

Homework 9

MATH 590

Due Wednesday, March 23, 2016

Instructions. Write up (in $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.

Exercise 1. Prove that \mathbb{S}^1 and $\mathbb{R}\mathbb{P}^1$ are homeomorphic.

Exercise 2. Let X be a topological space, \sim an equivalence relation on X such that the quotient map $\pi: X \rightarrow X/\sim$ is open.

- (a) Prove that $\pi \times \pi: X \times X \rightarrow (X/\sim) \times (X/\sim)$ is a quotient map.
- (b) Let \sim' be the equivalence relation on $X \times X$ defined by $(x, y) \sim' (x', y')$ if and only if $x \sim x'$ and $y \sim y'$. Prove that $(X \times X)/\sim' \cong (X/\sim) \times (X/\sim)$.

Exercise 3 (*). Let X be a topological space, \sim an equivalence relation on X .

- (a) Prove that if X/\sim is Hausdorff, then the graph G of \sim is closed in $X \times X$, that is, the set $\{(x, y) : x \sim y\} \subset X \times X$ is closed.
- (b) Now assume that the quotient map $\pi: X \rightarrow X/\sim$ is open. Prove that if the graph G of \sim is closed, then X/\sim is Hausdorff.

Exercise 4. Viewing \mathbb{Z} as a subgroup of $(\mathbb{R}, +)$, prove that \mathbb{R}/\mathbb{Z} is homeomorphic to \mathbb{S}^1 .

Exercise 5 (*). Viewing \mathbb{Z}^2 as a subgroup of $(\mathbb{R}^2, +)$, prove that $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. (Note: it is true in general that $\mathbb{R}^n/\mathbb{Z}^n \cong \prod_{i=1}^n \mathbb{S}^1$.)

Exercise 6 (*). Let G be a topological group and let $\alpha: G \times X \rightarrow X$ be an action of G on X . Prove that α induces a homomorphism $\alpha_*: G \rightarrow \text{Homeo}(X)$.

Exercise 7. Let G be a topological group and let H be a subgroup. Prove that G/H is homogeneous.

DISJOINT UNION

Definition 1. Let X and Y be topological spaces. The *disjoint union* (or *topological sum*) $X \sqcup Y$ of X and Y is defined as follows: As a set we have $X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$. Let $i_X: X \hookrightarrow X \sqcup Y$ be the inclusion given by $i_X(x) = (x, 0)$. Similarly define i_Y . We define a set U in $X \sqcup Y$ to be open if and only if both $i_X^{-1}(U)$ and $i_Y^{-1}(U)$ are open.

A common way of constructing topological spaces is by taking a disjoint union and identifying subspaces using a quotient. For instance, this is the method one uses for describing the classification of compact surfaces. An example of this type of construction is below.

Exercise 8. Let X and Y be topological spaces. Let i_X, i_Y be as in the definition above. Prove the following statements:

- (a) i_X and i_Y are homeomorphisms onto their images.
- (b) $X \sqcup Y$ is disconnected.
- (c) $i_X(X)$ and $i_Y(Y)$ are clopen.

Exercise 9 (*). Consider the subspace $Z = \partial B((-1, 0), 1) \cup \partial B((1, 0), 1) \subset \mathbb{R}^2$, where $B(x, r) \subset \mathbb{R}^2$ is the open ball of radius r centered at x . Let $X = \mathbb{S}^1$ and $Y = \mathbb{S}^1$ be two copies of \mathbb{S}^1 . Choose $x \in X$ and $y \in Y$ and set $A = \{i_X(x), i_Y(y)\} \subset X \sqcup Y$. Prove that $(X \sqcup Y)/A$ is homeomorphic to Z . This space is called a *wedge sum* of two circles. (You can do this with an any number of circles to get a *bouquet of circles* or *rose*.)

Exercise 10 (Extra credit). Let T^2 be the torus and let $x \in T^2$. Prove that $T^2 \setminus \{x\}$ retracts onto a wedge sum of two circles.