Homework 13

MATH 590

Due Thursday, April 28, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1. Let $F = \partial B(0,2) \subset \mathbb{R}^2$ be the circle of radius 2 centered at the origin. Let L be the closed line segment connecting $(0,-2)$ and $(0,2)$ along the second-coordinate axis. Define $\Theta = F \cup L$. Prove that Θ and a wedge sum of two circles (usually denoted $\mathbb{S}^1 \vee \mathbb{S}^1$) are homotopic, but not homeomorphic.

Exercise 2. Let $P = \mathbb{R}^2 \setminus \{p, q\}$ be the plane $\mathbb{R}^2$ with the points $p, q \in \mathbb{R}^2$ removed. Prove using pictures that P is homotopic to a wedge sum of two circles. The space P is called a *pair of pants*, can you see why? Note that the torus with a point removed is homotopic to a wedge sum of two circles (can you prove it?). This tells you that the torus with a point removed and P are homotopic and hence have the same fundamental group.

Exercise 3. Show that there is a continuous surjection from the torus to the 2-sphere.

Exercise 4. Let $p\colon E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\tilde{\alpha}$ and $\tilde{\beta}$ be liftings of them such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Prove that $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

Exercise 5. For $n \in \mathbb{Z}$, consider the map $g_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ given by $g_n(z) = z^n$, where we view

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Compute the induced homomorphism $(g_n)_*\colon \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{S}^1, 1)$.

Exercise 6. Let $p\colon E \to B$ be a covering map, with E path-connected. Prove that if B is simply connected, then p is a homeomorphism.

Exercise 7. Prove that if A is a retract of $\bar{\mathbb{D}}^2$, then every continuous map $f\colon A \to A$ has a fixed point.

Exercise 8. Show that if $h\colon \mathbb{S}^1 \to \mathbb{S}^1$ is nullhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.