Homework 4

MATH 590

Due Wednesday, February 10, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1. A *discrete-valued map* is a continuous function $f\colon X \to D$ with D a discrete topological space. Prove that a topological space X is connected if and only if every discrete-valued map on X is constant.

Exercise 2 (*)**.** For $n \in \mathbb{N}$, let C_n be a connected subspace of a topological space X. Prove that if $C_n \cap C_{n+1} \neq \emptyset$ for every $n \in \mathbb{N}$, then $\bigcup C_n$ is connected.

Exercise 3. Let $A \subset X$. Prove that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects ∂A.

Exercise 4 (The One-Dimensional Brouwer Fixed Point Theorem)**.** Let $f\colon X \to X$ be continuous. Prove that if $X = [0,1]$, then f has a *fixed point*, that is, there exists $c \in [0,1]$ such that $f(c) = c$. Is the same true for $X = [0,1)$?

Exercise 5 (*)**.** (A special case of the Borsuk-Ulam Theorem) Identify $\mathbb{S}^n$ with the unit sphere in $\mathbb{R}^{n+1}$ and let $f\colon \mathbb{S}^n \to \mathbb{R}$ be a continuous map. Prove that there exists a point x of $\mathbb{S}^n$ such that $f(x) = f(-x)$.

Assuming the surface of the Earth is a sphere and surface temperature is a continuous function, the above theorem tells us that at any given time there is a point on Earth that has the same temperature as the point directly opposite it. You can do the same for any continuous variable defined on the Earth.

Exercise 6. Justify your answers to the following:

(a) Is a product of path-connected spaces necessarily path connected?

(b) If $A \subseteq X$ is path connected, is $\bar{A}$ necessarily path connected?

(c) If $f\colon X \to Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected?

(d) If $\{A_\alpha\}$ is a collection of path-connected subspaces of X and if $\bigcap A_\alpha \neq \emptyset$, is $\bigcup A_\alpha$ necessarily patch connected?

Exercise 7 (*)**.** Let X be locally path connected. Show that every connected open set in X is path-connected.

Topological groups

Definition 1. A *group* is a nonempty set G together with a binary operation denoted $(a, b) \mapsto a \cdot b$ or abbreviated simply ab satisfying:

(i) *Closure*: If $a, b \in G$, then $ab \in G$;

(ii) *Associativity*: a(bc) = (ab)c for all $a, b, c \in G$;

(iii) *Identity*: There is an element $1 \in G$ such that $a1 = 1a = a$ for all $a \in G$;

(iv) *Inverse*: If $a \in G$, then there exists an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = 1$.

Definition 2. A *topological group* G is a group that is also a Hausdorff topological space such that the map $G \times G \to G$ defined by $(x, y) \mapsto xy$ and the map $G \to G$ defined by $x \mapsto x^{-1}$ are continuous.

Exercise 8. Prove that every topological group is *homogeneous*, that is, for every pair of points $x, y \in G$, there exists a homeomorphism $f : G \to G$ such that $f(x) = y$.

Exercise 9. Let G be a topological group and let C be the path component of G containing the identity. Prove that C is a topological group.

Exercise 10 (*)**.** (a) Let $\mathbb{C}$ be the complex numbers. For $z = x + iy \in \mathbb{C}$, let

$$|z| = \sqrt{x^2 + y^2}.$$

We can identify $\mathbb{S}^1$ with the set $\{z \in \mathbb{C} : |z| = 1\}$. Prove that $(\mathbb{S}^1, \cdot)$ is a topological group, where $\cdot$ refers to ordinary complex multiplication.

(b) Prove that the torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is a topological group.

Exercise 11 (Extra credit)**.** (a) Prove that the *general linear group* $\mathrm{GL}_n(\mathbb{R})$ is a topological group.

(b) Prove that the *orthogonal group* $O(n)$ is a topological group.