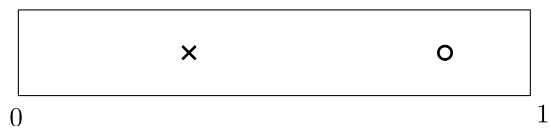
Wayne County Math Teachers' Circle (January 17 & 21) Nicholas Vlamis

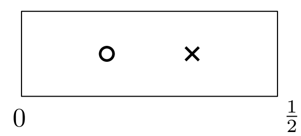
Stretching Fractions

(This activity was adopted from the following post on NRICH: `https://nrich.maths.org/4340`)

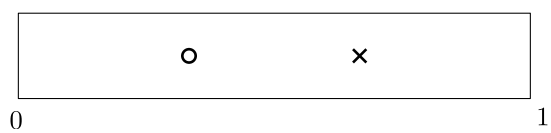
We start with a piece of dough of length 1 (in your favorite units). The dough is shown below with two points marked on it, one by an X and the other by an O:



We are going to complete a two-step process to the dough and show what happens to the two points in each step. First, we will fold the dough in half:



Now for our second step, we roll out (or stretch) the dough back to its original length:



In coordinates, the point marked X started distance $\frac{1}{3}$ from the left side of the dough and ended $\frac{2}{3}$ from the left side. The point marked O started at a distance of $\frac{5}{6}$ from the left and ended a distance $\frac{2}{3}$ from the left. Let us write this process down as a function. For the sake of choosing coordinates, our function will take in a number between 0 and 1 (inclusive), which measures the distance from the left hand side and will return a real number telling us the distance from the left hand side after the folding and rolling of the dough. The function describing this is as follows:

$$x \mapsto \begin{cases} 2x & \text{when } 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \text{when } \frac{1}{2} \leq x \leq 1 \end{cases}$$

The goal of this activity is to investigate what happens when this function is applied repeatedly to a number, i.e. we repeatedly fold and stretch a number. Let's do an example: we follow the number $\frac{2}{7}$

$$\frac{2}{7} \mapsto \frac{4}{7} \mapsto \frac{6}{7} \mapsto \frac{2}{7}$$

Notice that we have created a loop. If we continue the process we will just keep moving through this loop. Let's see what happens to $\frac{3}{7}$:

$$\frac{3}{7} \mapsto \frac{6}{7} \mapsto \frac{2}{7} \mapsto \frac{4}{7} \mapsto \frac{6}{7} \mapsto \frac{2}{7}$$

This time, we started at $\frac{3}{7}$ and entered into the same loop that we found above, but we never made it back to $\frac{3}{7}$. Let's introduce some definitions so we have a common language to use when discussing this problem.

Definition. Let x be a real number with $0 \leq x \leq 1$.

- x starts a **loop of length** n if after folding and stretching the number x exactly n times, we arrive back at x. For example, the number $x = \frac{2}{7}$ starts a loop of length 3.
- x **eventually enters a loop** if after some number of steps, we arrive at a number that starts a loop. For example, $x = \frac{3}{7}$ eventually enters a loop (of length 3).

In the teachers' circle, the teachers were asked to explore some examples and come up with questions and conjectures. After doing so, the rest of the time was spent trying to answer the questions and prove the conjectures. The act of creating the questions was an important part of the exercise; however, I include here a list of several of the questions that were posed.

- Does every rational number eventually enter a loop?
- Do irrational numbers enter a loop?
- Given a number, can you tell it if it eventually enters a loop? If so, can you tell how many steps it will take to enter a loop?
- Which numbers eventually end up at 0?
- Can any whole number be the length of a loop?