The Addition and Subtraction of Fractions in the Textbooks of Three Countries:
A Comparative Analysis

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Abstract

In this paper, we report on an analysis of the treatment of addition and subtraction of fractions in elementary mathematics textbooks used in Cyprus, Ireland, and Taiwan. For this purpose we developed and applied a framework to investigate the learning opportunities afforded by the textbooks, focusing on seven criteria that captured the presentation of the addition and subtraction of fractions and the textbook expectations as manifested in the pertinent tasks. We found key differences among the textbooks regarding the constructs, the representations, and the worked examples used, as well as the cognitive demands of the textbook tasks. Differences in the topics included and their sequencing were also identified. Overall, the differences identified in this cross-national textbook comparison highlight the importance of closely attending to the learning opportunities crafted by the textbooks in each country, through analytic frameworks that support a more in-depth investigation of curriculum materials.

Keywords: comparative study, curriculum, Cyprus, elementary grades, fractions, Ireland, Taiwan, textbook analysis.
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Establishing how mathematics textbooks in different countries structure learning opportunities for their students is an important question that has received different treatments from the research community. No consensus has yet emerged as to how learning opportunities in textbooks can be evaluated and compared. In the last two decades, researchers have expressed contrasting views regarding what can be learnt from analyzing mathematics textbooks. Some researchers have gone as far as to claim that textbook analysis can explain the differences in students’ performance in international comparative studies (Fuson, Stigler, & Bartsch, 1988; Li, 2000). Other researchers have argued that textbooks exert little influence on instruction and on what students learn (Freeman & Porter, 1989). Yet other researchers have looked at the textbook as a potential source for teacher learning, a goal that is frequently unfulfilled (Newton & Newton, 2006; Remillard, 2005).

A more balanced viewpoint posits that textbooks afford probabilistic rather than deterministic opportunities to learn mathematics (Mesa, 2004; Valverde, Bianchi, Wolfe, Schmidt, & Houang 2002). If one approaches textbook analysis from this stance, it becomes what Mesa (2004) has termed

a hypothetical enterprise: What would students learn if their mathematics classes were to cover all the textbook sections in the order given? What would students learn if they had to solve all the exercises in the textbook? Would they learn the particular mathematical notions that are presented in the textbook? Would that learning work well in their future work in mathematics? (p. 255-256)
The conditional tense of these questions reflects the challenge that is faced by those who review textbooks because textbooks are incomplete until they interact with students and teachers in the practice of teaching and learning. Although this perspective acknowledges that textbook analysis is limited to portraying the intended curriculum (the goals and objectives for mathematics intended for learning at a national or regional level\(^2\); Travers & Westbury, 1989, p. 6) and not the implemented curriculum, important insights can be gleaned from studying textbooks used in different countries by illustrating similarities and differences in the opportunities to learn mathematics offered to students around the world. Such insights relate to the different performance expectations that are made of students in different countries, the extent to which a textbook series in a country prioritizes conceptual understanding or procedural fluency, and how the level of mathematical content differs among countries (e.g. Li, 2000).

Our search in the literature revealed a variety of frameworks that have been used to analyze different levels of opportunities to learn that textbooks offer to students.\(^3\) We saw a need in the field for a synthesis of frameworks that could be a tool for pursuing analyses of textbooks that would attend to both general and particular aspects of the textbook and that could describe the opportunities to learn that the textbooks afford; that is a tool that could document and compare the intended curriculum in concrete terms. In this paper, we illustrate how a framework that integrates two types of textbook analyses can be used to characterize learning opportunities about a specific content topic in elementary school mathematics, namely the addition and subtraction of fractions, in three countries, Cyprus, Ireland, and Taiwan.

\(^2\) Or the publisher’s interpretation of those goals and objectives.

\(^3\) For the purposes of this paper, opportunity to learn refers to a description of the complexity of the mathematics that is present in the textbooks that students would use in their classrooms, thus limiting its definition to the mathematical content as opposed to the more general notion that includes school factors that most directly affect student learning. For a historical description of the evolution of this notion, see McDonnell (1995) and Tate and Rousseau (2006).
We chose addition and subtraction of fractions because it is an important component of children’s knowledge about operations with numbers that is addressed in elementary school mathematics throughout the world (Mullis et al., 2004), but that has not been as extensively studied as multiplication or division (Verschaffel, Greer, & Torbeyns, 2006, p. 65). Furthermore, operations with fractions, in general, is a difficult topic to learn and to teach (Lamon, 1999).

Looking at different countries was necessary because we were interested in determining whether the proposed framework would capture a broader range of differences and similarities than might be found in textbooks from a single country. The three countries, Cyprus, Ireland, and Taiwan, were selected because they all have a national curriculum, and thus approved textbooks are likely to reflect the curriculum that public school students in each country are expected to study. Moreover, the three countries have significant differences in history, size, language, economy, culture, and student attainment in international comparative studies. We expected that these differences would be reflected in how opportunities to learn are expressed in the mathematics textbooks. In addition, our experiences as students and teachers of elementary mathematics in these countries provided us with country-specific knowledge conducive to such an analysis.

Based on our choice of countries and our chosen topic, we identified two research questions:

(1) What similarities and differences can be observed in the presentation of addition and subtraction of fractions in elementary mathematics textbooks in Cyprus, Ireland, and Taiwan?

(2) What are the expectations about addition and subtraction of fractions afforded by tasks in elementary textbooks in Cyprus, Ireland, and Taiwan?
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We start by providing a summary of relevant literature, followed by the methods we used to respond to the questions, and our findings. We conclude with a discussion of the limitations of the study and research and practical implications.

1. Literature Review

We present here a summary of literature regarding students’ difficulties with addition and subtraction of fractions and literature related to textbook analyses, in comparative studies.

1.1. Addition and Subtraction of Fractions

Developing a deep understanding of fractions constitutes the bedrock for learning more complex mathematical ideas including decimals, probability, and proportion. However, elementary and middle school children encounter significant difficulties in learning fractions, which have been conceptualized as having roots in the complexity of the notion of fractions itself and in the instructional approaches employed when teaching fractions (Ball, 1993; Behr, Harel, Post & Lesh, 1993; Lamon, 1999; Tzur, 1999).

Fractions are among the first abstract mathematical ideas that elementary students encounter; meanings, models, and symbols that worked well when working on whole numbers may interfere with developing understanding of fractions (Lamon, 1999). Difficulties in learning fractions also stem from the fact that fractions comprise a multifaceted concept, consisting of five interrelated constructs: part-whole, measure, operator, quotient, and ratio (Kieren, 1976). Because each of these constructs captures different aspects of fractions, constructing a comprehensive schema of fractions requires developing a robust understanding of each of the five constructs and of their confluence (Behr, et al., 1993).

Research has shown that instruction may impede the learning of fractions, especially when it fails to build on students’ prior knowledge (Mack, 2001), emphasizes rote learning at
the expense of conceptual understanding (Ball, 1993; Mack, 2001), introduces formal symbols and algorithms before familiarizing students with the different aspects of the notion of fractions (Smith, 2002), or emphasizes only one of the constructs (usually the part-whole interpretation, Moss & Case, 1999).

Students’ difficulties in adding and subtracting fractions have concerned the educational community for years. For instance, given students’ difficulties in the area of the addition and subtraction of fractions, in the early 1930s, the Committee of Seven launched a study to examine the grade at which the addition and subtraction of fractions should be placed, in order to facilitate students’ mastery in this area (Raths, 1932). Researchers have also explored what makes the addition and subtraction of fractions so hard for school children, and particularly, what causes errors such as adding (or subtracting) the numerators and the denominators. This exploration suggested that several cognitive factors might explain such students’ errors. For instance, it was proposed that students often view fractions as two separate whole numbers rather than as quantities (Carpenter, Coburn, Reys, & Wilson, 1976), and that they are often misguided by wrong analogies (e.g., because in the multiplication of fractions one multiplies the numerators and the denominators, by analogy, when adding or subtracting fractions one can add or subtract the numerators and the denominators, Vinner, Hershkowitz, & Bruckheimer, 1981). The role of curriculum materials in contributing to students’ difficulties has, however, rarely been studied.

1.2. Textbook Analysis in Cross-National Studies

Studies show significant cross-national differences among textbooks. Two large-scale studies that compared and contrasted the textbooks used in almost 40 countries concluded that textbooks vary “in a myriad of ways” (Schmidt, McKnight, Valverde, et al., 1997, p. 22) and that “they exhibit substantial differences in presenting and structuring pedagogical situations”
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(Valverde et al., 2002, p 17). Small-scale studies also found major differences between the mathematics textbooks used in China, Japan, the former Soviet Union, Taiwan, and the United States (e.g., Fuson, et al., 1988; Mayer, Sims, & Tajika, 1995) or between the textbooks used in different European countries (Pepin et al., 2001).

In our search of literature, we found three broad categories of approaches to cross-national textbook-analysis that we named, horizontal, vertical, and contextual. In the horizontal analysis the textbook is examined as a whole, as a piece of technology in the educational system (Herbst, 1995) and the analysis focuses on general textbook characteristics (e.g., physical appearance, the organization of the content across the book). Critics have argued that this approach fails to illuminate fundamental differences in the learning opportunities offered to students in different countries, because topics are not treated in the same manner and with the same degree of emphasis in the different textbooks (Howson, 1995). The vertical analysis attends to the ways in which textbooks treat a single mathematical concept (Li, 2000; Mesa, 2006) and views the textbook as an “environment for construction of knowledge” (Herbst, 1995, p. 3). Such restriction overlooks how the treatment of the topic being examined relates to other topics presented in the textbook. The third approach that we call contextual analysis, attends to the ways in which textbooks are used in instructional activities by either instructors or students (Mesa, 2007; Remillard, 2005; Rezat, 2007) and sees the textbook as an “artifact in the broader sense … historically developed, culturally formed, produced for certain ends and used with particular intentions” (Rezat, 2007). The third approach deals directly with problems of implementing the curriculum, that is, of realizing the intentions of the textbooks. Because we are interested in understanding the intentions of the textbooks and in understanding how other researchers have proceeded to analyze such intentions, we did not include the contextual
approach in our discussion. We believe, however, that a framework that incorporates both horizontal and vertical analyses of textbooks is an important first step towards a subsequent contextual analysis of textbooks.

Studies that have analyzed textbooks pursuing both a horizontal and a vertical analysis illustrate that such a complementary approach is not only feasible but also worthwhile (Howson, 1995; Pepin et al., 2001) as it provides the means for explaining the opportunities to learn that students and instructors have as they engage with their mathematics textbooks. One difficulty with these studies, however, is that there is no consistency across the methods used. The framework that we used in this study (Charalambous, Delaney, & Hsu, 2005; Charalambous, Delaney, Hsu, & Mesa, 2007) integrates vertical and horizontal analyses and is explicit about what is being analyzed, facilitating consistency of comparisons across textbook series and countries.

2. Methods

In this section, we discuss the framework criteria that guided our inquiry, provide background information of the educational systems under consideration, and describe the data collection and analysis processes.

2.1. The criteria employed in the study

The theoretical framework that guided our inquiry integrated a horizontal and a vertical approach in analyzing textbooks (Figure 1). The framework was developed through an iterative process that combined results from the literature with a preliminary exploration of the textbooks used in the three countries.\(^4\) For the purposes of this paper, we focused on seven criteria.

\(^4\) More information on the development of the framework can be found in Charalambous et al. (2005).
From the horizontal analysis, we considered the *topics* covered with respect to the addition and subtraction of fractions and their *sequencing*. From the “what is presented” aspect of the vertical analysis, we considered the different *constructs* of fractions (i.e., part-whole, ratio, operator, quotient, and measure); the *representations*; and the *worked examples* employed in the textbooks. From the “what is expected” aspect, we focused on the *potential cognitive demands* of the tasks included in the textbooks and the *type of answer* expected from students. We used the first five criteria to analyze the *presentation* of the addition and subtraction of fractions, and the latter two criteria to analyze the textbook *expectations*.

We envisioned that altogether these seven criteria could help capture different aspects of the potential of these textbooks to scaffold student learning. Examining what topics are included in the textbooks is necessary, because, although the presence of a topic in the textbook does not ensure that it is covered, a topic is more likely to be covered if it is included in the textbook (Valverde, et al., 2002). We considered the sequencing of these topics in the textbooks, because the way in which different topics are ordered in the curriculum materials might reflect “important cultural differences as to how mathematical thinking develops” (Schmidt et al., 1997, p. 61).

We examined the presence of the different constructs of fractions in the textbooks because previous studies suggest that these constructs can help students understand different facets of the notion of fraction (Lamon, 1999) and differ in their potential to support students’ competence in adding and subtracting fractions (Charalambous & Pitta-Pantazi, 2007). We analyzed the representations and the worked examples of the textbooks, because these are important means of conveying meaning to students and scaffolding their understanding (Mayer, Sims, & Tajika, 1995). Research has shown that, worked examples can facilitate skill acquisition
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(cf., Renkl, 2002). Finally, we included an analysis of the potential cognitive demands and the performance expectations, because the cognitive complexity of the tasks can determine the possibilities for student engagement in cognitively challenging tasks during instruction (cf., Stein, Grover & Henningsen, 1996) whereas asking students to explain and justify their answers is also considered fundamental in reinforcing students’ understanding of mathematics (NCTM, 2000).

2.2. The setting of the study: Educational systems and textbook publishing

The educational system in Cyprus is characterized by a centralized administration. Primary education, which is compulsory, covers grades 1 to 6 and is under the authority of the Ministry of Education and Culture. The academic year in Cyprus averages 180 days on a five-day week timetable. Mathematics is taught five times a week in Grades 1 and 2, and six times a week in Grades 3 to 6 in 40-minute periods. In Cyprus, promotion to the next grade is automatic. Curricula for primary education in Cyprus are developed by the Ministry of Education and Culture, which is also responsible for publishing and delivering the majority of the school textbooks, including those used in mathematics. For each primary grade, only one textbook series is published per subject and is given to students for free at the beginning of the year. Thus, all public primary schools use the same mathematics textbook series. Developed along the lines of the NCTM Curriculum and Evaluation Standards (1989), the math textbook series currently used emphasizes problem solving, reasoning, communication and discourse around mathematical topics, and the use of manipulatives and group work (Christou, Eliophotou-Menon, & Philippou, 2004). For each grade, the series consists of four volumes of student textbooks, a teacher guide, and an assessment textbook with summative tests for each of the units included in the textbooks.
In this study, we focused on the fourth-grade textbook (Ministry of Education and Culture, 1998), hereafter referred to as the Cypriot Textbook (CT).

Ireland has a national school system with a national curriculum. Primary education is compulsory and covers grades Pre-K to 6. Schools are open for 183 days per year. The national curriculum recommends that 180 minutes per week be spent teaching mathematics from first to sixth grade. There are 120 minutes of discretionary time to be allocated to curriculum subjects at local level by teachers; part of this time may or may not be given to mathematics. There is no national test in Ireland during or at the end of primary education, although recently schools have been required to administer standardized tests at the end of first and fourth grades (or beginning of second and fifth grade). Several educational stakeholders, including teachers, parents, teacher educators, the Department of Education and Science, were involved in developing the 1999 mathematics curriculum currently used in Irish primary schools. This curriculum emphasizes quantitative and spatial thinking, problem solving, communication, building of connections, reasoning, and applying mathematics in real life and other areas of learning (Government of Ireland, 1999). When the curriculum was revised, three primary school publishing companies also published new textbook series. Decisions about which textbooks to use are made at school level by the principal, usually in conjunction with the teachers. Buying textbooks is a responsibility of parents; however, students whose parents cannot afford to buy the required textbooks are supported financially by grants paid to schools by the Department of Education and Science. The two textbook series analyzed in the present study (Barry, Manning, O’Neill, & Roche, 2003; Courtney, 2002 – hereafter referred to as Irish Textbook A [ITA] and Irish Textbook B [ITB], respectively) include a student book and a teacher manual.
Primary education in Taiwan is compulsory and covers grades 1 to 6. The Ministry of Education has a decisive role in shaping the mathematics curriculum. The academic year in Taiwan is 200 days long. Schools can decide the time allotted to mathematics (which ranges from 112 to 186 minutes per week) with most schools allotting about 180 minutes. Students in Taiwan take three examinations per semester. Because these examinations are closely related to the textbook content, the teachers in Taiwan are required to cover all the textbook content. The Taiwanese textbooks analyzed in the current study (Mou & Yang, 2005; Li, Ku, Su, & Chen, 2005 – hereafter referred to as the Taiwanese Textbook A [TTA] and Taiwanese Textbook [TTB], respectively) were developed using the Grade 1-9 Curriculum Temporary Guideline (Ministry of Education, 2000), which emphasize the development of competences such as problem solving, communicating, calculating, reasoning, and building connections among different ideas. Following these curriculum standards, five Taiwanese companies have published textbooks. Decisions about which textbook to use are made at the school level by the principal and teachers of mathematics. Like Ireland, textbooks in Taiwan are not supplied by the school; parents are expected to buy them, unless they have financial difficulties. Each of the textbook series analyzed in the present study includes student textbooks, student workbooks, and teacher guides.

Because in the present study we focused on the written curriculum (i.e. the objectively given structure of the textbook) and not on how the curriculum is presented by the teacher or enacted by students during instruction (i.e., the subjectively scheme, cf., Herbel-Eisenmann, 2007), we limited our analysis to the student textbooks. This approach also helped minimize inference-making, because analyzing the teacher guides as well would have forced us to make
interpretations of how teachers draw on these manuals and the student textbooks to structure their lessons.

2.3. Data collection and analysis

Selecting the grade levels on which to focus our analysis was a challenge. We believed that a direct comparison of textbook series across grades in the three countries would be problematic for several reasons. First, different amounts of time are devoted to mathematics teaching in the three countries. For example, Ireland’s curriculum allocates only three hours per week to teaching mathematics in classes from first grade to sixth whereas Cyprus allocates four hours. Taiwan is similar to Ireland but students attend school for almost twenty additional days per year. Similarly, students’ ages may vary across grades because students in Ireland spend eight years in primary school from the age of four or five whereas students in Cyprus and Taiwan spend six years from approximately age six.

Second, based on content included in earlier grade levels, textbooks in different countries may demonstrate different assumptions about the prior knowledge that students are expected to possess when a topic is presented in a particular grade. In our study, for example, we found that in Taiwan students are introduced to “composition and decomposition” of fractions in third grade; they are first expected to compose and decompose fractions with concrete materials (e.g., pizza) and then connect the activities using concrete materials with the algorithms for adding and subtracting fractions. The presentation of this topic in third grade might determine how it is presented in the textbook for the next grade level.

A third problem in selecting grade levels to analyze was that in Taiwan the curriculum was reformed twice between 1999 and 2006 and this led to changes in how the topic was presented in textbooks over this period. In Ireland, the curriculum was revised in 1999 and has
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not been revised since and textbooks produced in 2002 and 2003, based on the 1999 curriculum, have not been revised. Similarly, in Cyprus the current textbook has been used since 1998.

Given these difficulties, we decided to look at how addition and subtraction of fractions was taught in fourth grade in Cyprus and Taiwan because substantial, formal work with algorithms was done on the topic in this grade in both countries. In Ireland the topic is not formally introduced until fifth grade, so we looked at how the topic was treated in fifth grade there. By selecting these grade levels we would be able to observe how the topic was sequenced within each textbook, how worked examples were presented, the level of cognitive demand inherent in the tasks, and what was expected of students in terms of performance on the tasks. Such insights would be important for learning how the topic might be treated differently across countries and may offer benchmarks for comparison with other countries. Other countries may find that coverage of the topic in their textbook is similar or different to one or more of the examples that we analyze. Within each country the textbooks were chosen on a typical case sampling basis and therefore our findings are illustrative of how one mathematical topic is presented by textbooks in each country (Patton, 2002).

Having selected the topic and the grades to study, we identified all the pages in each textbook where the addition and subtraction of fractions were considered. We photocopied these pages, and, in the case of the Cypriot and Taiwanese textbooks, translated them into English to support the cooperative analysis of the content of all textbooks by all four authors and to gauge inter-rater reliability. To examine the topics introduced in these textbooks and their sequence, we listed the topics introduced in each textbook and their order of appearance, compiled all the topics into a table, and reexamined whether these were present or not in each textbook (see Table 1). We also included the sequence of these topics in all textbooks.
For the remaining five criteria, (worked examples, fraction constructs, representations, potential cognitive demands, and performance expectations we followed a combination of top-down and bottom up approaches for establishing the meaning and the coding for our categories. We relied mostly on the literature to derive meaning for our categories (a top-down approach) but we also relied heavily on the actual content in the textbooks (bottom-up approach) and went back and forth between the literature and the textbooks, in order to create a coding system that was consistent with the literature and with the content of the textbooks.

Drawing on Watson and Mason (2005), we considered as worked examples those portions of the textbook “that demonstrate the use of specific techniques” (p. 3), in our case the addition and subtraction of fractions. We avoided limiting our exploration to fully worked examples, which include solution steps and the final solution itself (cf. Renkl, 2002, p. 529), because an initial exploration of the textbooks of the three countries pointed to differences in the level of completeness of their worked examples. This approach resonates with how worked examples were considered in other studies (e.g., Valverde et al., 2002).

In analyzing the worked examples, we followed a grounded-theory approach (Strauss & Corbin, 1998). First, we considered the procedures for which a worked example was included in the textbooks and how these worked examples were sequenced in the textbooks. Comparing the worked examples presented in the textbooks, and particularly seeking to identify the ways in which these examples attempted to facilitate student learning also led us to generate a set of questions that guided subsequent analysis. Thus, in addition to considering the constructs and the representations that these worked examples employed, we asked: Do these examples state a rule or the steps for a procedure to follow? Do they present more than one method for the same procedure (e.g., for adding two mixed numbers), and if so, do they build connections between the
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different methods presented? Do these examples employ other graphical displays, in addition to
the representations identified above? And, are these examples situated in a mathematical, more
abstract, context or are they embedded in situations closer to students’ daily experiences (e.g.,
sharing food)?

Tasks were defined as the textbook segments that require students to answer one or more
questions, apply a procedure, or solve one or more problems. We used the higher level of
numbering of the exercises/problems designated by the textbook authors to identify a task.
Because this resulted in tasks that ranged considerably in their requirements, we further parsed
tasks into subtasks, each of which required students to address one question.

In analyzing the fraction constructs, we used the definitions provided in previous studies
(e.g., Behr et al., 1983; Lamon, 1999). In particular, the part-whole construct was considered to
represent a situation in which a continuous quantity or a set of discrete objects are partitioned
into parts of equal size (e.g., a pizza divided into \( n \) equal parts); the ratio construct was taken as a
comparison between two quantities (e.g., a comparison between the boys and the girls of a class);
the operator construct reflected a function that transforms line segments, figures, or numbers
(e.g., How many times should 9 be increased to get 15?); the quotient construct was considered
the presentation of a fraction as the result of a division of two whole numbers (e.g., How much
pizza would four friends get if they share five pizzas evenly?); and the measure construct related
to identifying fractions as numbers or associating fractions with the measure assigned to some
interval (e.g., locating fractions on number lines). After an initial inspection of the textbooks,
we included an additional category that pertained to a combination of the abovementioned
constructs, the most frequently identified being the part-whole and operator constructs. We

\[ \text{\textsuperscript{5}} \] A more detailed description of these constructs as used in the present study can be found in Charalambous and
Pitta-Pantazi (2007).
counted the construct(s) presented in each subtask but classified subtasks that provided no context that could support classification as “non-identified.” We also examined which constructs were used in the worked examples (see Table 1).

In analyzing the representations used in the textbooks, we excluded those representations that were unrelated to the content (i.e., decorative representations). Based on previous studies that examined the functions of representations in supporting student learning (Baturo, 2004; Boulet, 1998; Lamon, 1999), we classified the remaining representations into continuous representations, discrete representations, and number lines. Continuous representations were those representations for which “each part of the whole is a single continuous piece and contiguous to other parts” (Behr, Wachsmuth, & Post, 1988, p. 2), and could be one-dimensional (linear), two dimensional (circular, rectangular or otherwise shaped); and three dimensional (volumetric or pertaining to the weight of different commodities).\(^6\) Discrete-sets representations were defined as those depicting a set of discrete objects that comprise a unit. These objects (e.g., eggs, glasses) did not need to be of the same size or even the same shape, because what matters is the cardinality of the objects in the set (Behr et al., 1988). Finally, we included number lines because they emphasize the measure construct of fractions; these representations have been considered critical for supporting students’ understanding of the aforesaid construct of fractions (cf., Hannula, 2003; Lamon, 1999). When there were no accompanying representations we used the code “Representations: Not present.”

We analyzed the potential cognitive demands of each subtask using the *Task Analysis Guide* (Stein, Smith, Henningsen, & Silver, 2000, p. 16). According to this guide, *memorization* ...

\(^6\) See examples in Figure A1 in the Appendix.
tasks ask students to reproduce previously learned facts, rules, formulas, or definitions and can be solved without using procedures. Tasks requiring *procedures without connections* to meaning are usually algorithmic and have no connection to the concepts or meaning that underlie the procedures being used. Tasks requiring *procedures with connections* to meaning focus students’ attention on the meanings and concepts underlying the procedures needed to solve the task. In solving these tasks, the students cannot use a learned procedure mindlessly. According to the classification proposed by Stein et al., these tasks should not require the application of a well-established procedure. Tasks characterized as *doing mathematics* require complex and non-algorithmic thinking because of the unpredictable or not easily discernible nature of their solution process. The first two categories correspond to tasks of lower level demands, and the other two to tasks of higher cognitive complexity.

Finally, we classified each subtask according to its performance expectation. Following previous studies (e.g., Mayer et al., 1995; Schmidt, et al., 1997) we examined whether a subtask explicitly requires students to (i) provide only an answer (numerical answers or numerical expressions), (ii) explain their answer or the process they followed to get that answer, and (iii) justify the reasonableness of the approach they pursued in solving the task or the rationality of their answer. After the first round of analysis, we added a fourth category to account for some subtasks that asked students to provide both an answer and a mathematical sentence that can be used to get this answer (see Table 1).

2.4. Reliability of the coding system

We took several steps to ensure the reliability of the coding process. First, we selected particular examples from each country for understanding if the categorizations proposed in the framework were capturing the cultural differences among three countries and purposefully chose
tasks that we thought would be difficult to categorize. We selected 10 percent of the total pages from textbooks in each country and asked different pairs of the authors to code the tasks along the four dimensions of representations, cognitive demands, performance expectations, and fraction constructs. As part of the process, we also tested our definition of worked examples and of what counted as a (sub)task. We used Cohen’s $\kappa$ because it allows us to assess inter-rater reliability when there are two coders and the variables have several categories. According to Landis and Koch (1977) a $\kappa = 0.40$ to 0.59 is moderate inter-rater reliability, 0.60 to 0.79 is substantial, and 0.80 outstanding. In this first trial we found outstanding reliability for cognitive demands for Ireland and Taiwan ($\kappa = 1.0$ in both cases) and performance expectations for Cyprus and Ireland ($\kappa = 1.0$ in both cases). We had substantial reliability for representations for Cyprus and Ireland ($\kappa = 0.71$ and $0.65$ respectively) and construct for Taiwan ($\kappa = 0.73$). We had moderate reliability for representations in Taiwan ($\kappa = 0.58$). For the rest, cognitive demand in Cyprus, performance expectations in Taiwan, and construct in Ireland and Cyprus, we had low reliability.

The results revealed difficulties in determining what counted as a task, especially in the Taiwanese textbooks; it also highlighted different definitions of cognitive demands; difficulties categorizing what counted as a worked example; and difficulties attributing the construct used in a given task. We elaborated and revised some of our definitions for coding performance expectations, fraction constructs, and cognitive demands and to test them we selected another 5% of pages also searching for the cases in which we thought we would find the most difficulties in coding. In this second test, we had outstanding reliability for representations, performance expectations, and construct for all the countries ($\kappa \geq 0.90$) and low to moderate for cognitive demands ($0.36 \leq \kappa \leq 0.70$).
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The discrepancies emerged in distinguishing between procedures without connections and procedures with connections. Upon comparison, we realized that the definition for these categories assumed that the procedure was well-established (Stein et al., 2000). Because we could not decide when a procedure becomes well-established in the textbook, we decided to disregard this characteristic of the classification and instead made sure that a task would be connecting the procedures at stake to their mathematical meaning (i.e., addition and subtraction of fractions) for it to be coded as procedures with connections. We also agreed that when in doubt, we would err towards assuming that the procedure required connections. This helped bring consensus. In light of these results, we proceeded to code all tasks in the textbooks and held another meeting to decide on tasks for which raters had difficulties coding. We resolved these through consensus.

3. Findings

We present the findings of the study organized by the two research questions of the study, starting with the presentation of the addition and subtraction of fractions in the textbooks of the three countries followed by the findings regarding textbook expectations for student engagement with this content.

3.1. The Presentation of Addition and Subtraction of Fractions

3.1.1. Topics Covered and Sequence of these Topics

We found differences among the three countries. First, in the textbook from Cyprus and one of the textbooks in Taiwan (TTA) the addition and subtraction of fractions and mixed numbers with like denominators is present in the fourth grade textbooks, whereas the additive operations on fractions and mixed numbers with unlike denominators are considered in the fifth grade textbooks. Both Irish textbooks present the addition and subtraction of fractions and mixed
numbers with similar and different denominators. Only one of the Taiwanese textbooks analyzed, (TTB) includes the addition and subtraction of fractions with dissimilar denominators in fourth grade.

Second, as illustrated in Table 2, the Irish textbooks follow a slightly different sequence in the presentation of the content compared to the textbooks in the other two countries. The Irish textbooks start with the addition and subtraction of fractions with similar or unlike denominators. Then, both of them introduce additive operations on mixed numbers whose fractional part has different denominators; yet, none of the tasks that involved mixed numbers considers fractions with the same denominator. In contrast, the CT and one of the Taiwanese texts (TTB) both start with the addition and subtraction of fractions with similar denominators and then move to the addition and subtraction of mixed numbers whose fractional part has the same denominator. These two textbooks cover all possible cases of the addition and subtraction of fractions with the same denominator: addition and subtraction of proper fractions; addition (subtraction) of a proper fraction to (from) a whole number, a mixed number or an improper fraction; and addition and subtraction of two mixed numbers or improper fractions. After covering these cases, the TTB shifts to the addition and subtraction of fractions with dissimilar denominators. This is in stark contrast to the Irish textbooks, which both give more emphasis to the addition and subtraction of fractions with dissimilar denominators. Like TTB, TTA first presents the addition of proper fractions, but then considers the subtraction of a proper fraction from a mixed number or an improper fraction before considering the subtraction of a proper fraction from another proper fraction or from a whole number; the addition and subtraction of mixed numbers is introduced last.
In all the textbooks under consideration, addition and subtraction are interspersed, meaning that none of the textbooks first presents all possible operations on the addition of fractions and then moves to the subtraction of fractions. We also found that all five textbooks covered the same content before presenting the addition and subtraction of fractions or mixed numbers (concept of fractions, equivalent fractions, fraction simplification, converting improper fractions to mixed numbers or vice versa, and comparing and ordering fractions).

3.1.2. Fraction Constructs

A fraction construct may be identified by an accompanying representation or by the context in which a problem is set. In coding fraction constructs, we did not assign a construct to subtasks when the subtasks were not embedded in a context that would allow classification decisions (e.g., the subtask only asked students to add two fractions and there was not any accompanying context). Because of this, a large number of subtasks in the Irish textbooks (97% for ITA and 80% for ITB) and about half of the subtasks in the CT (47%) were coded as not assigned to a particular fraction construct (Table 3). In contrast, in the Taiwanese textbooks, it was possible to determine the fraction construct for most of the subtasks (more than 65% for TTA and 80% for TTB).

Table 3 shows that the part-whole interpretation of fractions is dominant in both the Cypriot and the Irish textbooks. This construct appears more than four times as frequently as the other constructs altogether in the Cypriot textbook (44%) and it is the only construct present in

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Although our analysis focuses on the subtasks in each textbook, the textbooks also differed in their number of subtasks within each task. The tasks in both Taiwanese textbooks tended to include only one or two subtasks (above 90% of the textbook tasks). In contrast, more than 40% of the tasks in the CT and more than 50% of the Irish textbooks included tasks with five or more subtasks. In the CT, there were two tasks that had more than 20 subtasks.
the ITA. The part-whole construct is as frequent as the measure construct in ITB (9%). In contrast, the part-whole construct appears less frequently in both Taiwanese textbooks (6% and 14%, for TTA and TTB, respectively). The dominant category in the Taiwanese textbooks was what we identified as the combination of part-whole and the operator constructs, occupying about 27% of TTA and 42% of TTB. In these subtasks there is information on the number of discrete objects within each unit, which allows using both constructs in solving the problem. Consider the following example:

One bag has 30 peaches. Feng-Hau has 3/10 bags of peaches and Shai-Yuh has 9/10 bags. How many fewer bags of peaches has Feng-Hau compared to Shai-Yuh? (TTB, p. 13)

Students can solve the problem by using the part-whole construct (i.e., Feng-Hau has 9/10-3/10=6/10 bags fewer than Shai-Yuh). The inclusion of the number of peaches in each bag also gives students the opportunity to solve the problem following a more complex, yet legitimate, approach, that draws on the operator construct (i.e., Feng-Hau has 3/10 bags of 30 peaches = 9 peaches and Shai-Yuh has 9/10 bags of 30 peaches = 27 peaches; thus, Feng-Hau has 18 peaches less than Shai-Yuh which is 18/30 bags of peaches or 6/10 bags). Such tasks appear less frequently in the CT and are absent from both Irish textbooks.

The measure construct was the second most frequently observed in four of the textbooks (about 5% for CT, 9% for ITB, 31% for TTA, and 6% for TTB). In most of the cases, these subtasks provided students with the measure of a line segment, the weight of different goods, or the volume of the liquid in volumetric glasses and did not require students to actually measure a given interval using a fractional number.
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The Taiwanese textbooks also differ from the Cypriot and the Irish textbooks in two important aspects. First, the Taiwanese textbooks include what we dubbed as twin questions, two subtasks that always appear in sequence and ask students to consider the “how many” (i.e., absolute amount) and the “how much” (i.e., relative amount) questions. Consider, for instance, the following example:

One bag has 12 tomatoes. Somebody used $\frac{2}{6}$ bags to prepare a soup. How much of a bag of tomatoes was left? How many tomatoes is this equal to? (TTA, p. 83)

The first question in this example prompts students to consider a relative amount, namely, how much of a bag of tomatoes was left, after using a portion of it ($1 - \frac{2}{6} = \frac{4}{6}$). The second question, which often requires the employment of the operator construct, prompts students to consider an absolute amount, in this case the number of tomatoes that were left ($\frac{4}{6}$ of 12 tomatoes). That the two questions appear together offers students the opportunity to link the relative to the absolute amounts, which is considered critical for the understanding of fractions (Lamon, 1999).

Second, both Taiwanese textbooks seek to familiarize their students with the concept of the unit fraction as it relates to addition and subtraction. For instance, after introducing the example mentioned above, TTB asks students to start with $\frac{1}{5}$ and add another $\frac{1}{5}$ each time, till they get to the whole unit. Other tasks also prompt students to consider a part of the whole as consisting of a number of fraction units. Take, for example, the task illustrated in Figure 2.

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Insert Figure 2 about here

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8 The concept of unit fraction provides some unity to the five constructs of fractions, because it brings together three fundamental ideas – equi-partitioning, unit, and quantity – that pervade all fraction constructs (cf., Carpenter, Fennema, & Romberg, 1993, p. 3). Brousseau (1997, pp. 104-105) also comments on the affordances of the unit fractions, as used by ancient Egyptians, for the addition of fractions.
Addition and Subtraction of Fractions

In this task, the white cube serves as a *pointer* for the unit fraction. The question “How much of the blue block is a white cube equal to?” may lead to establishing that a white cube is equal to 1/9 of the block of blue cubes. Then, the two questions “How much of the blue block are four white cubes equal to?” and “How much of the blue block are 4 of 1/9 of the blue block?” prompt students to connect the 4/9 of the block of blue cubes to 4 white cubes, or alternatively 4 of 1/9s (i.e., 4/9 = 4 x 1/9). The last set of twin questions (“If we have 4 of 1/9 of the blue block and add another 1/9 of the blue block, how many of 1/9s of the blue block will we get? How much of the blue block will this be equal to?”) prompts to consider the sum of adding blue cubes to the existing block of cubes as a part of the whole (e.g., 5/9 of a block of blue cubes) and as a set of unit fractions (e.g., a set of 5 1/9s).

Finally, in building situations to study the addition and subtraction of fractions, one of the Taiwanese textbooks (TTB) uses the quotient construct. None of the textbooks examined included addition or subtraction tasks that used the ratio construct of fractions.

3.1.3. Representations

A common pattern identified in the textbooks of all three countries is that many tasks are presented devoid of any accompanying representation. This is the case for all the subtasks in the Irish textbooks with only one exception, and for more than 60% of the subtasks in the Cypriot and Taiwanese textbooks (Table 4).

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Although the continuous representations (i.e., linear, rectangular, circular, other-area shaped, and volumetric-weight) are the most dominant in the Cypriot and Taiwanese textbooks, there are notable differences in the type of these representations in the textbooks examined. The CT employs a greater variety of continuous representations, compared to the Taiwanese
textbooks. The Cypriot textbook includes one-, two-, and three-dimensional continuous representations. The Taiwanese textbooks, in contrast, employ mostly linear representations (TTA) or circular two-dimensional representations (TTA and TTB).

Compared to the Cypriot and Irish textbooks, the Taiwanese textbooks, include a larger proportion of tasks that use discrete set representations (e.g., bag of cookies or eggs, boxes of stationery materials or fruits). These representations (about 15% and 11% in the TTA and TTB, respectively) appear in tasks that require students to consider the relative and absolute amount in tandem, as they depict both the unit (the objects as a whole) and the number of objects within the unit. None of these textbooks included tasks that had a number line for representing addition and subtraction of fractions.

3.1.4. Worked Examples

Each textbook contained roughly similar numbers of worked examples in the sections on addition and subtraction of fractions with four of the five textbooks having between 9 and 13. The exception was TTA which had only five. Although we found recurring patterns among the worked examples across all the textbooks the overall finding is one of more differences than similarities. The recurring patterns related to the procedures presented, the constructs, and representations employed, whereas the differences related to the completeness of the worked examples, the contexts in which the worked examples were placed, the frequency and type of other graphical displays, and the number of methods presented to solve problems.

The most common procedures demonstrated in the worked examples in every textbook series were adding two proper fractions, adding two mixed numbers (or improper fractions), subtracting two proper fractions, and subtracting two mixed numbers (or improper fractions). We also found some examples of adding a whole number to a proper fraction (2 each in CT and
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TTB) and some examples of subtracting a proper fraction from a whole number (1 each in CT, ITA, TTA, and TTB). We found no examples where proper fractions were added to or subtracted from improper fractions. The Irish textbooks and one of the Taiwanese textbooks, TTB, had worked examples where the denominators differed. All textbooks included examples of subtracting two mixed numbers where the fractional part of the subtrahend is greater than that of the minuend (Figure 3).

We identified many differences among the three countries in the worked examples. The predominant representation in the worked examples across textbooks in the three countries was an area model. This representation can be seen in Figures 3a (from CT), which features pizzas and apple pies, and 3c (from ITA), featuring a more abstract series of representations. In the CT every worked example was accompanied by an area model (two-thirds of them circular and one-third rectangular). In ITA, 12 worked examples were accompanied by a circular representation, all analogous to that reproduced in Figure 3c; and one had a rectangular area model. ITB was different in that 8 of the 10 worked examples had no pictorial representation (see Figure 3d) but the other two had area models, one circular and one rectangular. There were five worked examples in TTA, two had no representation, two had linear representations and one had a circular area representation. The greatest variety was found in TTB where there were 13 worked examples. Four of them had no representation, three had linear representations, two had circular area models and four consisted of discrete sets.

Fewer differences were observed among the fraction constructs. For example, the part-whole construct occurred in all textbook series. There were no exceptions to this in the Cypriot and Irish textbooks (see Figure 3a and 3c). In ITB no construct was specified in 8 of the 10
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worked examples (Figure 3d) but when a construct was used, it was part-whole. Although part-whole was the most common construct in both Taiwanese texts, in 2 out of 5 examples in TTA and in 8 out of 13 examples in TTB the part-whole construct was accompanied by the operator construct. Consider the following example:

One box has 15 jellies. There was one box of jellies and Xai-Guan ate 3/5 boxes. How much of a box of jelly is there now? How many jellies is this equal to? (TTA, p. 83).

The textbook suggests two ways of approaching this problem, the first corresponding to the part-whole interpretation of fractions (i.e., “One box of jellies equals 5/5 boxes of jellies. 5 of 1/5 minus 3 of 1/5 is equal to…”) and the second pertaining to the operator construct (“1/5 boxes is 3 jellies. 3/5 boxes are 3*3=9 jellies. The rest is …”). TTA also included the measure construct of fractions in two worked examples, the only textbook of the five to do so (see Figure 3b).

The Irish textbooks differed from those in the other two countries in the completeness of the worked examples. In both Irish textbooks every worked example included the solution (Figures 3c and 3d), whereas examples including the solution were uncommon in the textbooks from both Cyprus (2 out of 7) and Taiwan (1 out of 5 in TTA and 6 out of 13 in TTB). For example in the CT example, (Figure 3a), a diagram is presented but the students are required to color the pizzas and the apple pies themselves.

All the Irish worked examples in both textbooks were explicit in illustrating the steps to be followed when completing the procedure. In TTB two thirds of the examples were similarly explicit in illustrating the steps but in TTA and in CT only about half of the worked examples illustrated the steps of the procedure being explained. In the CT example in Figure 3a students must decide which pizzas to color in order to color 1 1/3. Similarly, the TTA example in Figure 3b does not remind students how to convert mixed numbers into improper fractions but simply
states it as something that was done. In contrast, the example in Figure 3c, from ITA, presents the student and teacher with a series of representations explaining each step of the procedure.

Another instance where the Irish textbooks differed from those in the other two countries was with regard to the context around which the worked example was built. Most worked examples in the Irish texts were set in exclusively mathematical contexts (11 out of 12 in ITA and 8 out of 10 in ITB, see Figures 3c and 3d). In the other two countries, worked examples were more likely to be embedded in “real world” contexts (6 out of 9 in CT; 5 out of 5 in TTA and 13 out of 13 in TTB).

The case of graphical displays other than mathematical representations in worked examples was another difference, with the exception being Taiwan. There were no graphical displays among the examples analyzed in the Irish textbooks and only 2 were found in the Cypriot text (e.g., Figure 3a). However every worked example in both Taiwanese textbooks had other graphical displays and therefore we looked at these more closely. In most cases, the graphics were cartoon pictures of students with speech bubbles from their mouths explaining the steps of a procedure (Figure 3b). Other graphics in TTB contained reproductions of writing on a chalkboard, possibly used to model what a teacher might write in class or what students might write in their notebooks. The other frequently used type of graphic in the Taiwanese worked examples seems to serve the purpose of enhancing the “real world” context of the example.

The Taiwanese textbooks also differed in the number of methods that were demonstrated in the worked examples. In all the worked examples from Cyprus and Ireland only one method was shown. However, only two Taiwanese worked examples (in TTB) restricted the procedure demonstrated to one method. All of the others presented at least 2 methods (Figure 3b), and 2 examples in TTB presented 3 methods. In this example, a student faces the problem of being
unable to subtract $4/5$, the fractional part of a mixed number, from $3/5$, the fractional part of another mixed number. Two possible solutions are presented. One is to change both mixed numbers to improper fractions. The second is to rename as a fraction one unit from the mixed number minuend and to add it to the fractional part of the minuend.

3.2. Textbook Expectations for the Addition and Subtraction of Fractions

3.2.1. Potential Cognitive Demands

The most notable difference across the textbooks of the three countries concerns the potential cognitive demands of their subtasks (see Figure 4). In both the Cypriot and the Irish textbooks over 85% of the tasks were classified as representing low cognitive demand (i.e., procedures without connections), whereas in about 71% of the subtasks in the TTA and 81% in the TTB were classified as having high cognitive demands (procedures with connections or doing mathematics).

The procedures-without-connections tasks in the Taiwanese textbooks were in general of a different nature than those included in the Cypriot and Irish textbooks. For example, besides asking students to find the sum or the difference of two fractions or mixed numbers (e.g., $6/7 + 4/6 = ?$), in some cases the tasks in the Taiwanese textbooks required students to find the missing addend or minuend/subtrahend (e.g., $8\ 5/12 + ? = 10\ 1/12$; $? + 2\ 6/7 = 5\ 3/7$; $20\ 3/16 - ? = 7\ 9/16$; $? - 1\ 23/30 = 3\ 19/30$ (TTB, p. 23 & 25). In the Cypriot textbook there were only two such tasks ($? + 1/5= 4/5$ and $3/7 - ? = 2/7$; vol. 3, p. 47). Previous studies (e.g., Fuson, 1992; Marshall, 1995) showed that tasks of the latter structure impose more demands on students than tasks in which the unknown quantity is the sum or difference. We also noticed that in both Irish textbooks there was an apparent progression from tasks of lower demands to tasks of higher
demands, whereas in the textbooks of the two other countries tasks of lower and higher demands were interspersed throughout the relevant sections.

3.2.2. Performance Expectations

Finally, we considered the performance expectations that the subtasks required from students. All the subtasks in the Cypriot and the Irish textbooks required students to provide a single answer (Figure 5). In contrast, about 29% of the subtasks in the TTA and 56% of the subtasks in the TTB expected students to additionally write the mathematical sentence to represent the underlying structure of the problem.

Explaining the solution approach was expected in about 8% of the tasks in the TTA, and it was not observed in any of the other textbooks. Consider the following example:

The length of the red rope is \(2 \frac{37}{100}\) meters. The length of the white rope is \(14/100\) meters. How long are the two ropes altogether? Write the mathematical sentence for this problem and give the final result in the form of a mixed number. Explain your thinking process in solving this problem (TTA, p. 76).

In this problem, students are not only expected to write a mathematical sentence and use it to figure out the answer to the problem, but they are also required to explain, presumably in words, their thinking in reaching a solution.

4. Discussion

In this study, we sought to examine the potential learning opportunities afforded to Cypriot, Irish, and Taiwanese students in working on the addition and subtraction of fractions. To this effect, we explored the pertinent content presented in the national textbook series used in
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Cyprus and two textbook series widely used in Ireland and Taiwan (first research question) and the expectations that these textbooks have for students when adding and subtracting fractions (second research question). In this section, we discuss the results that pertain to each research question. In the next section, we conclude with a discussion of the implications and limitations of the study.

4.1. The Presentation of the Addition and Subtraction of Fractions in the Textbooks

We used five criteria to explore the content that is presented in the textbooks under examination: the topics introduced in the textbooks and the way they are sequenced; the fraction constructs and the representations employed; and the worked examples presented to support students’ comprehension of the relevant content. We discuss the findings that correspond to these criteria starting from what we consider to be the most important differences.

Previous studies that focused on the teaching and learning of fractions (Baturo, 2004; Boulet, 1998; Lamon, 1999) have emphasized that engaging students with multiple fraction constructs catalyzes learning, not only because each of the fraction constructs captures part of the broader notion of fractions, but also because moving from one construct to another reinforces understanding. Affording students’ opportunities to consider constructs other than the part-whole interpretation of fractions has also been considered critical, because the latter construct has certain limitations (e.g., the whole cannot exceed the number of partitions) which, according to some researchers (e.g., Smith, 2002), might impede students’ understanding of and work with improper fractions. Our analysis pointed to salient differences in the way in which the textbooks in the three countries craft opportunities for their students to consider multiple fraction constructs when adding and subtracting fractions. Whereas the part-whole construct of fractions can be considered the inroad to presenting the addition and subtraction of fractions in the Cypriot and
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both Irish textbooks, in the Taiwanese textbooks, this construct is presented in tandem with the operator construct. With the exception of the ratio construct, which was not present in any of the textbooks studied, the other constructs (i.e., operator, measure, and quotient) were also more common in the Taiwanese textbooks than in the textbooks of the other two countries. The majority of the tasks in TTA emphasized the measure construct, a construct that is pivotal for understanding both the notion of fractions and their addition and subtraction (Lamon, 2001; Keijzer & Terwel, 2001). For instance, Behr and associates’ (1983) model presents the measure construct alongside the part-whole construct as the most “natural” to promote understanding of and competence with additive operations on fractions (p. 100).

The tasks in the Taiwanese textbooks drew largely on the concept of unit fractions, a notion that has been considered a tie among different constructs (e.g., part-whole, the measure, and the quotient) and a catalyst for students’ transition from whole to rational numbers (Carpenter, Fennema, Romberg, 1993; Mack 2001). Equally important for students’ transition to rational numbers is the notion of relative quantity (Lamon, 1999). In contrast to the Cypriot and Irish textbooks, the Taiwanese textbooks systematically included tasks that considered both the absolute and the relative amounts, by prompting students to respond to what we identified as twin questions: the “how much” and the “how many” aspects of a given situation.

Our analysis also showed salient differences regarding the frequency and the type of representations used to support students’ engagement with the addition and subtraction of fractions. With the exception of only one subtask, we found that the tasks in the Irish textbooks did not include representations; this seems a point of departure from the current consensus among researchers and educators about the role that representations have in supporting student learning of fractions (Ball, 1993; Lamon, 1999). We also found that whereas the Cypriot
textbook tends to use continuous representations (mostly area representations), the Taiwanese textbooks make use of both continuous and discrete representations, a combination found to promote a more profound understanding of fractions and their operations (Behr et al., 1988; Kyriakides & Charalambous, 2002). However, none of the textbooks examined included tasks on the addition and subtraction of fractions that made use of number lines, a powerful representation for supporting students’ understanding of fractions as quantities- numbers, which is critical for a conceptual understanding of the additive operations with fractions (Lamon, 1999).

Our analysis showed that in all three countries, the textbooks presented worked examples. However, there were notable differences in the examples in these textbooks. Whereas the Cypriot and the Irish textbooks limit their examples to the part-whole construct of fractions and to continuous area representations, the examples in the Taiwanese textbooks were more diverse in their use of fraction constructs and representations: they offered worked examples on the operator and the measure constructs which were accompanied by discrete set or continuous linear representations. We also found differences in the worked examples’ level of completeness and the number of methods modeled in them. In general, the most complete examples were found in the Irish textbooks which clearly spelled out a sequence of steps for students to follow when adding and subtracting fractions; the examples in the textbooks of the other two countries were partially complete and expected students to complete part of the process. Being less explicit may impose more demands on the student who is required to extrapolate from the information presented to complete a procedure. Alternatively it may be more demanding for a teacher who may have to do more explaining to make steps more explicit for students. Renkl (2002) considers this latter approach (integrating an “instructional explanation” with some level of student-contribution) more conducive to learning compared to the former approach. We also found that,
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contrary to the examples presented in the textbooks of the other two countries, the Taiwanese textbooks included worked examples that modeled more than one method to add and subtract fractions and prompted their students to build connections among these methods. The presence of these worked examples suggests that Taiwanese textbooks afford more opportunities for students to build connections between different approaches (cf., the Connections Principle, NCTM, 2000, pp. 64-66) than the other textbooks.

Finally, our analysis showed that the three countries differed in their grade placement of the addition and subtraction of fractions. Additionally, we found differences in how the content on the addition and subtraction of fractions was sequenced. Although we cannot suggest that these differences reflect cultural-specific theories or cultural dissimilarities on how students learn, we do consider that they are worthy of further investigation, because on the one hand, the grade placement of a topic determines the opportunities that students have to learn that topic at a particular age, and, on the other hand, the sequencing of the content pertinent to that topic might support or impede its learning.

4.2. Textbook Expectations

Cognitive demands expected of students can be considered to be either low, requiring memorization of facts presented or the use of procedures without connections, or high, requiring procedures with connections or doing mathematics. Based on our findings from the treatment of addition and subtraction of fractions in these textbooks, it might be claimed that the textbook tasks in Taiwan impose higher expectations on their students compared to the textbooks in Cyprus and Ireland. Even tasks that require procedures without connections in the Taiwanese textbooks include features (such as finding missing addends) not frequently found in the Cypriot or Irish textbooks. Research has illustrated that in setting up and implementing tasks, teachers are
more likely to either maintain or reduce the cognitive demands of a task than to increase it (Stein et al., 2000, p.25). This suggests that students from Cyprus and Ireland using the analyzed textbooks might be less likely to be challenged by tasks with high-cognitive demands than students in Taiwan, because substantially fewer tasks in textbooks from Cyprus and Ireland include high cognitive demands. Naturally, having tasks with high cognitive demands does not guarantee that they will be implemented as such in the classroom; but the larger proportion of these tasks in the Taiwanese textbooks suggest that it will be more likely for students to experience higher cognitive demand tasks when using those textbooks. Including more high cognitive demand tasks in the Cypriot and Irish textbooks might not lead automatically to students engaging with tasks at a higher cognitive level, but it sets up what appears to be a baseline for teaching that takes advantage of those tasks.

In terms of performance expectations we looked at whether textbooks requested students to provide just an answer to tasks or if an explanation was also required. Explanation is an important skill in mathematics related to the skills of reasoning and communicating (NCTM, 2000). In only one textbook series (TTA) did we find tasks that required explanations of the solution strategy. Most tasks in the textbooks from Ireland and Cyprus required students to simply supply the answer. Both Taiwanese textbooks placed an additional expectation on students, which was to write a mathematical sentence related to a problem statement. Writing such a statement can be a good discipline for students to help them clarify thoughts about how to understand and solve a problem. It is also something that teachers in Cyprus and Ireland may do in mediating the textbook for their students. The Taiwanese textbooks frequently reminded students (and teachers) to do this as a matter of course by the expectations built into the textbook tasks.
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5. Conclusions, Limitations, and Implications

This study explored how the *presentation* and *expectations* of addition and subtraction of fractions in the textbooks can comprehensively describe the learning opportunities that textbooks from three countries afford for their students. Our analysis of the *presentation* indicates that the three countries designed the mathematics content regarding addition and subtraction of fraction differently in their textbooks. The Taiwanese textbooks provide a more diverse set of tasks that expose students to many aspects related to the addition and subtraction of fractions (e.g., by using several different fraction constructs, by using discrete sets, by asking ‘twin’ questions, and by providing more than one method of solution in worked examples) than the textbooks from the other two countries. Simultaneously, our analysis also shows that the tasks in the Taiwanese textbooks tended to be of a higher cognitive demand than the tasks in the textbooks from the other two countries; even when the tasks were procedural without connections to meaning, tasks in the Taiwanese textbooks were more complex. Based solely on the content of these textbooks, it appears that the three countries offer their children different opportunities to learn addition and subtraction of fractions. Whereas a child in Ireland would be mostly exposed to examples with minimal representations and decontextualized repetitive tasks dealing mostly with no specified construct and requiring a single answer, children in Cyprus and Taiwan would be exposed to somewhat more complex tasks. Cypriot children would be exposed to a wider range of representations and contexts, would be given models for solving problems that they would need to finish by themselves, and would solve more demanding tasks. In addition, Taiwanese children would be exposed to qualitatively different tasks, tasks that are more demanding, require the use of multiple constructs simultaneously, ask students to provide more than an answer, and model multiple ways to approach a problem.
Attempting an explanation of why these textbooks have been designed in this way is beyond the scope of this study. Many historical, political, cultural, and curricular influences are at play. As mentioned initially, the countries have experienced curriculum reform in different ways and such reform has affected their textbooks differentially; we learned that, by the time we finished our analysis of textbooks, Taiwan had already experienced two substantial changes in their curriculum that moved a considerable portion of the 4th grade content, mainly regarding operations with fractions, to the 3rd grade. These changes, by necessity define different opportunities to learn for the students in a given country—and in some ways compromise the possible generalizations that we can make from these findings. In spite of this, we believe that the framework has been a particularly fruitful tool that has allowed us to uncover subtle differences and similarities that would not have been perceived without it, or by conducting simple horizontal or vertical analyses of the textbooks.

As stated at the outset, the textbook is only one factor of the many that affect instruction in a country and it is possible that teachers in some countries compensate for shortcomings in textbooks or in others they may conceivably reduce the demands of complex tasks. Therefore our findings beg the question: what actually happens in classrooms? Given the content students and instructors have at hand, which of these opportunities get realized? Which do not and why? How do instructors capitalize on the strengths these textbooks have and how do they compensate for their weaknesses? We can also ask, can teachers who may not have the benefit of studying textbooks across countries and who may have other subjects to teach recognize, or be expected to recognize, those strengths and weaknesses?

Remillard (1999) contends that curriculum materials can support teacher learning, if they are designed to “talk to teachers” and not to students through teachers. Thus a possible avenue
for further investigation regards the extent to which we can learn from these findings in order to design texts that are educative for teachers (cf., Ball & Cohen, 1996). At a basic level we can think about investigating the usefulness of using the framework to prepare teachers to make judgments about the textbooks they choose or use, thus taking a critical approach towards the selection and use of textbooks, which has proven to be useful in the teaching of science for understanding (cf., Davis & Krajick, 2005).

Given documentations of the marked differences in which Chinese and American teachers structure their instruction on fractions, with Chinese teachers relying mostly on the unit construct and American teachers on the part-whole construct (An et al., 2004), we can also ask, to what extent such disparities in the instructional approaches can be attributed to the content of the textbooks used in each country? Although strictly causal inferences cannot be made at this point, more analyses could be conducted to trace the root of these results. It seems plausible that the textbooks that teachers use day in and day out play a significant role in explaining the differences in sophistication of the instruction structure regarding fractions.

We believe that the analytical framework that we have developed can be expanded for use with other mathematical content. It can also be tested further through being applied in other countries, and with other grade bands. Such work would be useful in describing what aspects of presentation and expectations are due to the particular content analyzed, which are independent of the content analyzed, and which are dependent on contextual features of the country analyzed. This knowledge would be important for our understanding of the potential differences in structuring instruction for children in different countries.

The process of developing the framework made clear to us the value of discussing, revisiting, and revising several aspects of it. This was critical to increase its applicability, the
reliability in coding, and to make it more responsive to the cultural differences reflected in the textbooks that we analyzed. Our discussions were enriched by the multiple insiders’ perspectives brought to them by the team of authors, who have taught in these countries, who speak the language of the textbooks and who are also fluent in English.

On more practical grounds, we would like to propose areas that merit attention from the textbook authors’ point of view. Publishers often justify what is included in textbooks by conducting market research with teachers who are familiar with material structured and presented in a particular way (Wilson, 2003). By offering choices to teachers along the lines of the differences observed in this study teachers may be open to consider issues related to presentation and expectations of the topic as described below.

As teachers who have faced the need to teach fractions to children we welcomed the high complexity of tasks presented in the Taiwanese textbooks and the wide range of representations, constructs, and scaffolds presented. Including more than one construct in worked examples and in tasks, in particular by using twin questions, would be especially fruitful to gain a more robust understanding of operations with fractions. Likewise, including a wider range of representations could be important as children reify their understanding of addition and subtraction of fractions. We have seen the possibilities for including discrete representations, as they may invite more than one construct to emerge.

Finally, there is no reason to believe that children can’t deal with highly cognitive demand tasks at early grades; we underscore the importance of exposing students to activities that are beyond their ability level, much in line with the notion of the zone of proximal development (Vygotsky, 1986). We provided an existence proof for the claim that textbooks can be designed to provide students with the necessary tools to solve tasks that might be beyond their
ability level. In particular, we saw the potential for worked examples to serve as scaffolds for students’ thinking. We found it very powerful to have graphical displays of children’s thinking, using children’s language that also illustrated different ways to think about the problems. We see these as potentially useful additions to current textbooks.

When comparing textbooks, criteria need to be carefully chosen. Earlier we described why directly comparing the content of textbooks from different countries in order to make claims about content covered in particular grades needs to be approached cautiously; differences may be attributable to cultural or educational differences in particular settings. Nevertheless, studying similarities and differences of cognitive demands of tasks, constructs and representations used and worked examples can help to identify choices that have been made, for whatever reasons, by particular textbook publishers. Such an analysis highlights different opportunities presented to students in particular countries and is relevant to textbook designers, teachers, and those who select textbooks, as well as researchers. Looking at the textbooks across different countries can sharpen our understanding of the ways in which different educational systems structure opportunities for their students to learn mathematics and inform our current practices. As the world becomes flat (Friedman, 2005) and information and ideas travel across countries, the findings of cross-national studies such as the one reported here can provide important insights and inform mathematics education beyond the three countries studied in this paper.
Addition and Subtraction of Fractions

Authors Note

This paper is an application of a synthesis of research conducted by the first three authors while taking a curriculum in mathematics education class taught by the last author. We opt for an alphabetical listing of names, but contributions were at comparable levels.
Addition and Subtraction of Fractions

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Tate, W. F., & Rousseau, C. (2006). Engineering change in mathematics education. In F. K. Lester (Ed.), *Second handbook for research in mathematics teaching and learning* (pp. 1209-
Addition and Subtraction of Fractions


6. Figures

**Figure 1.** The framework used to analyze the mathematics textbook
Addition and Subtraction of Fractions

One block of blue cubes is equal to 9 white cubes. How much of the blue block are four white cubes equal to?

These have the same length.

How much of the blue block is a white cube equal to?

How much of the blue block are 4 of 1/9 of the blue block? If we have 4 of 1/9 of the blue block and add another 1/9 of the blue block, how many of 1/9s of the blue block will we get? How much of the blue block will this be equal to?

*Figure 2. The unit fraction in the Taiwanese Textbooks (TTB, p.112)*
Addition and Subtraction of Fractions

(a) Cypriot Textbook (CT, p. 4-47)

Constantinos’s father bought 3 pizzas for his son’s birthday party. The kids ate \(1\frac{1}{3}\) pizzas. How many pizzas were left?

Color the pizza that was eaten.

There were \(2\frac{1}{4}\) apple pies in the pan. The kids ate \(1\frac{3}{4}\) apple pies. How many apple pies were left?

Color the part that kids ate.

(b) Taiwanese Textbook (TTA, p. 2-85)

How many fewer meters of colorful belts did Gi-Wen use compared to Wen-Ting? Write the corresponding mathematical expression and then find the answer.

We cannot subtract \(\frac{4}{5}\) from \(\frac{3}{5}\). What should we do?

I converted all the mixed fractions into improper fractions.

(c) Irish Textbook (ITA, p. 50)

(d) Irish Textbook (ITB, p. 50)

Example 2: \(4\frac{1}{6} - 2\frac{3}{4}\)

\[
\begin{align*}
\frac{1}{6} &= \frac{2}{12} = \frac{9}{12} \\
\frac{3}{4} &= \frac{6}{8} = \frac{9}{12} \\
4\frac{1}{6} &= 4\frac{2}{12} = 3\frac{14}{12} \quad \text{(by renaming)} \\
-2\frac{3}{4} &= -2\frac{9}{12} = -2\frac{9}{12} \\
\end{align*}
\]

Figure 3. Worked examples on the subtraction of mixed numbers
Addition and Subtraction of Fractions

Figure 4. Percentages of subtasks exhibiting each cognitive demand

*Note.* No memorization subtasks were identified.

Figure 5. Percentages of subtasks exhibiting each performance

*Note.* No subtasks that asked students to justify their answer were identified.
### TABLE 1

Summary of the Criteria Used to Code the Content of the Textbooks

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Dimensions</th>
</tr>
</thead>
</table>
| **Topics**          | Adding proper fractions  
Adding a whole number to a proper fraction  
Adding a proper fraction to a mixed number or improper fraction  
Adding two mixed numbers or improper fractions  
Subtracting a proper fraction from a whole number  
Subtracting a proper fraction from a proper fraction  
Subtracting two mixed numbers  
Subtracting a proper fraction from a mixed number or improper fraction |
| **Sequence**        | Sequence of the topics  
Topics preceding the addition and subtraction of fractions and mixed numbers |
| **Worked examples** | Frequency  
Procedures outlined  
Methods per procedure  
Building connections  
Constructs  
Type of representation  
Context |
| **Constructs**      | Part-whole  
Ratio  
Operator  
Quotient  
Measure  
Combinations of the above |
| **Representations** | Discrete sets  
Continuous:  
One dimensional (linear)  
Two dimensional (rectangular, circular, other)  
Three dimensional (volumetric, weight)  
Number lines |
| **Potential cognitive demands** | Memorization Procedures without connections to meaning  
Procedures with connections to meaning  
Doing mathematics |
| **Performance expectations** | Only an answer  
Answer and mathematical sentence  
Explanation  
Justification |
### TABLE 2
Topics and Their Sequence in the Textbooks of the Three Countries

<table>
<thead>
<tr>
<th>Textbooks</th>
<th>Content on addition and subtraction of fractions (Tasks)</th>
<th>Cypriot</th>
<th>Irish</th>
<th>Taiwanese</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>CT</td>
<td>ITA</td>
<td>ITB</td>
</tr>
<tr>
<td></td>
<td>Adding proper fractions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Like denominators</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Unlike denominators</td>
<td>2</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Adding a whole number to a proper fraction</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Adding a proper fraction to a mixed number or improper fraction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Like denominators</td>
<td>5</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Unlike denominators</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Adding two mixed numbers or improper fractions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Like denominators</td>
<td>4</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>Unlike denominators</td>
<td>6</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Subtracting a proper fraction from a whole number</td>
<td>8</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Subtracting a proper fraction from a proper fraction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Like denominators</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Unlike denominators</td>
<td>5</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Subtracting two mixed numbers or improper fractions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Like denominators</td>
<td>6</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Unlike denominators</td>
<td>7</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Subtracting a proper fraction from a mixed number or improper fraction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Like denominators</td>
<td>7</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Unlike denominators</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Note.* The numbers indicate the order in which these topics appear in each textbook.
### TABLE 3

Frequencies and Percents of Subtasks Using Each Construct

<table>
<thead>
<tr>
<th>Construct</th>
<th>Cypriot textbook (CT) N = 211</th>
<th>Irish textbooks</th>
<th>Taiwanese textbooks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>f</td>
<td>%</td>
<td>f</td>
</tr>
<tr>
<td>Part-whole</td>
<td>93</td>
<td>44</td>
<td>7</td>
</tr>
<tr>
<td>Ratio</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Operator</td>
<td>5</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Quotient</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Measure</td>
<td>10</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Part-whole and operator</td>
<td>4</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Part-whole and quotient</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Part-whole and measure</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Not applicable</td>
<td>99</td>
<td>47</td>
<td>226</td>
</tr>
</tbody>
</table>

### TABLE 4

Frequencies and Percents of the Subtasks Using Each Representation

<table>
<thead>
<tr>
<th>Representations</th>
<th>Cypriot textbook (CT) N = 211</th>
<th>Irish textbooks</th>
<th>Taiwanese textbooks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>f</td>
<td>%</td>
<td>f</td>
</tr>
<tr>
<td>Discrete sets</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Continuous</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>One dimension</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Linear</td>
<td>5</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Two dimensions (area)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rectangular</td>
<td>34</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>Circular</td>
<td>10</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Other</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Three dimensions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volumetric-weight</td>
<td>7</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Not present</td>
<td>153</td>
<td>73</td>
<td>232</td>
</tr>
</tbody>
</table>

*Note.* No number lines were used.
Addition and Subtraction of Fractions

APPENDIX

Discrete sets

Continuous

One dimension (linear)

Two dimensions (circular)

Two dimensions (rectangular)
This category included fraction stripes, as well.

Two dimensions (other shapes)

Three dimensions (volumetric)

Three dimensions (weight)

Number lines

Figure A1. Examples of the coding of representations used in the textbooks
Addition and Subtraction of Fractions

Construct: Part-whole and operator; Representations: Decorative; Cognitive demands: Procedures with connections; Performance expectations: Answer and mathematical sentence.

One box has 22 kiwis. May-Chin store bought 8 boxes on Wednesday and sold out some of them. Now she has 3 5/11 boxes left. How many boxes of kiwis did May-Chin sell on Wednesday? This is equal to how many kiwis? Write a mathematical expression and write your answer as a mixed fraction (TTB, p. 2-23).

A box contains 8/10 kg of flour. Mrs. Georgia used 5/10 of the flour. What part of the flour is left? (CT, p. 3-45, task 3)

Construct: Measure; Representations: None; Cognitive demands: Procedures with connections; Performance Expectations: Answer only.

8. (a) \(\frac{1}{3} + \frac{2}{12} \Rightarrow \frac{12}{12} + \frac{2}{12} \Rightarrow \frac{14}{12} = \square\) (b) \(\frac{1}{6} + \frac{1}{12} \Rightarrow \frac{12}{12} + \frac{1}{12} \Rightarrow \frac{13}{12} = \square\) (c) \(\frac{1}{3} + \frac{5}{12} \Rightarrow \frac{12}{12} + \frac{5}{12} \Rightarrow \frac{17}{12} = \square\) (d) \(\frac{1}{4} + \frac{5}{12} \Rightarrow \frac{12}{12} + \frac{5}{12} \Rightarrow \frac{17}{12} = \square\)

Construct: None; Representations: None; Cognitive demands: Procedures without connections; Performance Expectations: Answer only.

This task consists of two subtasks (the question that the girl asks: first subtask; and the question the boy asks: second subtask).

First subtask: Construct: Part-whole; Representations: None; Cognitive demands: Procedures with connections; Performance Expectations: Answer only.

First subtask: Construct: Part-whole; Representations: None; Cognitive demands: Doing mathematics (the student is given the part and is expected to find the whole); Performance Expectations: Answer only.

Girl: Did you know that 7/10 of the human body consists of water? What part of the human body does not consist of water?

Boy: If the water in my body weighs 28 kg, how much do I weigh?

(TTA, pp. 86-87)

Construct: Measure; Representations: Linear; Cognitive demands: Doing mathematics; Performance Expectations: Answer and mathematical sentence.
Figure A2. Examples illustrating the coding of the subtasks