Homework 1

MATH 590

Due Wednesday, January 20, 2016

Instructions. *You are required to writeup (in LaTeX) and turn in all problems marked with an asterisks (*) at the beginning of class on the due date. You are expected to work through and understand all the problems on the sheet. You may turn in the extra credit problems at any time during the semester.*

In your solutions, you may use any material covered in class or in the homework. All claims must be sufficiently justified.

Exercise 1 (*). Let $A = \left\{ \frac{1}{n} : n \in \{1, 2, \dots\} \right\} \subset \mathbb{R}$.

(a) Prove that A is neither open nor closed.

(b) Prove that $\bar{A} = A \cup \{0\}$ is closed. (Note: $\bar{A}$ is the smallest closed set containing A and is called the *closure* of A.)

Exercise 2. Let X be a topological space.

(a) Prove that arbitrary intersections of closed sets are closed.

(b) Prove that finite unions of closed sets are closed.

(c) Show that an arbitrary union of closed sets need not be closed.

Exercise 3 (*). Let $a, b \in \mathbb{Z}, b \neq 0$. Denote

$$N_{a,b} := \{a + bn : n \in \mathbb{Z}\}.$$

Declare a non-empty subset $U \subseteq \mathbb{Z}$ to be open if it is a union of sets of the form $N_{a,b}$. Also declare the empty set to be open.

(a) Prove that the collection of open sets (as defined) forms a topology on $\mathbb{Z}$. (Note: this is not the standard topology on $\mathbb{Z}$ coming from $\mathbb{R}$.)

(b) Prove that the sets $N_{a,b}$ are closed.

(c) Prove that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ prime}} N_{0,p}\,.$$

(d) Prove that there are infinitely many primes.

Exercise 4. Prove that the standard topology on $\mathbb{R}^n$ is indeed a topology.

Exercise 5. Prove that the *open-half space* of $\mathbb{R}^3$ defined by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

is indeed an open subset in $\mathbb{R}^3$.

Exercise 6 (*)**.** Let X be a metric space with at most 3 points. Show that there is a subset Y of $\mathbb{R}^2$ and a bijection $f\colon X \to Y$ such that for any points u and v in X, one has $d_X(u,v) = d_Y(f(u), f(v))$, where d_Y denote the distance measured in $\mathbb{R}^2$ with respect to the Euclidean metric. (We say that X is *isometric* to Y.)

Exercise 7. For $x, y \in \mathbb{R}^n$ define $\rho(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

(a) Assuming the triangle inequality for $\mathbb{R}$, prove that $(\mathbb{R}^n, \rho)$ is a metric space.

(b) What shape is an open ball of radius r in this metric?

(c) Prove that the topology on $\mathbb{R}^n$ induced by ρ is the standard topology.

Note: ρ is called the *square metric* on $\mathbb{R}^n$.

Exercise 8. Let X and Y be topological spaces. Prove that $f\colon X \to Y$ is continuous if and only if for every closed set $U \subseteq Y$ the set $f^{-1}(U) \subseteq X$ is closed.

Exercise 9 (*)**.** Let (X, d) be a metric space.

(a) Let $a \in X$ and let $f_a\colon X \to \mathbb{R}$ be defined by
$$f_a(x) = d(a,x)$$
for all $x \in X$. Prove that f_a is continuous.

(b) Prove that the closed ball
$$\bar{B}(x,\delta) = \{y \in X : d(x,y) \leq \delta\}$$
is a closed subset of X.

Exercise 10 (Extra credit)**.** Let $p \in \mathbb{Z}$ be a fixed primed number. For an arbitrary nonzero integer $x \in \mathbb{Z}$ let
$$\mathrm{ord}_p(x) = \text{ the highest power of } p \text{ which divides } x,$$
while we define $\mathrm{ord}_p(0) = \infty$. (For example, $\mathrm{ord}_3(18) = 2$.) If $\alpha = \frac{x}{y} \in \mathbb{Q}^\times$, then we set
$$\mathrm{ord}_p(\alpha) = \mathrm{ord}_p\left(\frac{x}{y}\right) = \mathrm{ord}_p(x) - \mathrm{ord}_p(y).$$
Note that $\mathrm{ord}_p(\alpha)$ does not depend on the choice of x and y. $\mathrm{ord}_p(x)$ is called the *p-adic* order of x. For $\alpha, \beta \in \mathbb{Q}$ define
$$d_p(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha = \beta \\ \frac{1}{p^{\mathrm{ord}_p(\alpha-\beta)}} & \text{otherwise.} \end{cases}$$
This is called the *p-adic distance of* α *and* β.

Prove that $(\mathbb{Q}, d_p)$ is a metric space, which is in addition non-archimedean, that is, for every $x, y, z \in \mathbb{Q}$ one has
$$d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\}.$$
Conclude that in $(\mathbb{Q}, d_p)$ every triangle is isosceles.