Homework 7

MATH 590

Due Wednesday, March 9, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1. Let (X,d) be a metric space. Prove or disprove the following statements:

(a) X is first countable.

(b) X is second countable.

(c) If X is compact, then X is second countable.

Exercise 2. Let $f\colon X \to Y$ be an open continuous surjection. Prove that if X is first countable, then so is Y. Similarly, prove that if X is second countable, then so is Y.

Exercise 3. Prove that a connected manifold is path connected.

Exercise 4. Prove that $\mathbb{S}^n$ is a manifold.

Exercise 5 (*)**.** Give a two chart atlas for $\mathbb{S}^1$ and give explicit formulas for the two corresponding transition maps.

Exercise 6. Prove that there exists a continuous function $f\colon T^2 \to T^2$ such that $f^{-1}(x)$ contains two points for each $x \in T^2$.

Definition 1 (The line with two origins)**.** Let X be the union of the set $\mathbb{R} \smallsetminus \{0\}$ and the two-point set $\{p,q\}$. Topologize X by taking as a basis the collection of all open intervals in $\mathbb{R}$ that do not contain 0, along with all the sets of the form $(-a,0) \cup \{p\} \cup (0,a)$ and all sets of the form $(-a,0) \cup \{q\} \cup (0,a)$, for $a > 0$. The space X is called the *line with two origins.*

Exercise 7 (*)**.** Let X be the line with two origins. (You may assume that the basis above is indeed a topological basis.)

(a) Prove that both $X \smallsetminus \{p\}$ and $X \smallsetminus \{q\}$ are homeomorphic to $\mathbb{R}$.

(b) Prove that X is T_1, but not Hausdorff.

(c) Prove that X satisfies all the conditions for a 1-manifold except for the Hausdorff condition.

(d) Prove that there is no continuous injection $X \hookrightarrow \mathbb{R}^n$ for any $n \in \mathbb{Z}_+$.

Definition 2. Let i, j, k be symbols satisfying

$$ij = k,\ jk = i,\ ki = j,\ i^2 = j^2 = k^2 = -1,$$

and define the *quaternions* to be the set

$$\mathbb{H} = \{t + xi + yj + zk : t, x, y, z \in \mathbb{R}\}.$$

Given the relations above, we can easily define multiplication and addition in $\mathbb{H}$ in a similar fashion as the complex numbers. The *conjugate* of $q = t + xi + yj + zk$ is the quaternion

$$\bar{q} = t - xi - yj - zk.$$

Let $||\cdot||\colon\ \mathbb{H} \to \mathbb{R}$ be defined by

$$||q|| = \sqrt{q\bar{q}},$$

which gives rise to a metric $d\colon\ \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ defined by

$$d(q, q') = ||q - q'||.$$

With the metric topology, we easily see that $\mathbb{H} \cong \mathbb{R}^4$.

Exercise 8 (*)**.** (a) Prove that each quaternion $q \neq 0$ has a multiplicative inverse.

(b) Prove that $\mathbb{H}^\times = \mathbb{H} \smallsetminus \{0\}$ is a non-abelian topological group with respect to multiplication. (Note: associativity simply follows from the associativity of the real numbers.)

(c) Let $S = \{q \in \mathbb{H} : ||q|| = 1\} \subset \mathbb{H}$. Prove that multiplication restricted to S makes S into a topological group.

(d) Conclude that $\mathbb{S}^3$ can be made into a topological group.

The following is beyond the scope of this course, but interesting nonetheless:

Theorem 3. *$\mathbb{S}^0, \mathbb{S}^1$, and $\mathbb{S}^3$ are the only spheres that can be made into topological groups.*