LECTURE 38: APPLICATIONS OF CRITICAL POINT THEORY

1. Generalized Sphere Theorem

As a first application of the critical point theory of distance function, we shall prove

**Theorem 1.1** (Grove-Shiohama). *Let \((M, g)\) be a complete simply connected Riemannian manifold with \(K > \frac{1}{4}\) and \(\text{diam}(M, g) \geq \pi\), then \(M\) is homeomorphic to \(S^m\).*

**Remark.** According to Klingenberg’s estimate, if \(\frac{1}{4} < K \leq 1\), then 
\[
\text{diam}(M, g) \geq \text{inj}(M, g) \geq \pi.
\]

So Grove-Shiohama’s theorem implies the sphere theorem.

**Proof.** Let \(p, q \in M\) so that \(\text{dist}(p, q) = \text{diam}(M, g)\). In what follows we will prove that \(p\) has no other critical points. So from the Reeb type theorem we proved last time, \(M\) is homeomorphic to \(S^m\).

Suppose to the contrary, \(\bar{q} \neq q\) is a critical point of \(p\). Let \(\gamma\) be a minimal geodesic from \(q = \gamma(0)\) to \(\bar{q} = \gamma(l)\). By definition of critical points, there exists a minimal geodesic \(\sigma\) from \(\bar{q} = \sigma(0)\) to \(p = \sigma(l')\) so that \(\alpha = \angle(-\dot{\gamma}(l), \dot{\sigma}(0)) \leq \frac{\pi}{2}\). Similarly, there exists minimal geodesics \(\gamma_1, \sigma_1\) from \(p = \gamma_1(0) = \sigma_1(0)\) to \(q = \gamma_1(l'') = \sigma_1(l'')\) so that
\[
\beta = \angle(-\dot{\sigma}(l'), \dot{\sigma}_1(0)) \leq \frac{\pi}{2}, \quad \beta' = \angle(-\dot{\gamma}_1(l''), \dot{\gamma}(0)) \leq \frac{\pi}{2}.
\]

Since \(M\) is compact, there exists \(k > \frac{1}{4}\) so that \(K \geq k\). According to Toporogov comparison theorem (triangle version), there is a geodesic triangle in \(S^m(\frac{1}{\sqrt{k}})\) whose sides have length \(l, l', l''\) while all three angles \(\tilde{\alpha}, \tilde{\beta}, \tilde{\beta}'\) are all no more than \(\frac{\pi}{2}\). Since \(l'' = \text{dist}(p, q) \geq \pi\), the cosine law in \(S^m(\frac{1}{\sqrt{k}})\)
\[
0 > \cos(\sqrt{k}l'') = \cos(\sqrt{k}l) \cos(\sqrt{k}l') + \sin(\sqrt{k}l) \sin(\sqrt{k}l') \cos(\tilde{\alpha})
\]
implies exactly one of \(l\) and \(l'\), say \(l'\), is strictly greater than \(\frac{\pi}{2\sqrt{k}}\), and the other one, \(l\), is strictly smaller than \(\frac{\pi}{2\sqrt{k}}\). It follows that
\[
\cos(\sqrt{k}l'') = \cos(\sqrt{k}l) \cos(\sqrt{k}l') + \sin(\sqrt{k}l) \sin(\sqrt{k}l') \cos(\tilde{\alpha}) > \cos(\sqrt{k}l').
\]
In other words, \(l'' < l'\). This contradicts with the assumption that \(l'' = \text{diam}(M, g)\). \(\square\)
2. The Soul Theorem

As another application, we can prove a weaker version of the soul theorem.

**Lemma 2.1** (Gromov). Let \((M, g)\) be a complete non-compact Riemannian manifold with \(K \geq 0\). Let \(q\) be a critical point of \(p\) and \(\bar{q}\) a point in \(M\) satisfying
\[
\text{dist}(p, \bar{q}) > \lambda \text{dist}(p, q)
\]
for some \(\lambda > 1\). Let \(\gamma_1, \gamma_2\) be minimal geodesics from \(p\) to \(q, \bar{q}\) respectively. Then
\[
\angle(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) \geq \arccos(1/\lambda).
\]

**Proof.** Let \(\sigma_1\) be a minimal geodesic from \(q\) to \(\bar{q}\). Then there exists a minimal geodesic \(\sigma_2\) from \(q\) to \(p\) so that \(\angle(\dot{\sigma}_1(0), \dot{\sigma}_2(0)) \leq \frac{\pi}{2}\). We will denote
\[
\angle(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) = \alpha, \quad \angle(\dot{\sigma}_1(0), \dot{\sigma}_2(0)) = \beta
\]
and
\[
L(\gamma_1) = L(\sigma_2) = l, \quad L(\gamma_2) = l' > \lambda l, \quad L(\sigma_1) = l''.
\]
Applying Toporogov comparison theorem, we can get a triangle in the plane with sides \(l, l', l''\), and corresponding angles \(\tilde{\alpha} \leq \alpha, \tilde{\beta} \leq \beta \leq \frac{\pi}{2}\). It follows that
\[
(l')^2 \leq l^2 + (l'')^2
\]
and thus
\[
(l'')^2 - 2ll' \cos(\tilde{\alpha}) \leq (l')^2 - 2ll' \cos(\alpha) \leq 2l^2 + (l'')^2 - 2ll'' \cos(\alpha).
\]
This implies \(\cos(\alpha) \leq 1/\lambda\) and thus \(\alpha \geq \arccos(1/\lambda)\). \(\square\)

**Corollary 2.2.** Let \((M, g)\) be a complete non-compact Riemannian manifold with \(K \geq 0\), and let \(\lambda > 1\). Then there exists a number \(N(\lambda, m)\) so that if \(q_1, \cdots, q_N\) is a sequence of critical points of \(p\) satisfying
\[
\text{dist}(p, q_{i+1}) > \lambda \text{dist}(p, q_i), \quad i = 1, 2, \cdots, N - 1,
\]
then \(N \leq N(\lambda, m)\).

**Proof.** Let \(\gamma_i\) be a minimizing normal geodesic from \(p\) to \(q_i\). Then according to lemma 2.1, \(\angle(\dot{\gamma}_i(0), \dot{\gamma}_j(0)) \geq \arccos(1/\lambda)\) for all \(i \neq j\). In other words, if we denote \(r = \frac{1}{2} \arccos(1/\lambda)\), then \(B_r(\gamma_i(0)) \subset S_pM\) are disjoint balls in \(S_pM\). Obviously the number of such balls is bounded by a universal constant depending only on \(m\) and \(\lambda\). \(\square\)

Combining with the isotopy lemma, we get

**Corollary 2.3.** Let \((M, g)\) be a complete non-compact Riemannian manifold with \(K \geq 0\). Then given any \(p \in M\), there exists a compact set \(C\) such that \(p\) has no critical points lying outside \(C\). In particular, \(M\) is homeomorphic to the interior of a compact manifold with boundary.
Corollary 2.3 is a special case of the following structural theorem: (Recall that a subset in $A \subset M$ is called totally convex if for any $p, q \in A$, any geodesic $\gamma$ connecting $p$ to $q$ must be contained in $A$. See PSet 3 problem 5(a) for more details.)

**Theorem 2.4 (The Soul Theorem).** Let $(M, g)$ be a complete non-compact Riemannian manifold with $K \geq 0$. Then $M$ contains a closed totally convex submanifold $S$ with $\dim S < \dim M$ such that $M$ is diffeomorphic to the normal bundle over $S$.

The submanifold $S$ in the theorem is called a soul of the $M$. For example, the point $(0, 0, 0)$ is (the only) soul of the paraboloid $z = x^2 + y^2$, while any meridinal circle is a soul of the cylinder $S^1 \times \mathbb{R}$. Proof of the soul theorem will appear in student’s project.

**Remark (About History).**
- The Soul theorem was proved by Cheeger-Gromoll in 1972.
- Gromoll-Meyer 1969: If $K > 0$ on $M$, then $M$ is diffeomorphic to $\mathbb{R}^m$ (So $S$ is a single point).
- Perelman 1994: If $K \geq 0$ on $M$, and $K > 0$ at one point, then $M$ is diffeomorphic to $\mathbb{R}^m$.

**Remark.** The critical point technique, together with Toporogov comparison theorem, is also used to derive various finiteness theorems. c.f. Peterson “Riemannian geometry” for more details.