Homework 11

MATH 590

Due Wednesday, April 6, 2016

Instructions. *Write up (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date.*

Exercise 1 (*)**.** Identify $\mathbb{S}^1$ with the subset of $\mathbb{C}$ defined by $\{z \in \mathbb{C} : |z| = 1\}$. Define the map $f\colon \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ by

$$f(z, w) = \left(\frac{1}{z}, -w\right),$$

where $\hat{\mathbb{C}}$ is the Riemann sphere. Let $\iota\colon \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ denote the identity map, i.e. $\iota(z, w) = (z, w)$. Observe that $f \circ f = \iota$, so the set $G = \{\iota, f\}$ forms a discrete group with respect to function composition.

(a) Let $T^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Prove that $\alpha\colon G \times T^2 \to T^2$ defined by $\alpha(f, x) = f(x)$ is a proper and free action of G on T^2.

(b) Prove that $K = T^2/G$ is a surface, i.e. a 2-manifold. (K is called the *Klein bottle*. You should look up a picture of one if you haven't seen it before.)

Exercise 2. Prove that if $h, h'\colon X \to Y$ are homotopic and $k, k'\colon Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Exercise 3. A space X is called *contractible* if the identity map $i_X\colon X \to X$ is nullhomotopic (i.e. homotopic to a constant map).

(a) Prove that $\mathbb{R}^n$ is contractible.

(b) A *star domain* in $\mathbb{R}^n$ is a subset U of $\mathbb{R}^n$ such that there exists $x_0 \in U$ with the property that the line segment connected x_0 to any point $x \in U$ is contained in U. Prove that every star domain is contractible.

(c) Prove that every contractible space is simply connected.

Exercise 4 (*)**.** Let G be a topological group with operation $\cdot$ and identity element x_0. Let $\Omega(G, x_0)$ denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

(a) Prove that this operation makes the set $\Omega(G, x_0)$ into a group.

(b) Prove that this operation induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

(c) Prove that the two group operations $*$ and $\otimes$ on $\pi_1(G, x_0)$ are the same. [Hint: Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$.]

(d) Prove that $\pi_1(G, x_0)$ is abelian.

Exercise 5 (*)**.** Let $\mathbb{Z}^2$ act on $\mathbb{R}^2$ via $(n, m)\cdot(x, y) = (x+n, y+m)$ and let $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ be the associated quotient map. Let us identify the torus T^2 with $\mathbb{R}^2/\mathbb{Z}^2$.

(a) Let L be a line in $\mathbb{R}^2$ with rational slope. Prove that $\pi(L)$ is a loop in T^2.

(b) Suppose L and L' are two lines in $\mathbb{R}^2$ with the same rational slope. Prove that there is a continuous map $F\colon T^2 \times I \to T^2$ such that $F(\pi(L), 0) = \pi(L)$ and $F(\pi(L), 1) = \pi(L')$. (Here, we would say that $\pi(L)$ is *freely homotopic* to $\pi(L')$.)

Definition 1. Two topological spaces X and Y are said to be *homotopy equivalent* or simply *homotopic* if there exists $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$ where id_Y, id_X are the identity functions on Y and X, respectively. If X and Y are homotopic, we write $X \simeq Y$.

Remark. Though homotopic spaces are not necessarily homeomorphic, when it comes to the study of algebraic topology, two homotopic spaces will have all the same algebraic invariants, i.e. the tools of algebraic topology cannot tell homotopic spaces apart.

Exercise 6. Prove that a topological space is contractible if and only if it is homotopic to the one-point space.

Exercise 7. Prove that $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \simeq \mathbb{S}^n$, where $\mathbf{0}$ is the origin in $\mathbb{R}^{n+1}$.

Exercise 8. Prove that homotopy equivalence is an equivalence relation on the set of topological spaces.

Definition 2. Let X be a topological space and $A \subset X$ a subspace. A continuous map $F : X \times I \to X$ is called a *deformation retract* of X onto A if for every $x \in X$ and $a \in A$,

- $F(x, 0) = x$
- $F(x, 1) \in A$
- $F(a, t) = a$ for every $t \in I$.

Exercise 9. Let A be a subspace of a topological space X. Prove that if X deformation retracts onto A, then $X \simeq A$.

Exercise 10 (*)**.** Let $A = \mathbb{S}^1 \times I$ be the annulus and let M be the Möbius strip defined in HW 8 Exercise 3.

(a) Prove that A deformation retracts to a copy of $\mathbb{S}^1$ in A.

(b) Prove that M deformation retracts to a copy of $\mathbb{S}^1$ in M.

(c) Prove that $A \simeq M$.