

## Homework 5

MATH 590

Due Wednesday, February 17, 2016

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**Instructions.** Write up (in  $\text{\LaTeX}$ ) and turn in all problems marked with an asterisks (\*) at the beginning of class on the due date.

**Exercise 1.** Let  $X$  be a compact topological space,  $\{A_\alpha : \alpha \in I\}$  an arbitrary collection of closed sets, which is closed with respect to finite intersections (that is, given  $\alpha, \beta \in I$  there exists  $\gamma \in I$  such that  $A_\alpha \cap A_\beta = A_\gamma$ ). If for an open set  $U \subseteq X$  one has  $\bigcap_\alpha A_\alpha \subseteq U$ , then there exists  $\alpha \in I$  for which  $A_\alpha \subseteq U$ .

**Exercise 2.** Let  $X$  be a compact Hausdorff space. Prove that  $X$  is *normal*, that is, given disjoint closed subspaces  $A$  and  $B$  of  $X$ , there exist disjoint open sets  $U$  and  $V$  containing  $A$  and  $B$ , respectively. (Note: a normal space is also referred to as a  $T_4$  space.)

**Exercise 3.** Prove that if  $f: X \rightarrow Y$  is continuous, where  $X$  is compact and  $Y$  is Hausdorff, then  $f$  is a *closed* map (that is,  $f$  carries closed sets to closed sets).

**Exercise 4 (\*)**. Let  $f: X \rightarrow Y$ ; let  $Y$  be compact Hausdorff. Prove that  $f$  is continuous if and only if the *graph* of  $f$ ,

$$G_f = \{(x, f(x)) \in X \times Y : x \in X\}$$

is closed in  $X \times Y$ .

**Exercise 5.** Let  $f: X \rightarrow Y$  be a closed continuous function between topological spaces. Prove that for every  $y \in Y$  and every neighborhood  $U$  of  $f^{-1}(y)$  there exists a neighborhood  $W$  of  $y$  such that  $f^{-1}(W) \subseteq U$ .

**Exercise 6 (\*)**. (a) Prove that  $\text{GL}_n(\mathbb{R})$  is not compact.

(b) Prove that the orthogonal group  $O(n)$  is compact.

**Exercise 7.** (a) If  $f: X \rightarrow Y$  is a proper continuous function between locally compact Hausdorff spaces, prove that  $f$  extends to a continuous function  $\hat{f}: \hat{X} \rightarrow \hat{Y}$  of their one-point compactifications. This means that  $\hat{f}|_X = f$  and  $\hat{f}(\infty_X) = \infty_Y$ .

(b) If  $f: X \rightarrow Y$  is a homeomorphism between locally compact Hausdorff spaces, prove that  $f$  extends to a homeomorphism  $\hat{f}: \hat{X} \rightarrow \hat{Y}$  of their one-point compactifications.

**Exercise 8 (\*)**. Prove that the one-point compactification of  $\mathbb{R}$  is homeomorphic to  $\mathbb{S}^1$ . (The same ideas used in a proof here will also prove that the one-point compactification of  $\mathbb{R}^n$  is  $\mathbb{S}^n$ . You should convince yourself of this fact, but there is no need to write it up.)

**Exercise 9** (\*). Let  $\mathbb{C}$  be the complex plane. For  $z = x + iy \in \mathbb{C}$ , define  $\bar{z} = x - iy$  and  $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ . Then the function  $d : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$  defined by  $d(z, w) = |z - w|$  is a metric. We equip  $\mathbb{C}$  with the topology induced by this metric. It is clear that  $\mathbb{C} \cong \mathbb{R}^2$ .

Let  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  be the one-point compactification of  $\mathbb{C}$ .  $\hat{\mathbb{C}}$  is called the *Riemann sphere* (by Exercises 7 and 8, it is indeed a sphere). Let  $a, b, c, d \in \mathbb{C}$  such that  $ad - bc \neq 0$ . Prove that  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  defined by

$$f(z) = \frac{az + b}{cz + d}$$

is a homeomorphism. A function of this form is called a *Möbius transformation*; they are fundamental to the study of complex analysis, dynamics, hyperbolic geometry, and low-dimensional topology.