A \geq B \text{ if and only if } r_A \leq r_B\]
Poset Structure on ASM(3)
Each prism shape \((\lambda, d)\) defines a list of Grassmannian permutations.

\[ A_{\lambda, d} := \sup \{ [\lambda^{(1)}, d_1]_g, \ldots, [\lambda^{(k)}, d_k]_g \}. \]

**Theorem ([Wei17])**

\[ A_{\lambda, d} = \sum_{w \in \text{MinPerm}(A_{\lambda, d})} S_w. \]

\(\text{MinPerm}(A)\) is the set of the **minimum length permutations** above \(A\) in ASM\(\langle n \rangle\).
Example

\[ \lambda = ((1), (1)) \quad d = (1, 2) \]

\[ A_{\lambda, d} = x_1 x_2 + x_1^2 \]

\[ A_{\lambda, d} = \sup \{213, 132\} = ?? \]
Poset Structure on ASM(3)
Example

\[ \lambda = ((1), (1)) \quad d = (1, 2) \]

\[ \mathcal{A}_{\lambda, d} = x_1 x_2 + x_1^2 \]

\[ \mathcal{A}_{\lambda, d} = \text{sup}\{213, 132\} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ \text{MinPerm}(A_{\lambda, d}) = \{312, 231\} \]

\[ \mathcal{A}_{\lambda, d} = \mathcal{S}_{231} + \mathcal{S}_{312} \]
Poset Structure on ASM(3)
Bruhat Order as a Subposet

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Theorem (A. Lascoux-M.-P. Schützenberger, 1996)

ASM(n) is the smallest complete lattice which contains the strong Bruhat order on $S_n$ as a sub-poset.
Theorem (A. Lascoux-M.-P. Schützenberger, 1996)

ASM(n) is the smallest complete lattice which contains the strong Bruhat order on S_n as a sub-poset.

A permutation w is biGrassmannian if both w and w^{-1} have a unique descent.
For any \( A \in \text{ASM}(n) \),

\[ A = \sup \text{biGr}(A) \]

where biGr(A) is the set of maximal biGrassmannians below A in ASM(n).
Plot the 1’s of $A$ as black dots and the $-1$’s as white dots:

Write $D(A)$ for the diagram and $\mathcal{E}ss(A)$ for the essential set.
The Diagram and Essential Set of an ASM

Plot the 1’s of $A$ as black dots and the $-1$’s as white dots:

Write $D(A)$ for the diagram and $\text{Ess}(A)$ for the essential set.
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Write $D(A)$ for the diagram and $\mathcal{E}ss(A)$ for the essential set.
H. Rothe defined permutation diagrams in the 1800’s, [Rot00]. W. Fulton defined the essential set [Ful92].

A. Lascoux generalized diagrams to ASMs in [Las08] and with M. -P. Schützenberger identified the essential set [LS96].

http://www.e-rara.ch/zut/content/pageview/1109842
Theorem ([LS96, For08],...)  

\[ A \text{ is uniquely determined by the restriction of } r_A \text{ to } \mathcal{E}ss(A). \]

Upshot: You can read of biGr\((A)\) directly from the diagram.
BiGrassmannians are exactly the permutations with rectangular diagrams.
Example: \( w = 14235 \) and \( w^{-1} = 13425 \)
Finding Maximal BiGrassmannians below $A$
Finding Maximal BiGrassmannians below $A$
BiGrassmannian Prism Tableaux

$42513 = \sup\{41235, 23415, 14523\}$

$\lambda = ((3), (1, 1, 1), (2, 2))$

$d = (1, 3, 3)$
### BiGrassmannian Prism Tableaux

<table>
<thead>
<tr>
<th>In Prism$(\lambda, d)$</th>
<th>In Prism$(\lambda, d)$</th>
<th>Not minimal</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Tableau" /></td>
<td><img src="image2" alt="Tableau" /></td>
<td></td>
</tr>
<tr>
<td><img src="image3" alt="Tableau" /></td>
<td><img src="image4" alt="Tableau" /></td>
<td></td>
</tr>
<tr>
<td><img src="image5" alt="Tableau" /></td>
<td><img src="image6" alt="Tableau" /></td>
<td></td>
</tr>
</tbody>
</table>

Unstable triple  Unstable triple  Not minimal

---

Anna Weigandt

Prism Tableaux and ASMs
BiGrassmannian Prism Tableaux

\[ S_{42513} = \begin{array}{c}
1111 \\
221 \\
333 \\
\end{array} + \begin{array}{c}
1111 \\
221 \\
332 \\
\end{array} \]

\[ = x_1^3 x_2^2 x_3 + x_1^3 x_2^2 x_3 \]

This is the original prism tableaux model found in [WY15].
Given a permutation $w$ we can form a Grassmannian permutation $w^{(i)}$ by “drawing a line” after position $i$ and “sorting” entries on either side of the line.

Example: $42513 \leftrightarrow 42|513 \leftrightarrow 24|135$
Given a permutation $w$ we can form a Grassmannian permutation $w^{(i)}$ by “drawing a line” after position $i$ and “sorting” entries on either side of the line.

Example: $42513 \leftrightarrow 42|513 \leftrightarrow 24|135$

**Lemma**

$$w = \sup \{ w^{(i)} : w(i) > w(i + 1) \}$$
Parabolic Prism Tableaux

Given a permutation $w$ we can form a Grassmannian permutation $w^{(i)}$ by “drawing a line” after position $i$ and “sorting” entries on either side of the line.

Example: $42513 \mapsto 42|513 \mapsto 24|135$

**Lemma**

$$w = \sup\{w^{(i)} : w(i) > w(i + 1)\}$$

$$42513 = \sup\{4|1235, 245|13\}$$

$$\lambda = ((3), (1, 2, 2)) \quad d = (1, 3)$$
Parabolic Prism Tableaux

Not minimal $\text{In Prism}(\lambda, d)$ $\text{In Prism}(\lambda', d')$

$$\mathfrak{S}_{42513} = x_1^3x_2x_3^2 + x_1^3x_2^2x_3$$
Theorem (W.- 2017+)

*Minimal parabolic prism tableaux (for permutations) do not have unstable triples.*
Theorem (W.- 2017+)

Minimal parabolic prism tableaux (for permutations) do not have unstable triples.
Alternating Sign Matrix Varieties
An **ASM variety** $X_A \subseteq \text{Mat}(n)$ is defined by a list of rank conditions imposed on maximal northwest submatrices. The rank conditions come from the corner sum matrix of $X_A$.

When $w \in S_n$, then $X_w$ is the **matrix Schubert variety** studied by Fulton, Knutson-Miller, and others...

Example: rank 2 matrices in $\text{Mat}(n)$.
The lattice of ASMs determines the containment order of ASM varieties. In particular, if $A = \sup\{A_1, \ldots, A_k\}$ then

$$X_A = \bigcap_{i=1}^{k} X_{A_i}.$$
The lattice of ASMs determines the containment order of ASM varieties. In particular, if \( A = \sup \{ A_1, \ldots, A_k \} \) then

\[
X_A = \bigcap_{i=1}^{k} X_{A_i}.
\]

An ASM variety is irreducible if and only if it is indexed by a permutation [Ful92]. Otherwise, it decomposes as

\[
X_A = \bigcup_{w \in \text{Perm}(A)} X_w
\]

where \( \text{Perm}(A) \) is set of minimal permutations above \( A \).
The group of invertible diagonal matrices act on the space of $n \times n$ matrices by multiplication and endows the coordinate ring of $\text{Mat}(n)$ with a $\mathbb{Z}^n$ grading.

The multidegree is a function from $\mathbb{Z}^n$ graded modules to polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$.

$$C(M; \mathbf{x})$$

Whenever $X$ is stable under the action of $T$, its coordinate ring is a $\mathbb{Z}^n$ graded module. In this case, we write $C(X; \mathbf{x})$. 
The *multidegree* of a matrix Schubert variety is a Schubert polynomial [Ful92, FR03, KM05].

\[ C(\mathcal{X}_w; \mathbf{x}) = \mathcal{S}_w. \]

One goal of of [KM05] was to exhibit a geometrically natural explanation for previous combinatorial formulas for Schubert polynomials.
The multidegree of a matrix Schubert variety is a Schubert polynomial [Ful92, FR03, KM05].

\[ C(X_w; x) = \mathcal{S}_w. \]

One goal of [KM05] was to exhibit a geometrically natural explanation for previous combinatorial formulas for Schubert polynomials.

Pipe dreams label facets of the Stanley-Reisner complex of a degeneration of \( X_w \).
Computing $C(X; x)$

Let $X_\mathcal{I}$ be the coordinate subspace defined by setting the coordinate $z_{ij} = 0$ whenever $(i, j) \in \mathcal{I}$. By normalization

$$C(X_\mathcal{I}; x) = \prod_{(i,j) \in \mathcal{I}} x_i.$$
Computing $C(X; x)$

Let $X_\mathcal{I}$ be the coordinate subspace defined by setting the coordinate $z_{ij} = 0$ whenever $(i, j) \in \mathcal{I}$. By **normalization**

$$C(X_\mathcal{I}; x) = \prod_{(i,j) \in \mathcal{I}} x_i.$$ 

By **additivity**, if $X = \bigcup_{i=1}^n X_i$, then

$$C(X; x) = \sum_{i \in I} C(X_i; x),$$

where the sum is over $X_i$ so that $\text{codim}(X_i) = \text{codim}(X)$. 

**Degeneration** ensures that $C(X; x) = C(\text{init} \prec X; x)$. 

Anna Weigandt  
Prism Tableaux and ASMs
Let \( X_I \) be the coordinate subspace defined by setting the coordinate \( z_{ij} = 0 \) whenever \((i,j) \in I\). By normalization

\[
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\]

By additivity, if \( X = \bigcup_{i=1}^n X_i \), then

\[
C(X; x) = \sum_{i \in I} C(X_i; x),
\]

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Degeneration ensures that \( C(X; x) = C(\text{init}_{\prec} X; x) \).
By additivity,

$$C(X_A, x) = \sum_{w \in \text{MinPerm}(A)} C(X_w, x) = A_{\lambda, d}.$$

Minimal prism tableau for $(\lambda, d)$ label facets of the Stanley-Reisner complex of a degeneration of $X_{A_{\lambda, d}}$. 
M. Fortin.
The macneille completion of the poset of partial injective functions.

L. Fehér and R. Rimányi.
Schur and Schubert polynomials as Thom polynomials - cohomology of moduli spaces.

W. Fulton.
Flags, Schubert polynomials, degeneracy loci, and determinantal formulas.

A. Knutson and E. Miller.
Gröbner geometry of Schubert polynomials.
A. Lascoux.
Chern and Yang through ice.

A. Lascoux and M.-P. Schützenberger.
Polynômes de Schubert.

—.
Treillis et bases des groupes de Coxeter.

H. Rothe.
Uber permutationen.
A. Weigandt.
Prism tableaux for alternating sign matrix varieties.

A. Weigandt and A. Yong.
The prism tableau model for schubert polynomials.
Merci!