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Strategies for Controlling the Work in Mathematics Textbooks for Introductory Calculus

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Abstract. This study analyzes the availability of strategies for (a) deciding whether an action is relevant when solving a problem, (b) determining that an answer has been found, and (c) establishing that the answer is correct in 80 examples of Initial Value Problems (IVPs) in twelve calculus textbooks intended for first-year undergraduate calculus. Examples in textbooks provided explicit information about deciding what to do to solve the problem and determining the answer more frequently than they discussed establishing that a solution is correct or that it makes sense for the given situation. Strategies geared toward verification that the answer is correct or makes sense correspond to three aspects of verification: plausibility, correctness, and interpretation. Honors textbooks were more explicit than non-honors textbooks. Presenting examples as a collection of steps to solve problems – without consideration of what needs to be done to verify that the answer is correct – might obscure the need for verification in solving problems and suggest to students that reworking the solution is the only verification alternative. Implications for research and practice are discussed.

How, where, and when do students learn to (a) decide what to do when solving a problem, (b) determine that they have found an answer, and (c) establish that the answer is right? These are not trivial questions, as the large body of research on problem solving and metacognition has demonstrated (Brown & Walter, 1990; Kilpatrick, 1987; Nespor, 1990; Schoenfeld, 1985a, 1985b, 1992, 1994; Silver, 1987). These questions are at the core of reasoning and validation practices in mathematics and are closely related to metacognitive processes in mathematical problem solving. I approach these questions using Balacheff's theory of conceptions (Balacheff, 1998; Balacheff & Gaudin, 2003; Balacheff & Margolinas, 2005), which allows me to generate hypotheses about how instructors and students may react in hypothetical situations before observing real-life problem-solving. For generating hypotheses, I concentrate on textbook content, because textbooks' tasks and examples determine, to some extent, the kind of work both teachers and students do in the mathematics classroom (Doyle, 1988; Mesa, 2007; Stein & Smith, 1998). In this paper, I report the results of the analysis of 80 Initial Value Problem (IVP) examples taken from twelve textbooks intended for first-year calculus students used in a wide range of U.S. universities. I start by presenting the theoretical framework that guided the study, making connections to metacognition and validation in mathematics, as well

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as the rationale and the research questions. A description of the methods used and a report of the findings are followed by a discussion of the implications of the study for further research and for instruction.

Theoretical Framework and Prior Research

Balacheff's theory of mathematical conceptions defines a *conception* as the interaction between the cognizant subject and the milieu – those features of the environment that relate to the knowledge at stake (Balacheff & Gaudin, 2003; Balacheff & Gaudin, this volume; Balacheff & Margolinas, 2005). His basic proposition is that students' conceptions of mathematical notions are tied to particular problems in which those conceptions emerge. Thus Newton's conception of function was substantially different than Dirichlet's because each was working with a different phenomenon (Balacheff, 1998). As mathematics develops and we solve new problems, our conceptions get transformed. The combination of all these different conceptions is what constitutes a persons' knowledge (*knowing*) about a particular mathematics notion. Conceptions can be distinguished from each other because they require particular operations, particular systems of representations, and particular *control structures*: the organized set of criteria that "allows one to express the means of the subject to decide on the adequacy and validity of an action, as well as the criteria of the milieu for selecting a feedback" (Balacheff, 1998, p. 10). In operational terms, the control structure allows one to (a) decide what to do in a given situation, (b) determine that an answer has been found, and (c) establish that the answer is correct. Describing the control structure is fundamental because the control structure is closely related to processes of metacognition in problem-solving and to validation in mathematics. Research on these two aspects has been consistent: novice problem solvers do not use metacognitive strategies regularly (Schoenfeld, 1992), and they struggle with validation processes when doing proofs (Alcock & Weber, 2005).

The classical definition of metacognition states that the term refers to two separate but related aspects: "(a) knowledge and beliefs about cognitive phenomena, and (b) the regulation and control of cognitive actions" (Flavell, 1976, as cited in Garofalo and Lester (1985), 1985, p. 163; see also Schoenfeld, 1992, for a comprehensive review on the topic). Knowledge of cognition is categorized depending on whether certain factors – *person factors* like beliefs about self as cognitive being, *task factors* such as knowledge about scope, requirements, or complexity of the task, or *strategy factors* like knowledge of when strategies can be used – influence performance. On the other hand, regulation and control of cognitive actions refers to monitoring the processes in relation to the objectives sought. These categorizations generate four components of metacognition: orientation (strategic behavior to assess and understand a problem), organization (planning of behaviors and choice of actions), execution (regulation of behavior to conform to plans), and verification (evaluation of decisions made and of outcomes of executed plans). Two components, organization and verification, overlap to some extent with Balacheff's notion of control structure.

Determining whether proofs of statements are valid is not only a matter of knowing and using the appropriate format of the proof (Herbst, 2002; Martin & Harel, 1989) but also a matter of establishing the truth of each assertion and having the appropriate data and warrants for supporting each one (Alcock & Weber,

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2005; Stephan & Rasmussen, 2002). The literature tells us that undergraduate students generally have significant difficulties validating proofs (Martin & Harel, 1989; Selden & Selden, 1995) but that with appropriate guidance, they can learn to distinguish valid from invalid arguments. Learning to infer warrants in strings of assertions, recognizing what constitutes data that could be used, and explaining why the warrant has authority also overlap with the three elements of Balacheff's control structure as it applies to proving.

Thus, if learning about the control structure is important for mathematical learning, we must investigate the opportunities students have to get acquainted with it. One resource is students' mathematics textbooks. Prior research on textbooks, however, suggests that this structure might not be explicitly presented in students' textbooks. Three studies are pertinent: those by Mesa (2004), Raman (2002), and Lithner (2004).

In an analysis of over 2000 exercises on functions from 8^{th} grade texts from 18 countries, Mesa (2004) found that the control structure included three different types of criteria for verifying the correctness of an answer: *process*, *didactical-contract*, and *content*. Process criteria, found in 55% of the cases, included strategies tied to the procedure involved in solving a problem. Thus, repeating the procedure was necessary, and was suggested explicitly in many cases, as a means to verify that an answer was correct; finding the *same* answer was a warrant for correctness. Didactical-contract criteria, found in 28% of the cases, included strategies that used clues from the presentation of the text that deemed an action and the result appropriate; for example, obtaining an out-of-range ordered pair that could not be plotted in a given Cartesian plane would suggest an error in the solution. Finally, content criteria, found in 17% of the cases, included strategies by which definitions, theorems, or assumptions in the problem needed to be checked to substantiate a claim; thus, obtaining a positive number for a measure of time acted as a warrant that the answer was correct. These results are startling because they illustrate the possibility that part of students' reluctance to use strategies related to the problem itself (i.e., content criteria) to verify the correctness of an answer may be explained by the absence of these strategies in the textbooks. In other words, the opportunities to learn about the control structure are limited in these textbooks.

Raman (2002), in a study about the difficulties that students experience coordinating formal and informal aspects of mathematics, analyzed the role of textbooks in hindering or fostering such coordination. In particular, Raman noted that a pair of students, after working for about 45 minutes on a problem ("Is there a number that is exactly one more than its cube?") were unable to produce a solution until Raman mentioned that the problem was taken from the Intermediate Value Theorem section in their textbook. The students then solved the problem in five minutes. To the question of why was it so difficult to find the solution, a student replied,

> S3: Had you told us to open our books to page 89 and solve problem number 57, we would have done it in 5 min. Because it is, you know, after this section. So we know what we are supposed to do. But this just given like that, we don't know, you know, which part of our knowledge to access. (p. 147)

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This episode illustrates the strength of the didactical contract in determining the type of strategies that can be used in a given situation. For the student, the name of the section of the textbook acted as the trigger that helped her decide what to do in the situation, carry out the procedure, and then be sure that the obtained answer was correct. Textbooks can play an important role in helping students to make these decisions, especially if they include specific markers for students to follow, for example, labeling sections of the problem set using the same headings given in the main text.

Using a coding system designed to capture the structure of reasoning present in textbooks, Lithner (2004) demonstrated that about 90% of the exercises in a U.S. freshman calculus textbook were written in such a way that students could use superficial reasoning to solve them without the need to access the meaning of the notions at stake. There is an important parallel between Lithner's definition of reasoning structure and Balacheff's control structure. For Lithner, the reasoning structure has four elements: the *problematic situation*, *strategy choice*, *strategy implementation*, and *conclusion* (p. 406). *Strategy choice* refers to choosing "(in a wide sense: choose, recall, construct, discover, etc.) a strategy that can solve the difficulty... [i.e., it] can be supported by predictive argumentation." *Strategy implementation* refers to "verificative argumentation: Did the strategy solve the difficulty?" and *conclusion* refers to obtaining the result (p. 406). As a consequence, "reasoning" is defined by Lithner as

> The line of thought, the way of thinking, adopted to produce assertions and reach conclusions. The explicit or implicit *argumentation* is the substantiation, the part of the reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate. (p. 406)

Note that Lithner's predictive argument would correspond to aspects of deciding what needs to be done in a given situation in Balacheff's control structure, the verificative argument would correspond to processes of verification that the answer is indeed correct, and the conclusion could be mapped to deciding whether an answer has been found – that is, recognizing the solution – in Balacheff's theory. The control structure in Balacheff's theory, however, emphasizes the establishment of the criteria that govern the decisions students make, rather than analyzing the actual consequences of the decision or the type of reasoning obtained. In this sense, Balacheff's control structure connects to metacognitive strategies, what Schoenfeld (1985a) called *control*: "Global decisions regarding the selection and implementation of resources and strategies. Planning. Monitoring and assessment. Decision-making. Conscious metacognitive acts" (p. 15).

According to Lithner, in *superficial reasoning*, students may rely on recalling keywords, algorithms, or previous experiences rather than on using *plausible reasoning*, which is based on intrinsic aspects of the content at stake. This classification also parallels Mesa's (2004) findings of the types of control strategies suggested by exercises in middle school textbooks, which were based on process, the didactical contract, or the content. Moreover, both studies report an alarmingly low percentage of exercises that would require verification and reasoning strategies based on the content at stake (10% in Lithner's study, 17% in Mesa's study).

Taken together, these textbook analyses illustrate how textbooks have the potential to shape students' development of verification and reasoning strategies in unproductive ways. It is worth noting here that an important link, instruction, is

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still understudied. The larger project, from which this paper is derived, was undertaken with the purpose of understanding the link between textbook presentation and instruction regarding the manifestation of the control structure. However, the studies to date that have focused on textbooks have investigated problems or definitions, elements of the textbook that may not be suitable for disclosing what the control structure is. These studies also looked across topics, or across nations, or across levels of instruction. Different results might be possible when looking at a particular level of instruction, within a single educative system, and within a single topic. Therefore, it is important to further probe these findings about textbooks by looking at a specific mathematical notion in which intrinsic mathematical aspects of the content at stake are fundamental for the criteria in the control structure.

In this paper I present results regarding the stability of findings of the analysis of textbook material – specifically, sections in first-year calculus textbooks devoted to initial value problems. First-year calculus was selected for two reasons. First, for many students it is the first university course in which they start to learn important aspects of mathematical work; therefore, an analysis of the opportunity to learn about controlling work in mathematics could be started in this course. Second, calculus reform has emphasized the use of multiple representations and technology; these two tools can become important resources for verification processes in the control structure. Thus, it is possible to expect that they will be manifested in calculus textbooks.

Selecting initial value problems as the content for my analysis was a theoretical decision. Balacheff's theory of conceptions indicates that the control structure is particular to the problems in the subject–milieu system. I chose sections devoted to initial value problems because the topic interweaves integration and differentiation notions with physical and other real-life situations that could offer more opportunities for different types of control strategies to emerge. Thus, the first research question for the study reported here is:

(1) How do textbooks intended for introductory college calculus make their control structures explicit in examples devoted to Initial Value Problems (IVPs)? More specifically, how do these examples indicate
   (a) how to decide what to do to solve the problem,
   (b) how to determine that the answer has been found, and
   (c) how to establish that the answer is correct.

Because textbooks tend to differ depending on their audiences, I anticipated that textbooks intended for honors students would expose the control structure in a different way than would textbooks intended for non-honors students. The second research question was:

(2) Do textbooks intended for honors students differ from textbooks intended for non-honors students regarding their control structures? If so, in what ways?

These two research questions allowed me to: (a) determine the role that textbooks might play in establishing the control structures, (b) trace variations depending on the audience, and (c) suggest hypotheses about ways in which instructors might take advantage of what textbooks present.

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Methods

This study applied textual analysis techniques (Bazerman, 2006) to selected sections of a wide variety of textbooks intended for introductory calculus. The purpose of the methodology selected was to mine the *text* (i.e., the content within the textbook) in search of evidence of elements of the control structure. In what follows, I describe the decisions made, together with the rationale, regarding sampling and data analysis.

Sampling. There were two levels of sampling used in the study. They were sampling of textbooks and of text within textbooks.

Textbook Sampling. I chose 14 in-print calculus textbooks intended for introductory calculus at a wide range of postsecondary programs. To select the textbooks, I consulted the syllabi of various higher-education institutions in the U.S. Midwest and obtained information from instructors who were teaching or had taught calculus as a first-year undergraduate course. From over 30 suggestions, I selected the textbooks that were (a) most frequently mentioned and (b) used in post-secondary institutions within 100 miles of my university (to facilitate the continuation of the project). I obtained the latest edition of each of those textbooks. The student's solutions manuals, sold separately, presented the solutions without elaboration and none of the textbooks had a teacher's manual. For this reason, I did not analyze these texts. I visited the institutions' bookstores to make sure the textbooks were offered to the students. The majority of the textbooks were intended for a general audience of undergraduates; some were intended for more specialized audiences, such as math majors or science and engineering majors. Some institutions offered separate tracks of calculus for their students (e.g., applied, regular, honors) and used a different textbook for each. Textbooks intended for an applied or honors sequence in one institution might be used for a regular sequence in others. In the sample of 14 textbooks, five were intended for honors students.

Text Within Textbooks Sampling. All the textbooks followed the most common type of organization, namely: exposition–examples–exercises. In exposition, the text "directs the reader" and will "expound its subject matter in a discursive fashion, maybe use devices such as questions, visual materials, or tasks as assisting concept formation" (E. Love & Pimm, 1996, p. 387). The examples,

> [Are] intended to be 'paradigmatic' or 'generic', offering students a model to be emulated in the exercises which follow. The assumption here is that the student is expected to form a generalisation from the examples which can then be applied in the exercises. . . . some books similarly annotate examples, indicating particular points of difficulty and the reasons behind a choice of approach. (p. 387)

In the exercises, the students are encouraged to actively engage with the text, by working through tasks that mimic previous examples or with a more varied collection of problems. Of the three types of texts, I chose to analyze the examples because they were intended to be paradigmatic of the work to be done and would be most likely to contain explicit strategies for controlling that work (Watson & Mason, 2005). Another more pragmatic reason had to do with instructors' beliefs that students rely on examples in order to work out homework problems (Mesa, 2006b).

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Thus, I analyzed all examples provided in all sections devoted to initial value problems (IVPs) in these textbooks. The sections were located by finding the entries in the index corresponding to *initial value problems*, *initial condition problems*, or *introduction to differential equations*. Two honors textbooks lacked sections devoted to IVPs; therefore, those textbooks were not included in the analysis. The 12 textbooks analyzed are given in the reference list (Adams, 2003; Apostol, 1967; Goldstein, Lay, & Schneider, 2004; Hughes-Hallett, et al., 2005; Larson, Hostetler, & Edwards, 2006; MacCluer, 2006; Ostebee & Zorn, 2002; Simmons, 1996; R. T. Smith & Minton, 2002; Stewart, 2003; Thomas, Finney, Weir, & Giordano, 2001; Zenor, Slaminka, & Thaxton, 1999).

I examined 24 sections and 80 examples in total. I collected information on the definition of IVP and the contexts in which the examples were presented and on the control structure suggested in the example.

Definition and Context. Knowing what elements belong to the definition of an IVP was crucial for determining the internal logic of the arguments provided. A textbook that contains a definition that includes the order of the equation, for example, may be more likely to include "order of the equation" as part of the considerations in deciding what to do (e.g., in a second order equation, it will be necessary to solve two IVPs). The definitions were usually given in-line with the text (i.e., not under a definition heading or a box), as in the following examples:

> *Example 1.* A *differential equation* is an equation involving an unknown function and one or more of its derivatives. The *order* of such an equation is the order of the highest derivative that occurs in it. ... The arbitrary constant that appears in the general solution of a first-order equation is given a specific numerical value by prescribing, as an *initial condition*, the value of the unknown function $y = y(x)$ at a single value of x, say $y = y_0$ when $x = x_0$. (Simmons, pp. 178-180)
>
> *Example 2.* Sometimes we want to find a particular solution that satisfies certain additional conditions called *initial conditions*. Initial conditions specify the values of a solution and a certain number of its derivatives at some specific value of t, often $t = 0$. (...) The problem of determining a solution to a differential equation that satisfies given initial conditions is called an *initial value problem*. (Goldstein et al., p. 502)
>
> *Example 3.* In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$. This is called an initial condition, and the problem of finding a solution of the differential equation that satisfies the **initial condition** is called an **initial-value problem**. (Stewart, p. 626)

I highlighted the text containing these definitions and used the notions included to construct a concept map (Novak, 1998) for IVPs.

Because of the important role that context can play in establishing whether an answer is correct or not, I also categorized the examples depending on whether they were contextualized or not, and created a running list of the applications suggested. When the example provided an IVP that did not make reference to an application, the context was marked as not present.

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Control Structure. I selected all the examples that proposed and solved IVPs. When an example used a result that was found in another example (e.g., a general solution for a differential equation or a slope field), that example was included in the analysis. I read the solutions provided, the text preceding and the text following the example, and for each one answered the following three questions:

- How do I know how to solve the problem?
- How do I know that I got an answer?
- How do I know that the answer is correct?

This provided a holistic summary of what type of control structure was present, for which evidence was sought using a more detailed content analysis of the text in the example. I parsed the sentences in the example in order to classify them as belonging to one of four categories.

Category 1. What to do: when the sentence described a decision-making process listing actions that would lead to the solution of the particular problem,

Category 2. Answer found: when the sentence indicated that an answer to the problem had been obtained or described its characteristics without saying whether it was correct,

Category 3. Answer correct: when the sentence described ways in which the answer could be confirmed as making sense therefore deeming it correct or when the sentence verified the correctness of the answer, and

Category 4. Elaboration: when the sentence conveyed reasons or justifications for a given result, or described further implications of that result.

Sentences that did not fit any of these categories were coded as *other* and not considered in the analysis. Table 1 provides examples of the sentence categorization.

Data Analysis. The coding system was tested in two randomly selected textbooks with seven examples ($\sim 9\%$ of the sample of examples, 85 sentences, $\sim 11\%$ of the total number of sentences) by three other coders who were not familiar with the textbooks. I contrasted my coding to that assigned by the coders generating three contrasting pairs and calculated the Cohen's κ coefficient for each pair. The average was $\kappa = .72$, which is an acceptable level for inter-rater reliability (Landis & Koch, 1977). The remaining sentences and examples were coded using this process.

Definition and Context. I organized the definitions provided in the textbook by the common term being defined (e.g., *differential equation*, *initial condition*, *slope field*) and established the connections among those terms by common references; the terms and connections were organized into concept maps that illustrated how the definitions were organized in each textbook. I then produced a summary concept map combining all elements and connections within each individual concept map. I created a running list of all the contexts in which IVPs were presented and tallied them.

Control Structure. The holistic summaries were used to describe the contents of each of the categories; these summaries were useful for determining the extent to which the sentence parsing was capturing the essence of the summaries. The summaries also contained quotes (with page references) that provided evidence for particular categorizations.

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TABLE 1. Examples of Sentence Categorization

What to do	
	"We solve these problems in two steps. . . : (1) solve the differential equation, and (2) evaluate C"
	"We have already seen that the general solution is. . . The initial condition allows us to detemine the constant C"
	"The process of antidifferentiation amounts to solving a simple type of differential equation"
Answer Found	
	"The (unique) solution is"
	"Thus, . . . is the desired solution"
	"The temperature is decreasing at the rate of 20 degrees per second"
	"The result is a curve shown in Fig. . . "
Answer Correct	
	"It is easy to check by differentiation that every function of this form is a solution"
	"This translates to around 200mph—possible for an old–fashioned cannon"
	"Around the height of the tallest human–built structure"
	"We can visualize the solution curve by following the flow of the slope field "
Elaboration	
	"This DE simply asks for an antiderivative f of the function $6x+5$"
	"The differential equation (1) says that $y' + 2y$ equals zero for all values of t"
	"This DE is more challenging than the previous one."
	"First we observe that there is no finite time t at which $f(t)$ will be zero because the exponential e^{-kt} never vanishes."

For each textbook, I tallied the number of examples for which elements of the control structure were evident. For each example, within each textbook, I created a summary table tallying the number of sentences belonging to each category. Simple frequencies are reported and the aggregates are used to contrast the textbooks.

All sentences categorized as *What to do*, *Answer Found*, and *Answer Correct* were further analyzed using a combination of open coding and thematic analysis (Bazerman, 2006). The purpose was to produce descriptions of the kinds of strategies that were provided in the examples, attending to what Balacheff (1998) has described as means to decide adequacy (of both the decisions made regarding processes and of the answers), validity (of both processes and answers), and ways for the milieu to provide feedback (about both adequacy and validity).

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Results

The presentation of IVPs in these textbooks followed a consistent pattern: after providing the definition, textbooks then presented a non-contextual example with a relatively simple differential equation involved. The example would give the solution process (integrate and then evaluate the initial condition) accompanied by a figure containing some functions of the family of solutions, with the particular solution to the IVP highlighted in a different color. After the non-contextual example, applications with a more or less known context were provided. Only occasionally were the differential equations in non-contextual examples used in the contextual examples. Some textbooks indicated that most IVPs could not be solved analytically, then suggested Euler's method as a viable process to approximate a solution. Fewer textbooks indicated the origin of IVPs: in real life, what is observable is the *change* of a variable in relation to another variable; thus, the quest of the solver is finding or making explicit, in the best way possible, the nature of the relationship between the two (or more) variables – that is, finding a function whose change will be similar to the one that has been observed.

Figure 1 presents a concept map summarizing terms and connections among those terms associated with IVPs in these textbooks. The ovals and thicker lines indicate concepts and links that were common to all the textbooks studied.

The concept map suggested a basic conceptualization of IVPs in these textbooks: an IVP is a differential equation for which an initial condition is provided which generates a unique solution from among the infinite many that the equation has. In order to solve the problem, an antiderivative must be sought. Looking at more than one textbook, other information regarding the equation (separability, constant solution, slope field) and the solution to the IVP (domain of validity and the necessity of using Euler's method to find an approximation a solution) was garnered. The concept map allows us to anticipate possibilities for the control structure to emerge: (a) solutions can be verified through derivation, (b) slope fields can illustrate the behavior of the situation being modeled, (c) slope fields can also be used to verify a solution, and (d) applications can provide the means for assessing the plausibility of solutions. As we will see later, these elements were, in fact, present and referred to in the control structure of the textbooks analyzed.

Motion – rectilinear, vertical, and near the surface of the earth – was the only topic that was addressed by at least one example in all textbooks; other contexts frequently mentioned in the examples included growth/decay problems (24% of the examples), Newton's cooling or gravitational laws (17%), electric circuits (12%), and mixture problems (5%). Honors textbooks, on average, had more contextualized examples than non-honors textbooks (85% vs. 36%).

Control Structure. The initial reading of IVP examples revealed that, in general, there was substantial information about how to deal with particular problems and about recognizing an answer, but very few suggestions for how to establish that the solution was correct or that it made sense for the situation. Only four examples showed students how to test whether a certain function was a solution to an IVP, and it was illustrated through derivation. Only 20% of the examples (16 out of 80) made all three aspects explicit. The following excerpts illustrate such an example:

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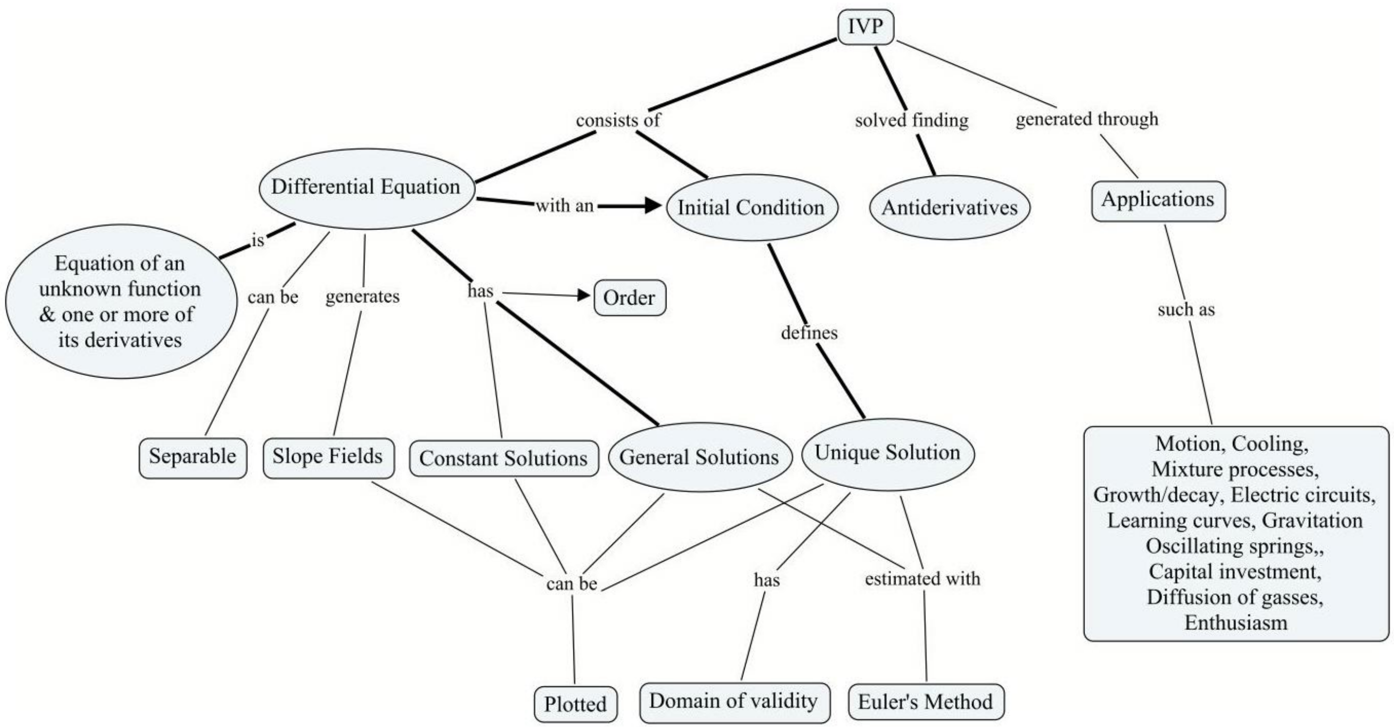

FIGURE 1. Concept map of elements of the definition of an IVP across textbooks. Oval boxes and thicker links refer to elements common to all the textbooks.

t restrictions may apply.

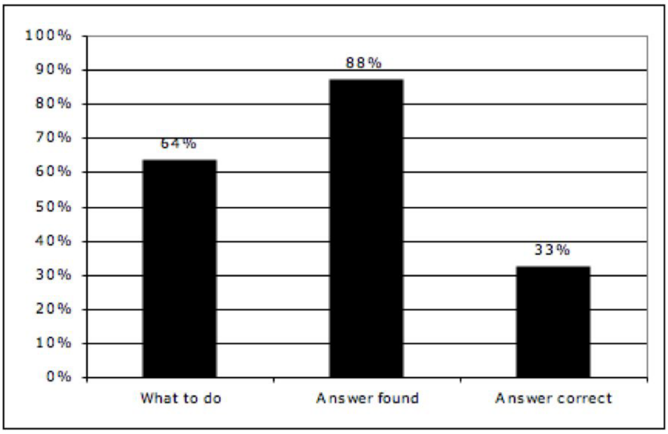

FIGURE 2. Percentage of examples ($n = 80$) in which elements of the control structure were made explicit across textbooks.

Example 4. Find an expression for the current in a circuit where the resistance is 12Ω, the inductance is $4H$, a battery gives a constant voltage $60V$, and the switch is turned on when $t = 0$. What is the limiting value? [An accompanying figure shows the circuit; the equation comes from an earlier example]
Solution: [a number of substitutions are made into the differential equation that models the current in the circuit given]... *we recognize this equation as being separable, and we solve it as follows:* [integration is carried out to yield a general solution for I]...Since I(0) = 0, we have ...[substitution in I] *and the solution is* $I(t) = 5 - 5e^{-3t}$ ...The *limiting current, in amperes, is*...5. Figure 6 shows how the solution in Example 4 (the current) approaches its limiting value. Comparison with Figure 11 in Section 10.2 *shows that we were able to draw a fairly accurate solution* curve from the direction field. (Stewart, 2003, p. 639-640, emphasis added)

The highlighted sentences are those in which elements that define the control structure were made explicit: what needed to be done (*equation [is] separable, solve it as follows*), that the answer had been found (*the solution is, the limiting current is*), and that it made sense (contrast with both the slope field plotted in a previous problem and with the graph of the current).

These observations were corroborated with the sentence categorization. I compared these data at the example level (strategies present or not) and at the sentence level (frequency observed). Figure 2 presents the percentage of examples across textbooks in which at least one element of the control structure was made explicit – that is, examples in which authors explicitly indicated how to decide what needed to be done, or how to recognize the answer, or how to establish its correctness.

Figure 2 indicates that about 6 out of 10 examples included information describing how to approach the solution of the given problem, 9 out of the 10 examples highlighted when the answer had been found, and 3 out of the 10 examples provided information that assessed the correctness of the answer or its appropriateness for

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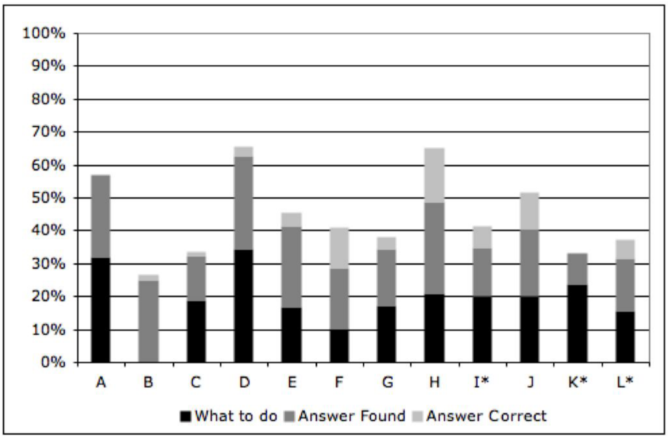

FIGURE 3. Percentage of sentences ($n = 763$) within textbooks in which elements of the control structure were made explicit. A starred letter indicates an honors textbook.

the situation. Thus, taken together, the examples in these textbooks were more explicit about how to solve the problems and about recognizing the answers than about establishing that the answers found were indeed correct.

Within textbooks, however, the situation changed. The examples in some textbooks provided more information regarding the control structure than in others and the strategies tend to carry over content across the textbook – that is, they were part of the writing style. Looking within textbooks may yield more practical implications than looking at the textbooks as a set because instructors and students tend to use a single textbook in their class work. Figure 3 presents the proportion of sentences in examples that were devoted to the different elements of the control structure, organized by textbook.Though textbooks are denoted with a letter in Figure 3, the letter was randomly assigned to avoid text identification.

Figure 3 illustrates that textbooks differed from each other in terms of the manifestation of the control structure. In general, the pattern observed across textbooks was found here: examples in these textbooks were more explicit regarding how to solve the problem or highlighting that the answer had been found than about establishing that an answer was correct. Notice that textbook H had about the same number of sentences for each element, making it the most balanced of all the textbooks analyzed. All textbooks explicitly showed how to recognize that an answer had been found, and most explicitly suggested ways to solve the problem. It is important to note that the coding system accounted for only explicit language used to highlight decision-making processes regarding the solutions. In the following example, no sentences were coded as *What to do* because the presentation consisted only of the steps taken, the actual actions leading to solving the problem:

Example. Find the function $f(x)$ whose derivative is $f'(x) = 6x^2 - 1$ for all real x and for which $f(2) = 10$.
Solution: Since $f'(x) = 6x^2 - 1$, we have $f(x) = \int(6x^2 - 1)dx = 2x^3 - x + C$ for some constant C. Since $f(2) = 10$ we have $10 = f(2) = 16 - 2 + C$. Thus $C = -4$ and $f(x) = 2x^3 - x - 4$.

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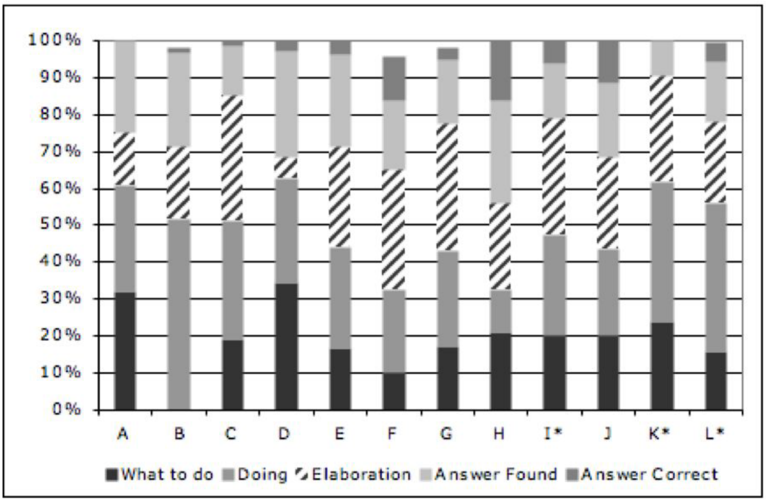

FIGURE 4. Percentage of sentences within textbooks corresponding to actions, elaborations, and control strategies.

(*By direct calculation we can verify that* $f'(x) = 6x^2 - 1$ and $f(2) = 10$). (Adams, Example 3, emphasis added)

The example tells what to do by doing the steps. This approach was fairly typical across all the textbooks observed and led me to include a new category, *Doing*, to categorize sentences of this type. Notice that this example includes an explicit verification sentence: by taking the derivative of the function that was found, one can obtain the original expression; also substitution of 2 in the function found yields the initial condition. The conclusion is, then, that the function found must be correct. However, as seen in Figure 3, verification statements were not very common.

A number of sentences that had relevant information for justifying the solution process also made up an important portion of the content of the examples. Figure 4 shows the percentage of sentences coded in each book considering both the *Doing* and the *Elaboration* sentences. Observe that for most of the cases, the five types of codes accounted for most of the content in the examples. *Doing* and *Elaboration* sentences accounted for about half of the sentences in most of the books (min 35%, max 71%).

The inclusion of *Doing* sentences, the sentences in which actual operations are carried out, showed that examples function as spaces for illustrating procedures. The demonstration is such that the reader is the spectator, similar to an apprentice, who observes the master carrying out the work and has to reproduce that work later on. In some textbooks the apprentice receives extra guidance from the master, with the *Elaboration* sentences serving the purpose of connecting ideas, justifying steps, or translating into similar language what has been or needs to be done. *Doing* and *Elaboration* sentences carry out the most important functions that examples are meant to fulfill; they provide the meat of the action for the apprentice to see how things work (Watson & Mason, 2005).

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Examples in the honors textbooks analyzed tended to be longer than those in other textbooks, to offer more explicit control strategies, to deal with fewer constraints and more general cases, and to mostly elaborate ideas, sometimes not following the pattern of problem-then-solution observed in all the non-honors textbooks (i.e., the example might consist of a discussion of a situation, for which no explicit questions were stated). Seven of the 16 examples that made explicit the three elements of the control structure were found in honors textbooks.[1] The following excerpts illustrate some of these points.

Excerpt 1.

Radioactive Decay. Although various radioactive elements show marked differences in their rates of decay, they all seem to share a common property–the rate at which a given substance decomposes at any instant is proportional to the amount present at that instant. If we denote by $y = f(t)$ the amount present at time t, the derivative $y' = f'(t)$ represents the rate of change of y at time t, and the "law of decay" states that $y' = -ky$, where k is a positive constant (called the 'decay constant') whose actual value depends on the particular element that is decomposing. The minus sign comes in because y decreases as t increases, and hence y' is always negative. The differential equation $y' = -ky$ is the mathematical model used for problems concerning radioactive decay. Every solution $y = f(t)$ of this differential equation has the form $f(t) = f(0)e^{-kt}$. Therefore, *to determine the amount present at time t, we need to know* the initial amount $f(0)$ and the value of the decay constant k. It is interesting to see *what information can be deduced from* [the general solution], without knowing the exact value of $f(0)$ or k. First we observe that there is no finite time t at which $f(t)$ will be zero... therefore, *it is not useful to study* the "total lifetime" of a radioactive substance. (Apostol, Example 1, emphasis added)

Excerpt 2.

A 0.1 kg ball [...] is thrown at an angle of 60 deg with the horizontal on a level field, with an initial speed of $50m/sec$. See Figure [it shows the ball's initial velocity and the forces acting on it]. The constant of resistance due to the atmosphere is known to be $k = 0.02$. How high will the ball go, and how far away will it land?

Solution: [some initial substitutions]...

Proceeding exactly as we did in the previous example, we get [$v_x(t)$, and $v_y(t)$, maximum altitude, time, $x(t)$].

Note that, even if the ball were going over a cliff and t could become quite large, *its range would be less than* $125m$. ... *To find the range, we must first find* the time $t \neq 0$ when $y(t) = 0$. (*Why can't we assume that it takes the same amount of time to get down as it took to get to the top of its trajectory?*)... Notice

[1] Numerical data is available. I chose not to present a summary table with this information, because the different sample sizes may lead to unwarranted conclusions about the differences between the textbooks.

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> *how much longer the ball takes to come down than it took going up. Why?* (Zenor et al., Examples 5 and 9, emphasis added).

Excerpt 3.

> Observe how the [...] problem above tracks exactly what some [...] call the "scientific method":
> *Step 1:* model the phenomenon as a differential equation.
> *Step 2:* Solve the differential equation
> *Step 3:* Impose the given data
> *Step 4:* Interpret the results.
> (MacCluer, Example 15, emphasis in original)

Excerpt 1 was taken from an example that did not pose a question to be answered; among the concepts that the example illustrates is that the "half-life is the same for every sample of a given material." Excerpt 2, on the other hand, was from an example on motion. It uses vectors for speed and position, which allows the problem to ask for the range of the motion and it also considers the effects of air resistance. None of the non-honors textbooks assumed air resistance (although one mentioned that the results would be "slightly" different if air resistance were considered). Finally, Excerpt 3 looks for extracting a general "method" illustrated by the previous example. Thus, the example is not only to be used as reference —a standard case that can be used to check understanding—but also as a model illustrating the general method behind the process (Michener, 1978).

Content of the Control Structure. In showing students how to decide what to do when solving an IVP, authors of these textbooks offered two options, depending on whether the differential equation could be solved with methods already introduced or not. That antiderivation is to be used is made clear in the presentation. The decision about how to antdifferentiate (or whether one should attempt it) depended on what the authors had included in the textbook. In principle, Euler's method could be used all the time, but the textbooks that include the method made a point of using it only when analytical strategies cannot be applied. Because this option was not available in all textbooks (see Figure 1), this decision was not always required. Similarly, when the textbooks introduced a section about separable equations, the decision about how to antidifferentiate involved recognition that the equation was separable. Without this section, that decision was not required either. When neither of these sections was included, the equations were simple enough that authors could refer to previous theorems to justify the antiderivation processes.

This approach to antiderivation suggests that deciding what needs to be done is in general driven by the didactical contract. Thus, when the section was labeled "Euler's method," the equation was solved using the approximation, even when the equation could have been solved analytically. In such cases, the analytical solution worked as a template against which the accuracy of the Euler's approximation could be measured.

Once a family of solution functions was found, the reader might be directed toward making sure that that family of functions fit the original differential equation (a verification step that will be discussed below). The next step consisted of using the initial conditions to fix the function, and therefore find the unique solution to the IVP. No decisions seemed to be needed for this step; rather, it is a matter

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TABLE 2. Description and Frequency of the Content Strategies

Strategy	Frequency
Establishing the goodness of fit: A given answer matches a previously known fact	18
Using the definition or a theorem: Use of analytical procedures to establish that the function found is a solution to the differential equation provided	15
Assessing the meaning of numerical values: Draw attention to the magnitude of the values in the situation	6
Analyzing Equations: Draw attention to the form of the expression of the solution	3
Issuing warnings: Draw attention to potential problems	3
Total	45

of substituting values, of applying a process. The outcome of this process is the answer. In showing students how to decide whether the answer is right, however, the textbooks seemed to offer more differentiated strategies.

The analysis of the 45 Answer Correct sentences from 23 examples generated five types of strategies that would be classified as content, following Mesa's (2004) study: establishing the goodness of fit, using the definition or a theorem, assessing the meaning of numerical values, analyzing equations, and issuing warnings.

Establishing the goodness of fit was the most common of the strategies (18 out of 45 sentences) and indicated the cases in which the authors showed how a given answer matched a previously known fact. For example, having found the solution to an IVP analytically, authors might use Euler's method to find an approximation and "use [the] exact solution as a basis for comparison for the performance of the Euler's method." The essence of the strategy was to rule out gross errors and to show that one could be certain the solution was accurate: "to make this correspondence more apparent, we have drawn a graph of the approximate solution superimposed on the direction field in [the figure]." This strategy used numbers, graphs, or symbolic expressions, and would be seen in both contextualized and non-contextualized situations. In one instance, the reader was encouraged to conduct an experiment to corroborate the findings.

Using the definition or a theorem was also common (15 out of 45 sentences) and corresponded to the use of analytical procedures to establish that the function found was indeed a solution to the differential equation provided: "(1) To decide whether the function $y = e^{2t}$ is a solution, substitute it into the differential equation: $d^2y/dt^2 + 4y = 2(2e^{2t}) + 4e^{2t} = 8e^{2t}$. (2) Since $8e^{2t}$ is not identically zero, $y = e^{2t}$ is not a solution." Theorems were invoked to justify the correctness of a statement: "By theorem 9 (...), these are the only possible derivatives."

Assessing the meaning of numerical values occurred in 6 of the 45 cases. In these sentences, authors called the reader's attention to the magnitude of the values in the situation. For example, one noted that $495m$, the height to which a cannonball rose, was "around the height of the tallest human-built structure." Another emphasized how the numerical values obtained were to be interpreted in the context provided: "note that, even if the ball were going over a cliff and t could

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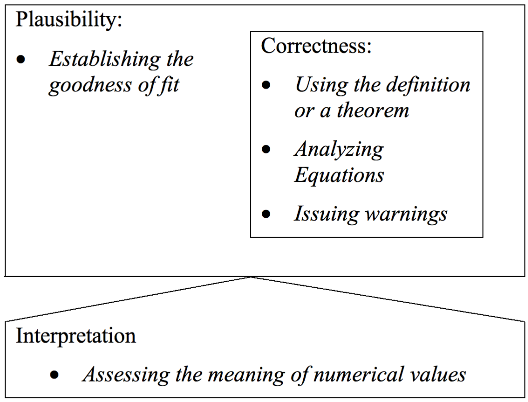

FIGURE 5. Three aspects of the process of verification.

become quite large, its range would be less than $125m$." Graphical representations were used in some cases.

Analyzing equations, seen in 3 of the 45 sentences, corresponded to instances in which the reader's attention was drawn to the form of the expression of the solution to establish the validity of the statement. For example, "this [$v = -32(t-4)$] tells us that for $t < 4$, the velocity is positive, so the stone is moving upward."

Finally, in 3 of the 45 sentences, authors *Issued warnings* that were meant to draw the reader's attention to potential problems. For instance, one textbook included the sentence, "we should be careful about the domain of our solution," suggesting that a mistake could be generated if the warning was not followed. A summary is presented in Table 2.

The five strategies seemed to address three different aspects of the process of verification, namely plausibility (whether what is obtained is possible), correctness (whether what is obtained is true), and interpretation (that what is obtained makes sense). In this reduced set of sentences, it was difficult to notice possible connections among these aspects, some of which could be conjectured. In some cases, it may not be possible to establish the correctness of a solution, and plausibility may be the best confirmation of accuracy that one can get (as when an IVP cannot be solved analytically).

By definition, "plausibility" suggests that there "appears to be merit for acceptance" (Plausible, 1989) and that possible discrepancies can be ignored. Correctness is grounded on the assumption that there are legitimate means for demonstrating or proving that a statement is true and that those means are valid – that is, that the process "contains premises from which the conclusion may logically be derived" (Valid, 2000). Thus, even though all correct statements may be plausible, not all plausible statements might be correct. Interpretation occurred only in contextualized examples, because context was what determined the meaning to be attributed to particular numbers. In this way, interpretation may be seen to support both plausibility and correctness (see Figure 5).

In summary, the types of strategies illustrated in these examples for deciding about what to do to solve a problem, determining that an answer has been found, and establishing that the answer is correct appeared to be grounded differently.

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"What to do" decisions seemed to be grounded mostly on the didactical contract that suggested the process to be followed. "Determining that an answer has been found" appeared to be a non-issue: the answer is either the function or the values that the situation asked for. Finally, "establishing whether the answer is correct or makes sense" seemed to be the only aspect that was grounded in the content at stake, and the strategies given in the textbooks seemed to address interrelated aspects of verification, plausibility, correctness, and interpretation.

Discussion

This study was conducted to determine (a) the ways in which textbooks intended for introductory college calculus were explicit about the control structure in examples devoted to Initial Value Problems (IVPs) and (b) the ways in which textbooks intended for honors students differ from textbooks intended for non-honors students regarding the control structure. I organize the discussion by these two topics.

The Control Structure for IVPs in Introductory Calculus Textbooks. Balacheff's theory of conceptions states that conceptions can be distinguished from each other because different problems require different operations, different systems of representations, and different control structures. In the case of IVPs undertaken in this study, and in the special case of what textbooks offer, it is interesting to note that how IVPs emerge—from real-life situations in which we can only trace the change of a variable in relation to another variable—is not prominently discussed across all textbooks. The concept map compiled from the content across the textbooks seems to emphasize the abstract nature of IVPs; even though these problems have originated from real-life situations, they have been conceptualized as models, and their treatment in textbooks emphasizes dealing with those models (antiderivating and finding a unique solution). This treatment seems related to the control structure that emerges.

The analysis of the sentences referring to the different aspects of the control structure, which are meant to speak about adequacy, validity, and the way in which the milieu provides feedback, revealed that these textbooks offer more opportunities to learn about what needs to be done to solve the IVPs and for recognizing the answer, than opportunities for making sure the answer is correct. The emphasis on *doing* and justifying the doing (*elaboration*) seems to reinforce the technical aspects of solving IVPs. Decisions about what needs to be done are tied to the didactical contract, and obtaining answers seems to be a consequence of doing a procedure, which is assumed to be correct by virtue of being an illustration. In the few sentences in which verification was carried out, the five identified strategies took advantage of the content at stake. But, as the content map and the coding reflect, these strategies did not appear in all textbooks analyzed.

Before discussing whether this situation could be different, especially regarding verification strategies, I will offer conjectures about why this is so. A look at calculus textbooks from different eras (e.g., De Morgan, 1842; Lacroix, 1797, 1837; Lamb, 1924; C. E. Love, 1916; C. E. Love & Rainville, 1962; Perry, 1897; R. H. Smith, 1897) and at reviews of some of those calculus textbooks shows that several changes have occurred since the first calculus textbook was written by the Marquis of L'Hopital in 1696 (Eves, 1990, p. 426). Until 1890, the textbooks appeared to be mostly compendia of all that was known at the time about calculus. The early

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1900s has been recognized as a period that profoundly affected the teaching of mathematics at the collegiate level (Kilpatrick, 1992), and which resulted in a new type of textbook available for calculus (Mesa, 2006a). Textbooks became sources by which instruction at universities could be delivered to larger numbers of students, who were not only interested in mathematics but also in engineering, physics, or chemistry. At the time, there were criticisms about the lack of verification in some textbooks. For example, Ransom (1917), in a review of a Clyde E. Love's *Differential and Integral Calculus* (1916), noted that in spite of C. E. Love's claim made in the preface that "A feature of the book is its insistence on the importance of checking the results of the exercises,"

> The reviewer found no attention given to the general question of checking, and except under double integration (where more than one order of work is often possible) very few problems call for solutions in two ways... No rough methods of checking are suggested, such as sketching derivative or integral curves or other graphical devices, nor is there any hint of checking limits by computing neighboring values, nor of checking differentials by computing small increments, nor of checking integrals by Simpson's rule. (Ransom, 1917, p. 175)

Textbooks intended for the applied disciplines contained step-by-step directions for solving problems, had long lists of exercises for practice, avoided theory when not related to practice, and limited the presentation to what was considered the most useful part of the subject. As in C. E. Love's textbook, there were few instances of verification strategies. It was not until the late 1970s and the calculus reform movement that attention to multiple representations and to technology use, among other things, created new opportunities for students to verify their work.

Although very few of the textbooks analyzed in this study claimed to be reform oriented, most of them included some reform features such as examples and exercises to be solved with technology (graphing calculators or computer algebra systems) and more references to graphical representations. These features make them different from earlier editions of the same textbooks. However, even though these features could be used for verifying answers, they were not prominently referred to in the textbooks.

Through the analysis of these textbooks, we see that even with a topic that provides many opportunities for establishing the correctness of answers, these opportunities are not made explicit. Perhaps the textbook writing tradition that emerged in the earlier 1900s still persists. Perhaps what E. Love and Pimm (1996) called the authority of the textbook is also playing a role: the textbook does not have mistakes, therefore what is presented is correct. Or perhaps carrying out the intentions of providing alternative solutions or rough methods of checking is difficult: how many alternatives can be proposed without making the textbook longer or cumbersome to read? Textbooks authors might also have wondered whether an introductory textbook was the place for such work. To respond to these questions, we need to weigh considerations of the potential benefit for students.

Research on students' use of verification strategies has suggested that what students can do in particular situations is limited. Eizenberg and Zaslavski (2004), in their study of students solving combinatorics problems, found five types of verification strategy: *reworking the solution*, *using a different solution method and*

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comparing answers, *adding justifications to the solution*, *evaluating the reasonability of the answer*, and *modifying some components of the solution. Reworking the solution*, *using a different solutions method*, and *modifying some components of the solution* map into what Mesa (2004) categorized as process-oriented strategies, as they depend on the procedures learned. The other two strategies identified by Eizenberg and Zaslavski (2004), *adding justifications* and *evaluating the reasonability of the answer*, could be categorized as content strategies because they require the content at stake to make decisions about the correctness of the solution. Eizenberg and Zaslavski found that some students were capable of producing verification strategies, but that this happened more frequently *when they were told* that their answers were incorrect; additionally, they found that:

> Many of the students who made attempts to verify their incorrect solutions, whether out of their own initiative or in response to the interviewer's prompts, *were not able to come up with efficient verification strategies* and were thus neither able to detect an error nor to correct their solution (p. 33, emphasis added).

Of the five strategies in their study, *reworking the solution* was the most frequently used and the most inefficient. It is a remarkable coincidence that just as textbooks emphasize decisions about how to deal with problems based mostly on procedures, they also emphasize the use of procedures with regard to verification strategies.

In the present study, the five verification strategies found were grounded in the content, and they referred to three different aspects of verification, namely: plausibility, correctness, and interpretation. But they appeared in a relatively reduced number of sentences (45 out of 763) in about one fourth of all the examples analyzed. Meanwhile, *all* examples were very explicit about the process to follow to solve an IVP. I argue that presenting examples as a collection of steps that suggest what needs to be done to solve the problem, without consideration of what needs to be done to verify that the answer is correct, leads, in the best case, to students who rely mostly on process-oriented strategies to assess whether their answers make sense, and, in the worst case, to students who do not attempt to verify the correctness of the solutions in the first place. This should be an area of concern for those who design and use textbooks.

As suggested by the Eizenberg and Zaslavski (2004) study, adding justification to the solution is useful. We see that in this study a number of textbooks provided elaboration sentences that explicitly justified intermediate steps. This is to be taken as a good sign, as an indication that it is feasible to include information that could potentially be used for justifying the correctness of the solution. It should be possible, then, to include statements that explicitly address why the justifications allow us to make sure that the answer is correct. The link between justification and verification needs to be made more explicit. Making this link more explicit might also be useful as students learn to prove.

Content strategies obviously depend on the content at stake. In this study, the availability of slope fields, of technology that can plot the differential equations, of a considerable amount of knowledge about functions and their behavior (these sections appeared after differentiation and after integration), and of a real-life context, all make this topic ideal for studying the uses of these devices in verification of the solution. As could be expected from Balacheff's theory of conceptions, different

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problems lead to different conceptions for a notion; in this case, it was possible to illustrate how aspects of the didactical contract, traditions regarding the content of examples that emphasize steps to follow, and the content itself interact to create a control structure that is, in many ways, dependent upon the types of problems considered here. It was crucial to find that the few strategies provided refer to different aspects of verification, but it is disappointing that in so few instances textbooks capitalized on the rich content associated with IVPs. Possibly, textbooks are written under the assumption that during class these opportunities will be realized. Having observed a wide range of instructors teaching introductory courses in different settings makes me doubt whether this assumption is often borne out.

The Control Structure in Honors Textbooks. The honors and non-honors textbooks differ. The prefaces of these textbooks indicate that it is assumed that the student who uses an honors textbook, or who is in an honors class, will read the textbook. Besides reading the textbook, the student is supposed to continually produce data and infer warrants so that the assertions made can be validated. As seen in one of the excerpts, questions such as "Why?" are introduced in the narrative, presumably as a reminder to the student that verification work is needed.

The discussion of why these textbooks are different is matter of a different analysis. The need for honors textbooks is predicated upon the idea of the honors student, one who is a markedly different person from the "average" ability student. However, for some scholars "ability" as a measure of innate capabilities or talent is a fictitious construct because mathematics performance, which is used to establish ability, is a consequence of structural practices that construct and reify the differences between high and low performers, therefore making the differences appear as "natural" (Dowling, 1998; Walshaw, 2001; Zevenbergen, 2005). I note here that instructors' possible assumption that non-honors students, in general, do not read the text in their textbook needs to be established empirically. Some of my data suggests that this might not be the case.

We can ask whether the differences that we observed in the two types of textbooks should be maintained, in particular, when we look at the control structure. I believe that there is some value in incorporating some of the strategies that were observed in the honors textbooks in the presentation of the non-honors textbooks. Here are a few examples:

- Pointing out that the number of initial conditions needed to find a unique solution depends on the order of the differential equation.
- Explaining the meaning and existence of constant solutions, an important characteristic of differential equations (it can be seen as a particular IVP) that can be addressed in tandem with the discussion of slope fields.
- Showing why the values obtained for time, distance, or height make sense in the proposed situation. In this study only one non-honors textbook explicitly addressed this issue, and another invited the reader to verify the reasonableness of the values by replicating the experiment.
- Discussing the kind of values that could be expected for a particular variable: IVPs grounded in a given context have the hypothetical advantage over abstract IVPs of providing other ways to verify answers and check procedures. The implicit assumption that most of the quantities are positive is pervasive. Beyond that, it would be important to know whether the quantities can range from 0 to infinity or are bounded.

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- Stating that finding solutions is very difficult whereas verifying whether a solution has been found is easy. The statement was made in only one of the non-honors textbooks, but it was made in all three honors textbooks. The contrast between verifying and finding is an important one, yet it is overlooked in most non-honors textbooks.
- Taking advantage of the slope fields to explain why the general and particular solutions are reasonable for the differential equation and IVP proposed. This was done in only 2 of the 9 non-honors textbooks.
- Making explicit the nature of the modeling involved in generating the differential equation and in finding a solution that would fit the IVP. The context was prominent in making decisions about the solution.

Conclusion

I am aware that these findings are limited by my selection of content and textbooks to analyze. The sections analyzed belonged to textbooks devoted to introducing the tools of calculus, rather than to presenting a theory of differential equations. We could not expect to have a complete treatment in a textbook that is by necessity introductory. However, I believe that what I found in the IVP sections is symptomatic of what happens across all textbooks. I chose this particular content because it affords the most potential for the control structure to emerge. The finding of strategies that address different aspects of the verification process corroborates that this was a good choice. However, given the few instances in which these strategies were made explicit across these textbooks, I wonder: Where should students learn to control their work? Further investigation is needed to determine whether and how these strategies should be made available during instruction and how the students who read these examples interpret and use these strategies.

The structure disclosed here does not look at regulatory aspects of metacognition that would be evident in an actual problem-solving session. Would students exposed to the different strategies described here exhibit them in a live problem-solving situation?

Assuming that students use the textbooks for doing exercises and that they read the examples in the text when in need, the study suggests that students will learn and will be able to recognize when they have found a solution, no matter what textbook they use. The study suggests, also, that depending on what textbook is used, students will be more or less aware of the need to verify that the solution is correct or that it makes sense. I believe that we should address this shortcoming. If examples in mathematics are to continue playing a crucial role for learning and understanding (Michener, 1978; Watson & Mason, 2005), they should illustrate not only the processes associated with solving problems, but also strategies for verifying that the solution is correct and that it makes sense. Doing so may provide a step in the right direction to improve students' understanding of the nature of validation and verification in mathematics.

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