Homework 3

MATH 590

Due Wednesday, February 3, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1. Let $f\colon X \to Y$ be a function between topological spaces. Prove that f is continuous if and only if

$$f(\bar{A}) \subseteq \overline{f(A)}$$

for every subset $A \subseteq X$.

Exercise 2. Let A, B, and A_α denote subsets of a topological space X. Prove the following:

(a) If $A \subset B$, then $\bar{A} \subset \bar{B}$.

(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(c) $\overline{\bigcup A_\alpha} \supseteq \bigcup \bar{A}_\alpha$; give an example where equality fails.

Exercise 3. (a) Prove that the product of two Hausdorff spaces is Hausdorff.

(b) Prove that a subspace of a Hausdorff space is Hausdorff.

Exercise 4. Prove that X is a Hausdorff space if and only if the *diagonal*

$$\Delta = \{(x, x) : x \in X\} \subset X \times X$$

is closed.

Exercise 5 (*)**.** Let Y be Hausdorff, X be an arbitrary topological space, $f, g\colon X \to Y$ continuous maps. If $f|_S = g|_S$ with $S \subset X$ satisfying $\bar{S} = X$ (i.e. S is *dense* in X), then $f = g$. (Hint: Use Exercise 4.)

Exercise 6. If $A \subset X$, we define the *boundary* of A by the equation

$$\partial A = \bar{A} \cap (\overline{X \setminus A}).$$

(a) Prove that $\mathring{A}$ and ∂A are disjoint, and $\bar{A} = \mathring{A} \cup \partial A$.

(b) Prove that $\partial A = \emptyset$ if and only if A is clopen (i.e. both open and closed).

(c) Prove that U is open if and only if $\partial U = \bar{U} \setminus U$.

Exercise 7 (*)**.** (a) Prove that no two of $(0, 1), [0, 1)$, and $[0, 1]$ are homeomorphic.

(b) Prove that $\mathbb{R} \ncong \mathbb{R}^n$ whenever $n > 1$.

Exercise 8. A space is *totally disconnected* if its only connected subspaces are one-point sets. Show that every discrete topological space is totally disconnected. Does the converse hold?

Exercise 9 (*)**.** From HW 2, recall the definitions of $O(n) \subset M_n(\mathbb{R})$ and the transpose A^T of an element $A \in M_n(\mathbb{R})$.

(a) Using the fact that $\det(A) = \det(A^T)$, prove that if $A \in O(n)$ then $\det(A) \in \{-1, 1\}$.

(b) Prove that $O(n)$ is disconnected.

(c) Describe $O(1)$.

Exercise 10 (*)**.** Let $\mathbb{C}$ denote the complex numbers and let $\mathbb{C}[t_1, t_2, \dots t_n]$ be the collection of polynomials in n-variables, with coefficients in $\mathbb{C}$. Given any subset of polynomials $S \subset \mathbb{C}[t_1, \dots, t_n]$, the *zero set* of S, denoted $Z(S)$, is the subset of $\mathbb{C}^n$ given by

$$Z(S) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0 \text{ for all } f \in S\}.$$

A *Zariski closed* subset of $\mathbb{C}^n$ is any subset $Z \subseteq \mathbb{C}^n$ of the form $Z = Z(S)$ for some subset $S \subset \mathbb{C}[t_1, \dots, t_n]$. The complement of a Zariski closed set is called *Zariski open.*

(a) Prove that for any pair of subset $S_1, S_2 \subset \mathbb{C}[t_1, \dots t_n]$, we have

$$Z(S_1) \cup Z(S_2) = Z(\{f_1 \cdot f_2 \in \mathbb{C}[t_1, \dots, t_n] : f_1 \in S_1 \text{ and } f_2 \in S_2\}).$$

(b) Prove that the set of all Zariski open subsets of $\mathbb{C}^n$ is a topology on $\mathbb{C}^n$.

(c) A topological space is said to be a T_1-*space* if every one-point set is closed. Prove $\mathbb{C}^n$ with the Zariski topology is T_1.

(d) Prove that $\mathbb{C}^n$ in the Zariski topology is not Hausdorff. (Another name for a Hausdorff space is a T_2-*space*; (c) and (d) show that $\mathbb{C}^n$ with the Zariski topology is T_1 but not T_2.)

Exercise 11 (Extra credit)**.** Prove that $\mathbb{C}$ and $\mathbb{C}^n$ are not homeomorphic in their respective Zariski topologies whenever $n > 1$.