

## Homework 13

MATH 590

Due Thursday, April 28, 2016

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**Instructions.** Write up (in  $\text{\LaTeX}$ ) and turn in all problems marked with an asterisks (\*) at the beginning of class on the due date.

**Exercise 1.** Let  $F = \partial B(0, 2) \subset \mathbb{R}^2$  be the circle of radius 2 centered at the origin. Let  $L$  be the closed line segment connecting  $(0, -2)$  and  $(0, 2)$  along the second-coordinate axis. Define  $\Theta = F \cup L$ . Prove that  $\Theta$  and a wedge sum of two circles (usually denoted  $\mathbb{S}^1 \vee \mathbb{S}^1$ ) are homotopic, but not homeomorphic.

**Exercise 2.** Let  $P = \mathbb{R}^2 \setminus \{p, q\}$  be the plane  $\mathbb{R}^2$  with the points  $p, q \in \mathbb{R}^2$  removed. Prove using pictures that  $P$  is homotopic to a wedge sum of two circles. The space  $P$  is called a *pair of pants*, can you see why? Note that the torus with a point removed is homotopic to a wedge sum of two circles (can you prove it?). This tells you that the torus with a point removed and  $P$  are homotopic and hence have the same fundamental group.

**Exercise 3.** Show that there is a continuous surjection from the torus to the 2-sphere.

**Exercise 4.** Let  $p: E \rightarrow B$  be a covering map. Let  $\alpha$  and  $\beta$  be paths in  $B$  with  $\alpha(1) = \beta(0)$ ; let  $\tilde{\alpha}$  and  $\tilde{\beta}$  be liftings of them such that  $\tilde{\alpha}(1) = \tilde{\beta}(0)$ . Prove that  $\tilde{\alpha} * \tilde{\beta}$  is a lifting of  $\alpha * \beta$ .

**Exercise 5.** For  $n \in \mathbb{Z}$ , consider the map  $g_n: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  given by  $g_n(z) = z^n$ , where we view

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Compute the induced homomorphism  $(g_n)_*: \pi_1(\mathbb{S}^1, 1) \rightarrow \pi_1(\mathbb{S}^1, 1)$ .

**Exercise 6.** Let  $p: E \rightarrow B$  be a covering map, with  $E$  path-connected. Prove that if  $B$  is simply connected, then  $p$  is a homeomorphism.

**Exercise 7.** Prove that if  $A$  is a retract of  $\mathbb{D}^2$ , then every continuous map  $f: A \rightarrow A$  has a fixed point.

**Exercise 8.** Show that if  $h: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is nullhomotopic, then  $h$  has a fixed point and  $h$  maps some point  $x$  to its antipode  $-x$ .