Homework 12

MATH 590

Due Wednesday, April 13, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1 (*)**.** Let $p\colon E \to B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a k-fold covering.

Exercise 2. (a) Prove that a covering map is an open quotient map.

(b) Prove that a 1-fold covering is a homeomorphism.

Exercise 3. Prove that the torus is a k-fold cover of itself for any $k \in \mathbb{Z}_+$.

Exercise 4 (*)**.** Prove that the annulus is a 2-fold cover of the Möbius band.

Exercise 5 (*)**.** Let $p\colon E \to B$ be a covering map.

(a) If B is Hausdorff, then so is E.

(b) If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

Exercise 6 (*)**.** Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the covering map $p(t) = (\cos(2\pi t), \sin(2\pi t))$, so that $p \times p\colon \mathbb{R}^2 \to T^2$ is a covering map of the torus. Consider the path

$$f(t) = (\cos(2\pi t), \sin(2\pi t)) \times (\cos(4\pi t), \sin(4\pi t))$$

in T^2. Sketch what f looks like on the T^2 when you view T^2 as the surface of a doughnut in $\mathbb{R}^3$. Find a lifting of $\tilde{f}$ of f to $\mathbb{R}^2$ and sketch it. (You should hand draw your images, do not spend countless hours trying to do this with a computer!)

Exercise 7. Let X, Y be path-connected topological spaces. Prove that if $X \simeq Y$ (X is homotopy equivalent to Y), then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ for any $x_0 \in X$ and $y_0 \in Y$. (*Remark: the invariants coming from algebraic topology cannot distinguish homotopy equivalent spaces.*)

Exercise 8 (Extra credit)**.** Prove that a proper local homeomorphism between manifolds is a covering map.