Homework 2 MATH 590

Due Wednesday, January 27, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1 (*). Let $(a,b) \subset \mathbb{R}$ and equip this set with the metric topology coming from restricting the Euclidean metric to (a,b). Prove that (a,b) is homeomorphic to $(0,1)$. Similarly, prove that $[a,b]$ is homeomorphic to $[0,1]$.

Exercise 2. Let X be a set. Let $\mathcal{S} \subset 2^X$ be a collection of subsets satisfying

$$\bigcup_{U \in \mathcal{S}} U = X$$

and define τ to be the collection of arbitrary unions of finite intersections of elements of $\mathcal{S}$. Prove that τ is a topology of X. This topology is called the *topology generated by* $\mathcal{S}$.

Exercise 3. Let $f\colon X \to Y$ be a function between topological spaces and let $\mathcal{S}$ be a subbasis for Y. Prove that f is continuous if and only if for every $U \in \mathcal{S}$ the set $f^{-1}(U)$ is open in X.

Exercise 4. (Extra credit) How many non-pairwise homeomorphic topologies are there on the set $X = \{a,b,c\}$?

Exercise 5. (a) Show that the function $s, p\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$s(x,y) = x + y$$
$$p(x,y) = xy$$

are continuous. (If you are new to ϵ-δ proofs, see Exercise 12 in §21 for hints.)

(b) Prove that $\mathbb{R}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^n$.

(c) Let X be a topological space and let $f, g\colon X \to \mathbb{R}$ be continuous functions. Show that $f \pm g$ and fg, are continous; if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is continuous as well.

(d) Let $P\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial in n variables. Prove that P is continuous.

Exercise 6 (*). (a) Prove that $m\colon M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $m(A,B) = AB$ is continuous. (*Hint: You should think of $M_n(\mathbb{R})$ as $\mathbb{R}^{n^2}$.*)

(b) Prove that

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A \text{ is invertible}\}$$

is open in $M_n(\mathbb{R})$.

(c) Given $A \in M_n(\mathbb{R})$, the *transpose* of A, denoted A^T, is defined by

$$[A^T]_{i,j} = [A]_{j,i}$$

for all $1 \leq i, j, \leq n$. Prove that

$$O(n) = \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$$

is closed in $M_n(\mathbb{R})$.

Exercise 7. Prove the following statements.

(a) If $f\colon X \to Y$ is a map, $A \subseteq X$ a subspace, then

$$f|_A\colon A \to Y$$

given by $f|_A(x) = f(x)$ whenever $x \in A$ is a continuous function.

(b) Let X, Y be topological spaces, $Y_1 \subseteq Y_2 \subseteq Y$ subspaces, $f\colon X \to Y_2$ a continuous function. Then f as a function $X \to Y$ is also continuous. If $f(X) \subseteq Y_1$, then f as a function from $X \to Y_1$ is continuous as well.

Exercise 8. Equip $\mathrm{GL}_n(\mathbb{R})$ with the subspace topology coming from $M_n(\mathbb{R})$. Let $A \in \mathrm{GL}_n(\mathbb{R})$ and let $f_A, g_A\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ be given by

$$f_A(B) = AB$$
$$g_A(B) = BA$$

for all $B \in \mathrm{GL}_n(\mathbb{R})$. Prove that both f_A and g_A are homeomorphisms.

Exercise 9 (*)**.** An n-sphere $\mathbb{S}^n$ is a topological space homeomorphic to the subspace $\partial B_{\mathbb{R}^{n+1}}(0,1)$ of $\mathbb{R}^{n+1}$ given by

$$\partial B_{\mathbb{R}^{n+1}}(0,1) = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\right\}.$$

For this exercise, identify $\mathbb{S}^n$ with $\partial B_{\mathbb{R}^{n+1}}(0,1)$.

(a) Prove that $\mathbb{S}^n$ is a closed subset of $\mathbb{R}^{n+1}$.

(b) Prove that the the upper hemisphere

$$H = \mathbb{S}^n \cap \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_n \geq 0\}$$

is homeomorphic to the closed n-ball $\bar{D}^n$. Here, the closed n-ball $\bar{D}^n$ refers to any topological space homeomorphic to the closed unit ball in $\mathbb{R}^n$:

$$\bar{B}_{\mathbb{R}^n}(0,1) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 \leq 1\right\}.$$