Homework 6

MATH 590

Due Wednesday, February 24, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Theorem 1. *$\mathbb{Q}$ is dense in $\mathbb{R}$ (i.e. $\bar{\mathbb{Q}} = \mathbb{R}$).*

We will not prove this theorem, but you may assume it from here on out.

Exercise 1 (*). Prove that $\mathbb{Q}$ is not locally compact. (The topology on $\mathbb{Q}$ is the subspace topology coming from $\mathbb{R}$.)

Exercise 2. Let (X, d) be a metric space; let $A \subseteq X$ be nonempty.

(a) Prove that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Prove that $B(A, \epsilon) = \bigcup_{a \in A} B(a, \epsilon)$.

(d) Let $A \subseteq X$ be compact and let $U \subseteq X$ be open with $A \subseteq U$. Show that there exists $\epsilon > 0$ such that $B(A, \epsilon) \subseteq U$.

(e) Show the result in (d) need not hold if A is closed but not compact.

Exercise 3. Prove that a connected metric space having more than one point is uncountable.

Exercise 4. Let (X, d) be a metric space. If $f\colon X \to X$ satisfies the condition

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$, then f is called an *isometry* of X.

(a) Prove that an isometry is a continuous injection.

(b) Prove that an isometry $f\colon X \to X$ need not be surjective.

(c) Prove that if X is compact and f is an isometry of X, then f is a homeomorphism.

Exercise 5 (*). Let (X, d) be a metric space. If $\alpha < 1$ is a real number and $f\colon X \to X$ satisfies

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$, then f is called a *contraction*. A *fixed point* of f is a point $x \in X$ such that $f(x) = x$. Prove that if X is compact and f is a contraction, then f has a unique fixed point.

Definition 2. Let X be a topological space; let

$$\mathrm{Homeo}(X) = \{f\colon\, X \to X \mid f \text{ is a homeomorphism}\}.$$

$\mathrm{Homeo}(X)$ is called the *homeomorphism group* of X. (You can easily check that it forms a group under function composition.) If C is a compact subspace of X and U is an open subset of X, define

$$S(C,U) = \{f \in \mathrm{Homeo(X)} \mid f(C) \subseteq U\}.$$

The sets $S(C,U)$ form a subbasis for a topology on $\mathrm{Homeo}(X)$ called the *compact-open topology.*

Exercise 6 (*)**.** (a) Recall that a compact Hausdorff space is normal. Prove that if $C \subset X$ is compact and $U \subset X$ is open such that $C \subset U$, then there exists an open set V containing C such that $\bar{V} \subset U$. (You may need this in (b).)

(b) Prove that if X is compact and Hausdorff, then $\mathrm{Homeo}(X)$ equipped with the compact-open topology is a topological group. (For continuity arguments, recall Exercise 3 on HW 2.)

Exercise 7. Let X be a compact Hausdorff space; let $\mathrm{MCG}(X)$ denote the collection of path components of $\mathrm{Homeo}(X)$. Given $f \in \mathrm{Homeo}(X)$, let $C_f \in \mathrm{MCG}(X)$ denote the path component of $\mathrm{Homeo}(X)$ containing f. We now define a binary operation on $\mathrm{MCG}(X)$ as follows: Given $C, C' \in \mathrm{MCG}(X)$, write $C = C_f$ and $C' = C_g$ for some $f \in C$ and $g \in C'$. Define $C \cdot C' = C_{f \circ g}$.

(a) Prove $m\colon\, \mathrm{MCG}(X) \times \mathrm{MCG}(X) \to \mathrm{MCG}(X)$ defined by $m(C,C') = C \cdot C'$ is well-defined, that is, it does not depend on the choose of f and g.

(b) Prove that $(\mathrm{MCG}(X), \cdot)$ is a group.

(c) (Only if you are familiar with group theory) Let $\mathrm{Homeo}_0(X)$ be the path component of $\mathrm{Homeo}(X)$ containing the identity. Prove that

$$\mathrm{MCG(X)} = \mathrm{Homeo}(X)/\mathrm{Homeo}_0(X)$$

(In a previous homework, you proved that $\mathrm{Homeo}_0(X)$ is a topological group.)

Definition 3. The group $\mathrm{MCG}(X)$ is a called the *mapping class group*. It is a major object of study in mathematics. Many people, including myself, spend their time thinking about this group.

Exercise 8 (Extra credit)**.** Let

$$\mathrm{Sym}(\mathbb{Z}_+) = \{f\colon\, \mathbb{Z}_+ \to \mathbb{Z}_+ \mid f \text{ is a bijection}\}.$$

Equip $\mathbb{Z}_+$ with the subspace topology coming from $\mathbb{R}$. Let $\bar{\mathbb{Z}}_+$ be the one-point compactification of $\mathbb{Z}_+$. Prove that $\mathrm{Homeo}(\bar{\mathbb{Z}}_+)$ is isomorphic to $\mathrm{Sym}(\mathbb{Z}_+)$ as groups, that is, there exists a bijection

$$\Phi\colon\, \mathrm{Homeo}(\bar{\mathbb{Z}}_+) \to \mathrm{Sym}(\mathbb{Z}_+)$$

satisfying $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in \mathrm{Homeo}(\bar{\mathbb{Z}}_+)$. (*This came up in my research.*)