Homework 9

MATH 590

Due Wednesday, March 23, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1. Prove that $\mathbb{S}^1$ and $\mathbb{RP}^1$ are homeomorphic.

Exercise 2. Let X be a topological space, $\sim$ an equivalence relation on X such that the quotient map $\pi\colon X \to X/\sim$ is open.

(a) Prove that $\pi \times \pi\colon X \times X \to (X/\sim) \times (X/\sim)$ is a quotient map.

(b) Let $\sim'$ be the equivalence relation on $X \times X$ defined by $(x, y) \sim' (x', y')$ if and only if $x \sim x'$ and $y \sim y'$. Prove that $(X \times X)/\sim' \cong (X/\sim) \times (X/\sim)$.

Exercise 3 (*). Let X be a topological space, $\sim$ an equivalence relation on X.

(a) Prove that if $X/\sim$ is Hausdorff, then the graph G of $\sim$ is closed in $X \times X$, that is, the set $\{(x, y) : x \sim y\} \subset X \times X$ is closed.

(b) Now assume that the quotient map $\pi\colon X \to X/\sim$ is open. Prove that if the graph G of $\sim$ is closed, then $X/\sim$ is Hausdorff.

Exercise 4. Viewing $\mathbb{Z}$ as a subgroup of $(\mathbb{R}, +)$, prove that $\mathbb{R}/\mathbb{Z}$ is homeomorphic to $\mathbb{S}^1$.

Exercise 5 (*). Viewing $\mathbb{Z}^2$ as a subgroup of $(\mathbb{R}^2, +)$, prove that $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. (Note: it is true in general that $\mathbb{R}^n/\mathbb{Z}^n \cong \prod_{i=1}^n \mathbb{S}^1$.)

Exercise 6 (*). Let G be a topological group and let $\alpha : G \times X \to X$ be an action of G on X. Prove that α induces a homomorphism $\alpha_*\colon G \to \mathrm{Homeo}(X)$.

Exercise 7. Let G be a topological group and let H be a subgroup. Prove that G/H is homogeneous.

Disjoint Union

Definition 1. Let X and Y be topological spaces. The *disjoint union* (or *topological sum)* $X \sqcup Y$ of X and Y is defined as follows: As a set we have $X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$. Let $i_X\colon X \hookrightarrow X \sqcup Y$ be the inclusion given by $i_X(x) = (x, 0)$. Similarly define i_Y. We define a set U in $X \sqcup Y$ to be open if and only if both $i_X^{-1}(U)$ and $i_Y^{-1}(U)$ are open.

A common way of constructing topological spaces is by taking a disjoint union and identifying subspaces using a quotient. For instance, this is the method one uses for describing the classification of compact surfaces. An example of this type of construction is below.

Exercise 8. Let X and Y be topological spaces. Let i_X, i_Y be as in the definition above. Prove the following statements:

(a) i_X and i_Y are homeomorphisms onto their images.

(b) $X \sqcup Y$ is disconnected.

(c) $i_X(X)$ and $i_Y(Y)$ are clopen.

Exercise 9 (*)**.** Consider the subspace $Z = \partial B((-1,0),1) \cup \partial B((1,0),1) \subset \mathbb{R}^2$, where $B(x,r) \subset \mathbb{R}^2$ is the open ball of radius r centered at x. Let $X = \mathbb{S}^1$ and $Y = \mathbb{S}^1$ be two copies of $\mathbb{S}^1$. Choose $x \in X$ and $y \in Y$ and set $A = \{i_X(x), i_Y(y))\} \subset X \sqcup Y$. Prove that $(X \sqcup Y)/A$ is homeomorphic to Z. This space is called a *wedge sum* of two circles. (You can do this with an any number of circles to get a *bouquet of circles* or *rose.*)

Exercise 10 (Extra credit)**.** Let T^2 be the torus and let $x \in T^2$. Prove that $T^2 \setminus \{x\}$ retracts onto a wedge sum of two circles.