Fast Control of Linear Systems Subject to Input Constraints

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Efficient design of high performance automatic control systems is extremely important for high technology systems. To get the best hardware cost-to-performance ratio, it is desirable to design a controller that takes full advantage of actuator capabilities, but this can lead to nonlinear behavior due to actuator saturation. The saturation nonlinearities in the system may have severe effects on system performance due to, for example, integrator windup. In this paper, a new design method is presented based on Lyapunov stability theory. By incorporating the actuator constraints directly in the design method, better utilization of the available control effort can be ensured in achieving desired system behavior. [S0022-0434(00)01801-3]

1 Introduction

Every physical control system has some kind of actuator limitation or nonlinearity (e.g., a motor can only supply limited torque and limited power). The actuator limitation ultimately limits the achievable closed-loop performance. Often, actuator constraints impose more severe performance limitations than other sources such as modeling uncertainties [1]. The effects of amplitude saturation have been extensively studied for decades, but it is not the only type of actuator saturation. Recently, increasing work has been done on the control of systems where the output rate of change of the actuator is limited (e.g., [2–4].) Power saturation depends on the product of two states (e.g., voltage and current). Other constraints are internal to the plant. A plant state variable could, for example, be limited for safety reasons.

The design method presented in this paper deals with the design of linear systems with input amplitude saturation. However, it can be extended to handle other types of constraints.

The traditional design approach for systems with amplitude saturation has two-steps: A linear controller is designed first, ignoring the effects of actuator saturation, and then an ad hoc anti-windup (AW) scheme is designed to reduce the problems that arise due to saturation nonlinearities. Since the constraints are not considered in the linear design it may not take full advantage of the available actuator capacity. Since the most severe wind-up problems often come from integral control, that is where much attention has been focused. See, e.g., the research on proportional-integral-derivative (PID) controllers in [5–9]. Another approach is to use guaranteed domains of attraction (GDA) (invariant sets) to design a reference governor that restricts the reference signal to avoid saturation, see [10]. See also [11], which uses GDA:s to avoid saturation by switching between increasingly ''faster'' controllers as the states approaches the set point.

The design procedure presented in this paper can be applied to both full-state feedback and full-state feedback plus integral control for single-input single-output (SISO) systems. When integral control is used, a simple AW scheme is added to improve system performance.

First, the general design concept will be described. Second, several design examples are presented. They include simulation and experimental results that support the design approach. Additional examples that support the feasibility of the design method can be found in [12].

The ideas presented in this paper are closely related to work presented in [13] and [14], where the goal is to design a saturating controller that fully utilizes available control effort while optimizing performance. See [15] for a method that results in a guaranteed cost in the presence of saturation. The design method presented in this paper differs in that the emphasis is on achieving a desired linear behavior, instead of robustness to saturation.

2 Assumptions

The plant is given by the discrete-time system

$$\mathbf{x}(t+1)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\text{sat}(u(t)), \tag{1}$$

$$\mathbf{y}(t)=\mathbf{C}_1\mathbf{x}(t), \tag{2}$$

$$z(t)=\mathbf{C}_2\mathbf{x}(t), \tag{3}$$

with state $\mathbf{x}\in\Re^n$, controlled output $z\in\Re$, output measurement $y\in\Re^n$, and actuator output $u\in\Re$. The saturation function is defined by

$$\text{sat}(u)=\begin{cases} u, & \text{if } \text{abs}(u)\leq u_{\max} \\ \text{sign}(u)u_{\max}, & \text{if } \text{abs}(u)>u_{\max} \end{cases} \tag{4}$$

It is assumed that $u_{\max}$, is known. $\mathbf{C}_1$ is the identity matrix so that full state feedback,

$$u=-\mathbf{K}\mathbf{x}=-\mathbf{K}\mathbf{y} \tag{5}$$

can be used. The reference input is assumed to be zero. If $\mathbf{C}_1 \neq \mathbf{I}$, the design method can still be used by designing a state observer. See [16] for a discussion on the effect of the design of the observer on the proposed design method.

3 The Design Concept

In most research on saturating control it is assumed that a good linear design already exists, and the emphasis is on finding a method to minimize the effect of the saturation element on the system performance. The emphasis here is different, in that there is no a priori linear design. The goal of the design is to come up with a linear controller that *with saturation* minimizes the settling time of the system. A key idea is the scaling of the linear design by a parameter $\beta>0$ which is chosen so that the system performance improves as β increases.

It was shown in [17] how such a dependency on β can be achieved using linear quadratic regulator (LQR) design. To make things specific, let the performance be measured by the settling time and choose the scaling parameter by pole placement. Specifically, suppose β is determined by the following procedure:

1 Define vector of complex pole locations $\mathbf{p}_b$ that results in a desirable system behavior.

2 Introduce a real scalar scaling parameter β such that the final

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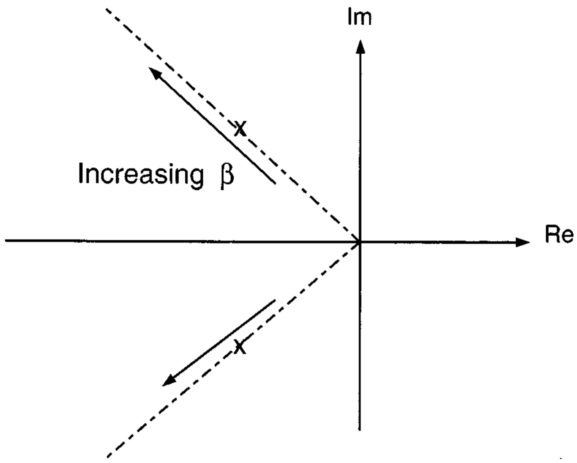

Fig. 1 Scaling of poles

pole locations $\mathbf{p}_c$ are defined by $\mathbf{p}_c=\beta\mathbf{p}_b$ (see Fig. 1), or if discrete time control is used $\mathbf{p}_c=e^{-\beta\mathbf{p}_b\Delta T}$, where ΔT is the sampling time. This way of scaling maintains the damping of the poles, while changing the speed of the response.

3 Choose the feedback gains $\mathbf{K}(\beta)$, so that the closed loop poles are given by $\mathbf{p}_c$.

This simple scaling idea is fine for many plants. Indeed, the performance cost (settling time) will often be reduced as β increases. In general, though, a more advanced form of scaling will be needed, such as the LQR based method in [17]. For example, if the plant transfer function has poorly placed zeros, the overshoot and hence the settling time might increase as β is increased. The vector $\mathbf{p}_b$ is assumed given and the nontrivial problem of how to choose it are not addressed here. The method of scaling is not critical to the design method as long as the performance cost is a monotone decreasing function of β.

Clearly, if the control signal does not saturate, it is desirable to increase β. In the presence of saturation there will be a limit to the achievable performance, but that limit is generally not given by the onset of saturation. Figure 2 shows the 1 percent settling time, as a function of β, for the digital state feedback control of a double integrator plant: $\mathbf{p}_b=[-1,-1]$ (i.e., a critically damped design), for an initial position displacement of 1, and a sampling time of 0.01 s.

From Fig. 2 it is clear that the performance can be improved by increasing β outside the interval $(0,\beta_{\text{lin}})$ where the control does not saturate. However, as β is increased beyond β_{opt} the behavior of the system starts to deteriorate due to saturation effects. Similar results can be obtained using other performance measures, such as the integral of the squared error. The extent to which the behavior can be improved relative to the nonsaturating case are highly dependent on the dynamics of the open loop plant.

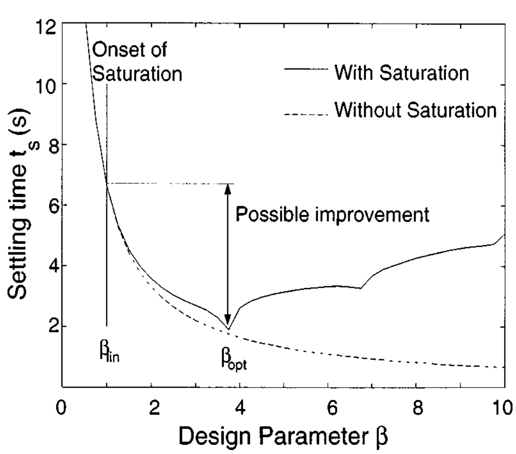

Fig. 2 Scaling design for PD control of a double integrator plant

The success of a controller designed in this manner may vary. The 1 percent settling time for the time optimal bang-bang control for the above example is 1.8 s, and, as can be seen from Fig. 2, it is possible to get quite close to that time by just using a linear feedback and letting it saturate. Of course, the time-optimal control reaches the set-point in finite time, whereas the linear saturating controller asymptotically approach the set-point.

Similar conclusions have been drawn in [18] for the proportional-derivative (PD) control of second order systems. They show it is possible to increase the closed-loop bandwidth and decrease the root mean square (RMS) error by choosing PD feedback gains so that the controller saturates.

Since the design has been parameterized by a single design parameter β it is easy to do a search over β to find the optimal value for a particular initial condition (or step input size). In general, it is desirable to have a good response for a set of initial conditions, not just one particular initial condition. In this case, it is natural to minimize the worst case settling time, i.e.,

$$\beta_{\text{opt}}=\arg\min_{\beta}(\max_{x_0\subset \text{IC}} t_s), \tag{6}$$

where IC is the specified set of initial conditions. While the optimal solution to Eq. (6) may be achieved numerically for systems of very low order by repeated simulation, it is generally intractable. Another disadvantage of trying to solve Eq. (6) is that nothing is known about the robustness of the design.

Another approach, which circumvents Eq. (6) is to construct a guaranteed domain of attraction, $\Omega_s(\beta)$. The "size" of $\Omega_s(\beta)$ generally becomes smaller as β increases. The idea is to maximize β under the constraint that $\text{IC}\subset\Omega_s(\beta)$. The resulting value of β is

$$\beta_{\text{sat}}=\max_{\text{IC}\subset\Omega(\beta)}\beta. \tag{7}$$

While it is generally true that β_{sat} is less than β_{opt}, solving for β_{sat} is numerically feasible and it can produce efficient designs. Moreover, designs can also be made robust to disturbances and parameter uncertainties (see [16]). A method for constructing ellipsoidal GDA's is described in the following section. Notice that as long as a monotone dependence between a scalar design parameter and a performance cost can be identified any linear design technique (state space or frequency domain) can be used to do the scaling. In this paper, pole placement is used to perform the scaling for the initial examples, and LQR design is used to do the scaling for the experimental setup.

4 A Guaranteed Domain of Attraction

First, we state a theorem that for fixed $\mathbf{K}$ determines an ellipsoidal domain of attraction for the saturated feedback system determined by Eqs. (1)–(5). The proof depends on a simple application of the discrete-time Lyapunov equation. The proof for the continuous-time case can be obtained by using the continuous-time Lyapunov equation instead of the discrete time Lyapunov equation.

Given an $m\times m$, positive definite symmetric matrix $\mathbf{P}$, and a feedback gain $\mathbf{K}$ that stabilizes the system $\mathbf{x}(t+1)=(\mathbf{A}-\mathbf{BK})\mathbf{x}(t)$, let $0<\alpha_{\min}<1$, be a number such that the matrix inequality

$$(\mathbf{A}-\alpha\mathbf{BK})^T\mathbf{P}(\mathbf{A}-\alpha\mathbf{BK})-\mathbf{P}<-\epsilon\mathbf{I} \tag{8}$$

is satisfied for all $\alpha\in[\alpha_{\min},1]$, for a fixed $\mathbf{P}$, and for an arbitrarily small constant $\epsilon>0$. Note that $\alpha_{\min}$ will depend on $\mathbf{K}$ and $\mathbf{P}$. A necessary condition for the existence of a matrix $\mathbf{P}$ that satisfies Eq. (8) for all $\alpha\in[\alpha_{\min},1]$ is that all the eigenvalues of the matrix $\mathbf{A}-\alpha\mathbf{BK}$ are placed inside the unit disk for $\alpha\in[\alpha_{\min},1]$.

Theorem 4.1. Define

$$\gamma \triangleq (\mathbf{K}\mathbf{P}^{-1}\mathbf{K}^T)^{-1} u_{\max}^2 \alpha_{\min}^{-2} \tag{9}$$

and

$$\Omega_s(\mathbf{K},\mathbf{P}) \triangleq \{\mathbf{x} \in \Re^n | \mathbf{x}^T\mathbf{P}\mathbf{x} \leq \gamma\} \tag{10}$$

Then the system

$$\mathbf{x}(t+1) = \mathbf{A} - \mathbf{B}\text{sat}(\mathbf{K}\mathbf{x}(t)) \tag{11}$$

has an asymptotically stable equilibrium point at $\mathbf{x}=\mathbf{0}$ *and* Ω_s *is a guaranteed domain of attraction. Specifically,* $\mathbf{x}(0) \in \Omega_s$ *implies* $\mathbf{x}(t) \in \Omega_s$ *for all* $t>0$ *and*

$$\mathbf{x}(t) \to 0 \quad \text{as} \quad t \to \infty \tag{12}$$

Proof: First we show that $\mathbf{x} \in \Omega_s$ implies $\text{sat}(\mathbf{Kx}) = \alpha(\mathbf{x})\mathbf{Kx}$ where $\alpha_{\min} \leq \alpha(\mathbf{x}) \leq 1$. It is clear that $\alpha(\mathbf{x})=1$ when $|\mathbf{Kx}| \leq u_{\max}$ and $\alpha(\mathbf{x})$ is minimum when $|\mathbf{Kx}|$, $\mathbf{x} \in \Omega_s$, is maximum. The condition $\alpha_{\min} \leq \alpha(\mathbf{x})$ is achieved by choosing γ appropriately. The $\max_{\mathbf{x} \in \Omega_s} \mathbf{Kx}$ exists since Ω_s is compact. Moreover, a maximizing state $\mathbf{x}^*$ must satisfy the multiplier rule for inequality constraints: $\lambda \nabla \mathbf{Kx} = \nabla(\mathbf{x}^T\mathbf{P}\mathbf{x} - \gamma)$ at $\mathbf{x}=\mathbf{x}^*$, $\lambda \geq 0$. Equivalently $\mathbf{x}^* = -(1/2\lambda)\mathbf{P}^{-1}\mathbf{K}^T$.

Substituting $\mathbf{x}^* = -(1/2\lambda)\mathbf{P}^{-1}\mathbf{K}^T$ into $\mathbf{x}^{*T}\mathbf{P}\mathbf{x}^* = \gamma$ and $\alpha_{\min}\mathbf{K}\mathbf{x}^* = u_{\max}$ and solving for γ results in Eq. (9). Now let $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ and form

$$V(\mathbf{x}(t+1)) = \mathbf{x}^T(t+1)\mathbf{P}\mathbf{x}(t+1). \tag{13}$$

Since $\mathbf{x}(\mathbf{0}) \in \Omega_s$, $\alpha(\mathbf{x}(\mathbf{0})) \geq \alpha_{\min}$ is guaranteed to hold. Consequently, the inequality in Eq. (8) holds and

$$V(\mathbf{x}(1)) \leq V(\mathbf{x}(0)) - \epsilon|\mathbf{x}(0)|^2 \leq \gamma. \tag{14}$$

Repeating the argument for increasing t shows that

$$V(\mathbf{x}(t)) \leq \gamma \quad \forall t \geq 0. \tag{15}$$

Thus, $\mathbf{x}(t) \in \Omega_s$, $t \geq 0$, also,

$$V(\mathbf{x}(t+1)) - V(\mathbf{x}(t)) \leq -\epsilon|\mathbf{x}(t)|^2 \quad \forall t \geq 0. \tag{16}$$

Result (12) is true if $\mathbf{x}(t)=0$ for some $t \geq 0$. Thus, suppose $|\mathbf{x}(t)| > \epsilon \forall t \geq 0$. Since $V(\mathbf{x}(t))$ is strictly decreasing, and bounded from below it has to converge to a limit, say V^*. Using

$$V(\mathbf{x}(t)) - V(\mathbf{x}(t-1)) \geq \epsilon|\mathbf{x}(t)|^2 \quad \forall t \geq 0 \tag{17}$$

it follows that $|\mathbf{x}(t)| \to 0$. □

Theorem 4.1 is illustrated schematically for a second order example in Fig. 3. Theorem 4.1 does not say anything about how to obtain the matrix $\mathbf{P}$. Any choice of $\mathbf{P}$ that satisfies Eq. (8) for $\alpha = 1$ can be used; it then follows that there exists an $\alpha_{\min} < 1$.

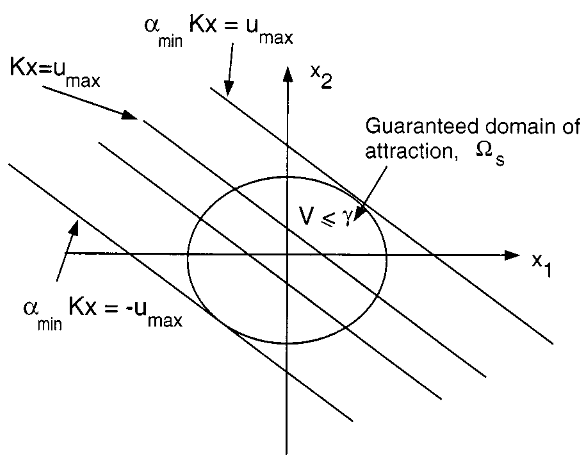

Fig. 3 Two-dimensional example

However, the "size" and "shape" of the GDA Ω_s will depend on the choice of $\mathbf{P}$. In the design procedure a search will be made to find a matrix $\mathbf{P}$, that results in $\text{IC} \subset \Omega_s$.

The stability result obtained in Theorem 4.1 is sufficient, but not necessary to guarantee asymptotic stability, and therefore the obtained GDA will be conservative.

A similar method was used by [11] to define domains of attraction, but they wanted to avoid saturation. In [19] the use of higher order Lyapunov functions were investigated to obtain a less conservative estimate of the set for which it can be guaranteed that the controller will not saturate. In [12] it was shown how the obtained GDA can be made robust to bounded input disturbances and parameter uncertainties that satisfies the matching condition.

If the inequality in Eq. (8) holds for all $\alpha \in (0,1]$, then the system in Eq. (11) is globally asymptotically stable (GAS). This was used in [20–22] to ensure global stability for systems with no right-hand poles or repeated poles on the imaginary axis. Since the GDA is bounded, there is more freedom in changing the feedback gains compared to when global stability is required. The method can also be applied to open-loop unstable systems. Different $\mathbf{P}$ will result in different values for $\alpha_{\min}$, and different "sizes" for the set Ω_s. It is desirable to find a Lyapunov function that results in a sufficiently large Ω_s, which in general means a small $\alpha_{\min}$ (see Eq. (9)).

Satisfaction of the criterion $\text{IC} \subset \Omega_s$ depends on $\mathbf{K}(\beta)$, and the matrix $\mathbf{P}$. As discussed in Section 3 it is desirable to maximize β to improve performance, while being able to guarantee stability.

By defining

$$\alpha_{\min} = \inf_{(\mathbf{A}-\alpha\mathbf{B}\mathbf{K})^T\mathbf{P}(\mathbf{A}-\alpha\mathbf{B}\mathbf{K})-\mathbf{P} < -\epsilon I \quad \forall \alpha \in [\alpha_{\min},1]} \alpha, \tag{18}$$

the problem of finding the "best" controller for a given $\mathbf{P}$ can be formulated as

$$\beta_{\text{sat}} = \max_{\text{IC} \subset \Omega_s(\mathbf{P},\mathbf{K}(\beta))} \beta. \tag{19}$$

Thus, $\Omega_s(\mathbf{P},\mathbf{K}(\beta))$ replaces $\Omega(\beta)$ in Eq. (7).

Since Ω_s is convex it follows that $\text{IC} \subset \Omega_s \Rightarrow \text{coIC} \subset \Omega_s$. In addition, since Ω_s is symmetric around the origin $\text{IC} \subset \Omega_s \Rightarrow \text{co}(\text{IC} \cup -\text{IC}) \subset \Omega_s$, where $-\text{IC} = \{\mathbf{x} \in \Re^n | -\mathbf{x} \in \text{IC}\}$. Consequently, it is sufficient to define a set IC* consisting of a finite number of points, where IC* is chosen so that $\text{IC} \subset \text{co}(\text{IC}^* \cup -\text{IC}^*)$. Then, $\text{IC}^* \subset \Omega_s \Rightarrow \text{IC} \subset \Omega_s$ will hold. This means that the condition in Eq. (19) only has to be checked for a finite number of points, IC*, even if the set of initial conditions, IC, has an infinite number of possible initial conditions.

A software that performs the design has been developed in Matlab. In the developed software a numerical search is performed to find the matrix $\mathbf{P}$ that results in the largest β_{sat}. The basic structure of the algorithm is described informally by the following five steps:

1. Choose an initial β.
2. Repeat steps 3–5 until β is determined within a specified tolerance.
3. Search for a $\mathbf{P}$ that satisfies $\text{IC} \subset \Omega_s(\mathbf{P},\mathbf{K}(\beta))$.
4. If a $\mathbf{P}$ can be found such that $\text{IC} \subset \Omega_s(\mathbf{P},\mathbf{K}(\beta))$, then increase β and go to 3.
5. If a $\mathbf{P}$ cannot be found such that $\text{IC} \subset \Omega_s(\mathbf{P},\mathbf{K}(\beta))$, then reduce β and go to 3.

The key step is step 3 where the $\mathbf{P}$ matrix is obtained. The $\mathbf{P}$ matrix is obtained by defining the function

$$S(\mathbf{P},\beta) = \gamma(\mathbf{P},\mathbf{K}(\beta),u_{\max})^{-1} \max_{\mathbf{x} \in \text{IC}} \mathbf{x}^T\mathbf{P}\mathbf{x}. \tag{20}$$

When $S(\mathbf{P},\beta) < 1$ holds, it follows from Eq. (10) that $\mathbf{IC} \subset \Omega_s(\mathbf{P},\mathbf{K}(\beta))$ holds. Since any $S(\mathbf{P},\beta) < 1$ guarantees the stability, it is desirable to minimize $S(\mathbf{P},\beta)$ with respect to $\mathbf{P} = \mathbf{P}^T > 0$. Since $S(\mathbf{P},\beta)$ is a well-defined objective function, any

Table 1 Comparison of design methods, full state feedback

	$\beta=\beta_{lin}$	$\beta=\beta_{opt}$	$\beta=\beta_{sat}$
β	0.79	3.23	2.29
$max(t_s)\ \forall \mathbf{x}\in IC$	12.0	4.5	5.9
$\mathbf{K}$	[0.37 0.87 0.49]	[7.47 29.29 32.43]	[4.77 14.36 11.66]

of a number of numerical optimization techniques can be used to minimize $S(\mathbf{P},\beta)$ for a fixed β, and thereby find a $\mathbf{P}$ such that

$$\mathrm{IC}\subset\Omega_s(\mathbf{P},\mathbf{K}(\beta)). \tag{21}$$

There is no guarantee that there exists a pair $\mathbf{P},\beta$ such that Eq. (21) is satisfied for an open-loop unstable system, in fact, if a system is open-loop unstable and the input control is bounded, there exists initial states from which it is impossible to stabilize the system. To reduce the computation time, the search for the matrix $\mathbf{P}$ that minimizes the function $S(\mathbf{P},\beta)$, can be interrupted as soon as $S(\mathbf{P},\beta)<1$ holds, since that is sufficient to guarantee that $\mathrm{IC}\subset\Omega_s(\mathbf{P},\mathbf{K}\beta))$. Most of the computation time is spent on searching for the matrix $\mathbf{P}$, and, in general, the higher β is the harder it is to find a $\mathbf{P}$ such that $\mathrm{IC}\in\Omega_s(\mathbf{P},\mathbf{K}(\beta))$.

5 Example 1

In this example a controller is designed for the plant

$$\dot{\mathbf{x}}=\begin{bmatrix}-2 & -1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\0\\0\end{bmatrix}\mathrm{sat}(u) \tag{22}$$

$$z=[0\quad 0\quad 1]\mathbf{x}.$$

The plant has open-loop eigenvalues $[-1,\ -1,\ 0]$, and $u_{\max}=1$. A sample time of 0.01 s is used, and a critically damped pole structure ($\mathbf{p}_b=[-1,-1,-1]$) is used for the scaling of the design. The set of initial conditions IC are chosen to be seven points on the unit sphere ($\mathbf{x^Tx}=1$), given by

$$\mathrm{IC}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}0.71\\0.71\\0\end{bmatrix},\begin{bmatrix}0.71\\0\\0.71\end{bmatrix},\begin{bmatrix}0\\0.71\\0.71\end{bmatrix},\begin{bmatrix}0.58\\0.58\\0.58\end{bmatrix}\right\}. \tag{23}$$

Notice that even if just a few points on the unit sphere were chosen, by using the stability based design, the stability will also be guaranteed for the much larger set given by $\mathrm{co}(\mathrm{IC}\cup-\mathrm{IC})$. By choosing the set IC to be just a few points in state space, the numerical solution to Eq. (6) can be computed approximately without too much effort. Thus, making it possible to compare the result obtained using the stability based design to the optimal design. However, when the optimal solution is obtained numerically the result will only hold for the set of initial conditions that were used in the optimization, and not for the larger set defined by $\mathrm{co}(\mathrm{IC}\cup-\mathrm{IC})$.

Table 1 shows the result of using the stability based design approach, compared to the case when the controller is not allowed to saturate, and to the optimal simulation based solution to the min/max problem in Eq. (6). $\mathbf{P}$ and γ resulting from the stability based design are given by $\gamma=8.09$, and

$$\mathbf{P}=\begin{bmatrix}1.0000 & 2.0761 & 1.2067\\ 2.0761 & 5.3015 & 3.6290\\ 1.2067 & 3.6290 & 3.1865\end{bmatrix}. \tag{24}$$

The values for $V(\mathbf{x}_0)$ for the seven initial conditions are: 1.0, 5.3, 3.2, 5.2, 3.3, 7.9, and 7.8, i.e., all of them satisfy the condition $V(\mathbf{x}_0)\leq\gamma$, and are contained in the set Ω_s.

The optimal solution is obtained numerically and, consequently, is only approximate. However, since the set of initial conditions is a limited number of points in state space, it is possible to obtain β_{opt} quite accurately. The accuracy is determined by the size of the change in the parameter β used when searching for β_{opt}.

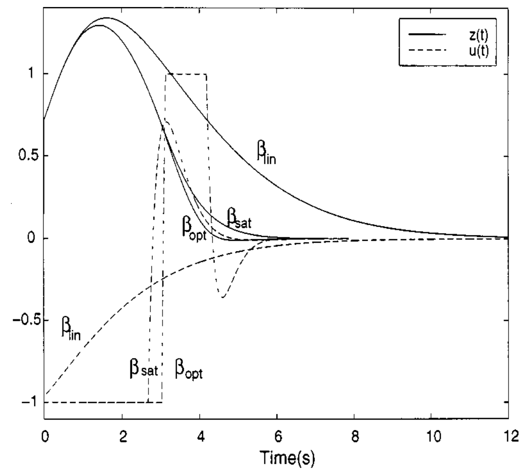

Fig. 4 Simulation results for $\mathbf{x}(0)=[0\quad 0.71\quad 0.71]^T$ and β=0.79, 2.29, and 3.23

Figure 4 shows the simulation results for $\mathbf{x}(0)=[0\ \ 0.71\ \ 0.71]$ for the different designs. The computational effort required to find an approximate solution to Eq. (6) is not insignificant (about 400 simulations were made to find an approximation to β_{opt}), even though there are only seven initial conditions. For this example it took 2 min 18 s to find β_{sat}, using a Pentium 166 processor, compared to 6 min 30 s to find β_{opt}.[1]

As can be seen from the simulations the stability based design resulted in improved performance relative to the controller that was designed to avoid saturation, and the result was relatively close to the optimal, with $\beta_{\mathrm{sat}}<\beta_{\mathrm{opt}}$.

6 Integral Control

It is often desirable to use integral control to reduce steady-state errors, due to disturbances or model uncertainty, but integral control leads to integrator windup. Integrator windup may have severe effects on the system performance and can even render the system unstable. To reduce this effect an anti-windup (AW) scheme can be used. There are many different AW schemes available in the literature, here a relatively simple AW method will be used, that is not necessarily better than other available AW methods, but is effective and has a simple structure suitable for the following analysis.

The AW scheme, which is used here, was first suggested in [23]. Later, it was shown in [8] that for a second order plant with no right-hand poles, the suggested AW method achieves global stability. To keep the equations simple no reference input are considered. The basic idea behind this AW scheme is to always ensure that the value of the integral state is consistent with the output generated from the saturated controller. This is done by resetting the integral state so that $\mathrm{sat}(u(t))=u(t)=-\mathbf{Kx}(t)-K_ix_i(t)$ holds after the AW is applied, i.e., if $(u(t)>u_{\max})$ the integral state is reset to

$$x_i(t)=(-u_{\max}-\mathbf{Kx(t)})/K_i \tag{25}$$

and, $u(t)$ updated. This is equivalent to the control implemented by

$$\tilde{u}(t)=-\mathbf{Kx}(t)-K_ix_i(t)$$

[1]The programs used to find β_{opt} and β_{sat} are still in their early stages and improvements can definitely be made in terms of the required computation time. These results are cited for comparison purposes only.

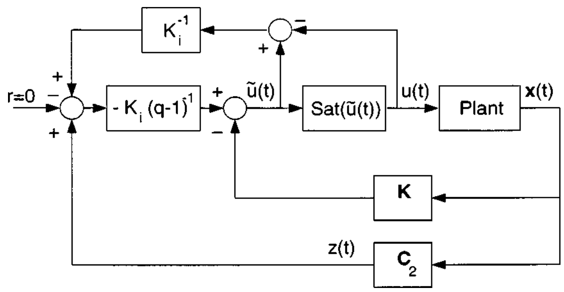

Fig. 5 Block diagram of AW

$$u(t)=\text{sat}(\tilde{u}(t))$$

$$x_i(t+1)=x_i(t)+z(t)+(\tilde{u}(t)-\text{sat}(\tilde{u}(t)))/K_i, \qquad (26)$$

with plant state $\mathbf{x}\in\Re^n$, integral state $x_i\in\Re$, and controlled output $z\in\Re$. The AW is shown in block diagram form in Fig. 5, where q denotes the unit delay operator. This structure is very typical for many AW methods; what is special about this method is the special choice of the AW gain that results in the integral state becoming a function of the state $\mathbf{x}$ during saturation.

Assuming that $-\mathbf{K}\mathbf{x}(t)-K_i x_i(t)>u_{\max}$ and applying Eq. (25) to reset the integral state results in

$$x_i(t+1)=\mathbf{C_2}\mathbf{x}(t)+(-u_{\max}-\mathbf{K}\mathbf{x}(\mathbf{t}))/K_i. \qquad (27)$$

If the control strategy in Eq. (26) is used for the same case the integral state at time $t+1$ is given by

$$x_i(t+1)=x_i(t)+\mathbf{C_2}\mathbf{x}(t)+(-\mathbf{K}\mathbf{x}(t)-K_i x_i(t)-u_{\max})/K_i. \qquad (28)$$

In the above equation the state x_i can be eliminated and the equation reduces to Eq. (27). In other words, the two control strategies applied the same control to the plant at time t, and they result in the same states $\mathbf{x}(t+1)$ and $x_i(t+1)$. The same result holds for $\mathbf{K}\mathbf{x}(t)+K_i x_i(t)<-u_{\max}$, and in both cases the control are not affected by the AW if $|\mathbf{K}\mathbf{x}(t)+K_i x_i(t)|<u_{\max}$, i.e., the two methods results in exactly the same control.

Later, a slightly modified version of this AW method will be used, where the modification makes it possible to find a GDA, Ω_s, similar to what was done for the full state feedback controller without integral control in Sec. 4.

The closed-loop system with the controller given by Eq. (26) can be written in the form of a single linear plant with a single nonlinearity sat$(\tilde{u})$ in the feedback path according to

$$\begin{bmatrix}\mathbf{x}(t+1)\\ x_i(t+1)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & 0\\ \mathbf{C_2}-\mathbf{K}/K_i & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ x_i(t)\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ -1/K_i\end{bmatrix}\text{sat}(\tilde{u}(t)) \qquad (29)$$

$$\tilde{u}(t)=-[\mathbf{K}K_i]\begin{bmatrix}\mathbf{x}(t)\\ x_i(t)\end{bmatrix}. \qquad (30)$$

The system is now in a form where Theorem 4.1 could be applied to find a GDA. However, doing that will, in general, result in very conservative results. This is because during saturation $x_i(t+1)$ is a function only of the state $\mathbf{x}(\mathbf{t})$, and therefore behaves very differently compared to when the control does not saturate and $x_i(t+1)$ depends on both $x_i(t)$ and $\mathbf{x}(t)$. This fact means that it is hard to find a Lyapunov function that is valid both when the controller saturates, and when it doesn't. Using Theorem 4.1 often results in $\alpha_{\min}\approx 0.99$, which means that hardly any saturation can be tolerated.

Instead of directly applying Theorem 4.1 a combination of describing function and Lyapunov function analysis will be used to obtain a GDA. First, a Lyapunov function will be used to obtain a set Ω_s inside which it can be shown that the system is stable, and then the describing function technique will be used to argue that the system is not only stable, but asymptotically stable inside the set, Ω_s.

As mentioned earlier during saturation the integral state depends only on the state $\mathbf{x}$. The following Lemma expresses the form of this result.

Lemma 6.1. Suppose the control strategy in Eq. (26) is applied and $\mathbf{KB}>0$. If $u(t)=u(t+1)=u_{\max}$ or $u(t)=u(t+1)=-u_{\max}$, then

$$u(t)=\text{sat}(-\mathbf{K_e}\mathbf{x}(t)) \qquad (31)$$

with

$$\mathbf{K_e}=-(\mathbf{K}-\mathbf{K}\mathbf{A}-K_i\mathbf{C_2})(\mathbf{KB})^{-1} \qquad (32)$$

Proof: If $u(t)=u_{\max}$ holds it follows that

$$\tilde{u}(t)=-\mathbf{K}\mathbf{x}(t)-K_i x_i(t)\geqslant u_{\max} \qquad (33)$$

$$\tilde{u}(t+1)=-\mathbf{K}(\mathbf{A}\mathbf{x}(t)+\mathbf{B}u_{\max})-K_i(\mathbf{C_2}\mathbf{x}(t)+(-u_{\max}-\mathbf{K}\mathbf{x}(t))/K_i) \qquad (34)$$

$$=(\mathbf{K}-\mathbf{K}\mathbf{A}-K_i\mathbf{C_2})\mathbf{x}(t)+(1+\mathbf{KB})u_{\max}. \qquad (35)$$

For $\tilde{u}(t+1)\geqslant u_{\max}$ to hold

$$\mathbf{K}-\mathbf{K}\mathbf{A}-K_i\mathbf{C_2})\mathbf{x}(t)+\mathbf{KB}u_{\max}\geqslant 0 \qquad (36)$$

has to be satisfied. If $\mathbf{KB}>0$ then the above condition is equivalent to

$$\mathbf{KB}^{-1}(\mathbf{K}-\mathbf{K}\mathbf{A}-K_i\mathbf{C_2})\mathbf{x}(t)\geqslant u_{\max}. \qquad (37)$$

The above condition can be restated as

$$-\mathbf{K_e}\mathbf{x}(t)\geqslant u_{\max}, \qquad (38)$$

where $\mathbf{K_e}$ is given by Eq. (32). Since $\mathbf{K_e}\mathbf{x}(t)>u_{\max}$ is a condition for saturation to occur at time t+1, the control u(t) has to be given by Eq. (31). Exactly the same result holds for $-\mathbf{K_e}\mathbf{x}(t)\leqslant -u_{\max}$. □

The condition $\mathbf{KB}>\mathbf{0}$ is restrictive, however, it is satisfied for most designs. In particular, as will be shown, it is satisfied when LQR design is used to obtain $\mathbf{K}$ and K_i for a continuous time-controller. The fact that $\mathbf{KB}>0$ holds for a continuous-time controller implies that it is likely to hold for a discrete time controller obtained using sufficiently small sampling time.

Let

$$\mathbf{A}'=\begin{bmatrix}\mathbf{A} & 0\\ \mathbf{C_2} & 0\end{bmatrix} \qquad (39)$$

and $B'=[\mathbf{B}^T 0]^T$ be the augmented plant matrices obtained by including the integral state in the plant dynamics of a continuous time plant. The augmented matrices can be used to solve the continuous time Ricatti equation and obtain $\mathbf{K}'=[\mathbf{K}K_i]=R^{-1}\mathbf{B}'\mathbf{P}$, where $\mathbf{P}$ is the positive definite solution to the continuous time Ricatti equation.

Partitioning the matrix $\mathbf{P}$ according to

$$\mathbf{P}=\begin{bmatrix}\mathbf{P_1} & \mathbf{P_2}\\ \mathbf{P_3} & \mathbf{P_4}\end{bmatrix}, \qquad (40)$$

$\mathbf{KB}$ can be obtained as

$$\mathbf{KB}=R^{-1}\mathbf{B^T}\mathbf{P_1}\mathbf{B}>0 \qquad (41)$$

since $\mathbf{P_1}>0$ follows from $\mathbf{P}>0$.

Lemma 6.1 states that during saturation, the control applied to the plant is equivalent to the control that would be applied by a full state feedback controller with feedback gain $\mathbf{K_e}$. Notice that $\mathbf{K_e}$ is the result of a particular choice of $\mathbf{K}$ and K_i, it is not a feedback gain being chosen by some design method. The Lemma means that, provided that the matrix $\mathbf{A}-\mathbf{B}\mathbf{K_e}$ has all its eigenvalues inside the unit disk, it is possible to use Theorem 4.1 to obtain a positive definite Lyapunov function that will be strictly decreasing whenever the plant state is inside Ω_s and the controller saturates.

In other words, if the initial state are inside Ω_s and the controller saturates it is guaranteed that the controller will be brought out of saturation and stay inside Ω_s while it is saturating. The condition that $\mathbf{A}-\mathbf{B}\mathbf{K_e}$ has all its eigenvalues inside the unit disk is crucial in obtaining the set Ω_s, this condition should be checked before using this AW method. However, this is not sufficient to guarantee stability since it does not guarantee that the states stay inside the set Ω_s once the controller no longer saturates. To overcome this the AW method in Eq. (26) is slightly modified according to

$$\begin{aligned}
&\tilde{u}(t)=-\mathbf{K}\mathbf{x}(t)-K_i x_i(t)\\
&\text{if}\quad V(\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathrm{sat}(\tilde{u}(t)))\leq\gamma\ \text{ then}\\
&u(t)=\mathrm{sat}(\tilde{u}(t))\\
&x_i(t+1)=x_i(t)+z(t)+(\tilde{u}(t)-\mathrm{sat}(\tilde{u}(t)))/K_i \qquad (42)\\
&\text{if}\quad V(\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathrm{sat}(\tilde{u}(t)))>\gamma\ \text{ then}\\
&u(t)=\mathrm{sat}(-\mathbf{K}_e\mathbf{x}(t))\\
&x_i(t+1)=x_i(t)+z(t)+(\tilde{u}(t)+\mathrm{sat}(\mathbf{K}_e\mathbf{x}(t)))/K_i
\end{aligned}$$

where $\mathbf{P}$ and γ are the parameters that defines the set $\Omega_s(\mathbf{K}_e, P)$ in Theorem 4.1. $\mathbf{P}$ and γ are obtained in exactly the same manner as if the control were given by a full state feedback controller with feedback gain $\mathbf{K_e}$. The above modification to the AW scheme ensures that the states will stay inside the set Ω_s, and that the integral state will always be consistent with the control applied to the plant. If a set Ω_s is obtained using Theorem 4.1 using state feedback gain $\mathbf{K}_e$ from Eq. (32) the following theorem holds.

Theorem 6.1. A plant given by Eq. (1) being controlled by the control strategy in Eq. (42) with initial conditions inside the set Ω_s will stay inside the set Ω_s for all times.

Proof: Suppose, the theorem is not true. Then there exists $\mathbf{x}(t)\in\Omega_s$ and $\mathbf{x}(t+1)\notin\Omega_s$. Thus,

$$\begin{aligned}
V(\mathbf{x}(t+1))&=V(\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t))>\gamma\Rightarrow u(t)\\
&=-\mathrm{sat}(\mathbf{K}_e\mathbf{x})\Rightarrow V(\mathbf{x}(t+1))<\gamma\\
&\Rightarrow\mathbf{x}(t+1)\in\Omega_s
\end{aligned}$$

which is a contradiction. □

Figure 6 schematically illustrates Theorem 6.1 for a second order example. Notice that the modification to the AW scheme never affects the control when the controller saturates since Lemma 6.1 together with the fact that Ω_s is obtained as for a full state feedback controller with feedback gain $\mathbf{K}_e$ ensures that $V(\mathbf{A}\mathbf{x}(\mathbf{t})+\mathbf{B}\mathbf{u}(t))<\gamma$ holds.

The set Ω_s determines an invariant set for the plant state, $\mathbf{x}(t)$, regardless of $x_i(0)$. This follows from the fact that even if $x_i(0)\in\Re$, and hence is unbounded, $x_i(t), t>0$ belongs to a bounded set since once the AW is applied the integral state $x_i(t)$ is bounded by

$$-u_{\max}-\mathbf{K}\mathbf{x}(t)\leq x_i(t)\leq u_{\max}-\mathbf{K}\mathbf{x}(t)\forall t<0 \qquad (43)$$

and $\mathbf{x}(t)$ belongs to the bounded set Ω_s. The fact that $x_i(t)$ for $t>0$ is bounded means that an invariant set for $\mathbf{x}$ and x_i can be obtained from Ω_s.

Using this approach we have not been able to show rigorously that the system is asymptotically stable, i.e., that $\mathbf{x}\in\Omega_s$ implies $\mathbf{x}(t)\rightarrow 0$ and $x_i(t)\rightarrow 0$. However, this result can be supported strongly by describing function analysis. As long as $V(\mathbf{A}\mathbf{x}(\mathbf{t})+\mathbf{B}u(t))<\gamma$ holds, the proposed AW is the same as the original AW in Eq. (26). By using the describing function technique the possibility of a nonconvergent oscillation will be made highly unlikely. A rigorous result cannot be obtained because the describing function approach is based on sinusoidal approximations of the oscillating motion.

The linear system used for the describing function analysis that relates $u(t)$ to $\tilde{u}(t)$ is given by

$$\begin{bmatrix}\mathbf{x}(t+1)\\ x_i(t+1)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & 0\\ \mathbf{C_2}-\mathbf{K}/K_i & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ x_i(t)\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ -1/K_i\end{bmatrix}u(t) \qquad (44)$$

$$\tilde{u}(t)=-[\mathbf{K}K_i]\begin{bmatrix}\mathbf{x}(t)\\ x_i(t)\end{bmatrix}. \qquad (45)$$

The equivalent gain η obtained using the describing function for the saturation function is given in [24]

$$\eta(\delta)=\left\{\begin{matrix}(2/\pi)(\sin^{-1}(\delta)+\delta\sqrt{1-\delta^2}) & \text{if }\ \delta\leq 1\\ 1 & \text{if }\ \delta>1\end{matrix}\right\}, \qquad (46)$$

where δ is the ratio between the saturation limit and the largest amplitude of the sinusoidal signal, $\tilde{u}$. For details on describing function analysis see [24]. Assuming that the linear design is stable when the control does not saturate the Nyquist path will encircle the -1 point as many times as there are open loop poles in $\tilde{u}=G(s)u$, where $G(s)$ is the continuous time approximation of the system in Eq. (44).

Figure 7 shows a sample Nyquist plot for a system $G(s)$. The value of η that results in $-1/\eta(\delta)$ being equal to largest value where the Nyquist path intersects the real axis in the interval $(-\infty, -1)$ will be referred to as $\eta_{\lim}$. If $\eta>\eta_{\lim}$ the number of encircle-

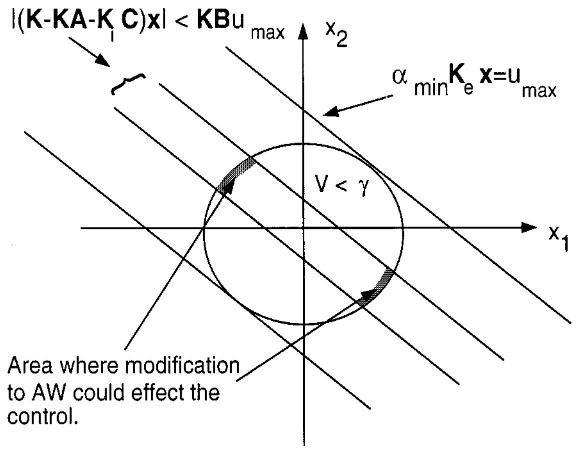

Fig. 6 Two-dimensional example with AW

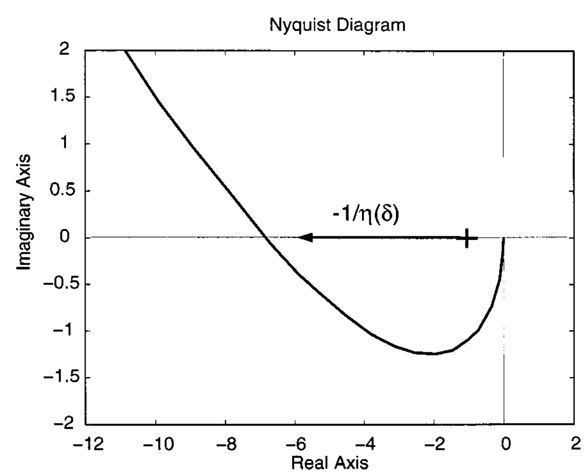

Fig. 7 Nyquist plot of $\tilde{u}=G(s)u$

ments of the -1 point will not have changed compared to the stable design. Consequently, a sinusoidal signal with $\eta > \eta_{\lim}$ cannot exist and the system will be asymptotically stable. On the other hand, if $\eta < \eta_{\lim}$ the number of encirclements will have changed and the system will not be asymptotically stable. If it will grow exponentially or exhibit a limit cycle behavior will depend on if there are additional intersections between the Nyquist path and the negative real axis that will change the number of encirclements of the point $-1/\eta(\delta)$ when the amplitude of the sinusoidal signal $\tilde{u}$ increases, and thereby changes the value of $-1/\eta(\delta)$. For open-loop unstable systems with a stabilizing feedback there will always exist an intersection between the Nyquist path and the interval $(-\infty, -1)$, therefore describing function analysis alone can never be sufficient to conclude closed-loop stability for an open-loop unstable system in the presence of saturation. However, given the invariant set Ω_s the amplitude of any sinusoidal $\tilde{u}$ that can exist can be bounded.

The maximum amount of saturation (ratio between the saturation limit, $u_{\max}$ and the amplitude of $\tilde{u}$) for any sinusoidal signal $\tilde{u}$ that can exist when $\mathbf{x} \in \Omega_s \forall t$ can be expressed as

$$\delta_{\min} = \min_{\mathbf{x} \in \Omega_s} \frac{u_{\max}}{\tilde{u}} = \min_{\mathbf{x} \in \Omega_s \& u \leq u_{\max}} \frac{u_{\max}}{(\mathbf{K} - \mathbf{KA} - K_i \mathbf{C_2})\mathbf{x} + u(1 + \mathbf{KB})} \tag{47}$$

since $\tilde{u}(t+1)$ can be obtained from Eq. (35).

The value of δ can easily be obtained from Eq. (47), since $\mathbf{x}$ belongs to the compact set Ω_s and u belongs to compact set $[-u_{\max}, u_{\max}]$. If $\eta(\delta_{\min}) > \eta_{\lim}$ the describing function indicates that any periodic sinusoidal signal $\tilde{u}$ will result in an asymptotically stable closed-loop response if $\mathbf{x}(0) \in \Omega_s$.

This means that the same procedure as for the full state feedback control can be used to obtain the GDA, Ω_s, with the additional condition that

$$\eta(\delta_{\min}) > \eta_{\lim} \tag{48}$$

has to be satisfied. It has been found that the inequality in Eq. (48) is, in general, satisfied with a ''large'' margin (so far this has always been the case). If the above inequality does not hold, the set Ω_s has to be reduced by reducing γ in Eq. (10). The obtained GDA for the case with integral control can now be used in exactly the same way as for the case of full state feedback to find the solution to Eq. (19).

7 Example 2

In this example a controller is designed for the same plant and the same initial conditions of the plant as in Section 5 and with, but with integral control, AW, and $x_i(0)=0$ (the value of $x_i(0)$ does not matter in the design since the GDA is expressed in terms of the plant state only). A sample time of 0.01 s was used with a critically damped pole structure ($\mathbf{Pb}=[-1,-1,-1,-1]$). During the design it turned out that the constraints imposed by Eq. (48) never limited the choice of Ω_s. For the particular IC set used in the simulation the modification to the AW was never activated and the response is, therefore, identical to the response that would have been obtained with the original AW scheme in Eq. (26). The results are shown in Table 2 and Fig. 8. To evaluate the performance the 1 percent settling time are compared for the different designs as well as the velocity error constant, K_v, with respect to a disturbance input. $\mathbf{P}$ and γ resulting from the stability based design are given by $\gamma=7.5$, and

$$\mathbf{P} = \begin{bmatrix} 1.0000 & 1.899 & 1.0506 \\ 1.899 & 4.9604 & 3.2859 \\ 1.0506 & 3.2859 & 3.3001 \end{bmatrix}. \tag{49}$$

The initial value for $V(\mathbf{x}_0)$ for the seven initial conditions are 1.0, 5.0, 3.3, 4.9, 3.20, 7.4, and 7.2, i.e., all of them satisfy the condition $V(\mathbf{x}_0) \leq \gamma$.

Table 2 Comparison of design methods, full state feedback plus integral control

	$\beta=\beta_{lin}$	$\beta=\beta_{opt}$	$\beta=\beta_{sat}$
β	0.93	4.0	3.2
$max(t_s)\ \forall \mathbf{x} \in IC$	11.5	4.5	5.5
K_v	0.74	239	99
$\mathbf{K}$	[1.71 4.16 3.20]	[13.4 89.8 241.2]	[10.4 57.9 125.2]
K_i	0.0074	2.39	0.99

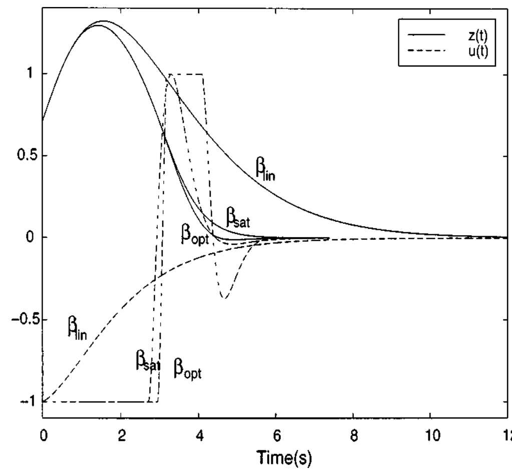

Fig. 8 Simulation results for $\beta=0.93$, 3.2, and 4.0

8 Experiment

The design method presented in this paper has also been applied to the position control of a 1 d.o.f robot arm with a DC motor. The equations of motion for the robot arm are given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -4.2039 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 9.681 \\ 0 \end{bmatrix} \mathrm{sat}(u), \tag{50}$$

$$\Theta = [0 \quad 0.63]\mathbf{x}$$

where u is the input in voltage to the DC motor and Θ is the output in rad measured using a variable potentiometer. The plant has open loop eigenvalues at -4.2039 and 0. The plant has dynamics which are not modeled by Eq. (50). They exist somewhere in the frequency range $120-140$ rad/s. Actuator saturation is introduced by the amplifier that can supply voltages between -8 and $+8$ V, i.e., the input voltage to the DC motor is limited by ± 8 V.

The controller is implemented using a sample time of 0.001 s. Since not all the states are measured an observer is used to obtain estimates of the states. See [12] for a discussion about the effect of using an observer rather than full state feedback, as assumed when obtaining Ω_s. Instead of using an observer to obtain $\dot{\Theta}$, a differentiator could also have been used to obtain $\dot{\Theta}$. Notice that it is important to include the saturation when the observer is implemented to get correct estimates of the states. To do the scaling of the design, the method based on LQR design described in [16] are used, with the scaling parameter β being given by the inverse of the cost on the control in the quadratic cost function.

The set of initial conditions is defined by

$$\frac{\theta^2}{1.9^2} + \frac{\dot{\theta}^2}{11.8^2} \leq 1, \tag{51}$$

or equivalently

Table 3 Comparison of design methods for experiment

	$\beta=\beta_{lin}$	$\beta=\beta_{sat}$	$\beta=\beta_{opt}$
β	2.67	1201	3906
$max(t_s)\ \forall \mathbf{x}\in IC$	1.73	0.56	0.51
K	[0.37 1.63]	[2.70 34.18]	[3.75 61.3]

$$\frac{x_1^2}{3.0^2}+\frac{x_2^2}{18.7^2}\leqslant 1. \qquad (52)$$

Table 3 shows the settling times for the optimum, the nonsaturating, and the stability based designs. **P** and γ resulting from the stability based design are given by $\gamma=840$,

$$\mathbf{P}=\begin{bmatrix}1.0000 & 4.99\\ 4.99 & 74.34\end{bmatrix}. \qquad (53)$$

Figure 9 shows the set Ω_s resulting from the design and the set of initial conditions IC.

Figure 10 shows the simulation and the experimental result of using the stability based design method.

Finally, the stability based method is applied to the case of full state feedback plus integral control for the same plant as in the previous example. In this case it turns out that the limiting factor is not the saturation, but the unmodeled dynamics. This means that the design parameter β has to be limited to avoid exiting the

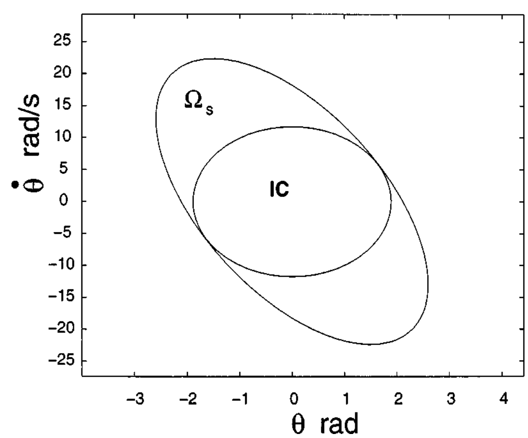

Fig. 9 Guaranteed domain of attraction Ω_s

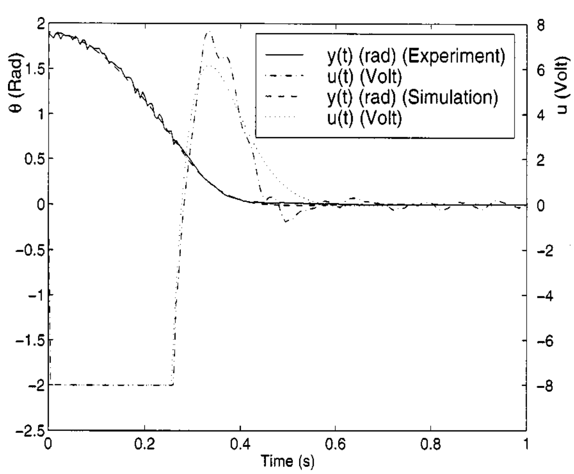

Fig. 10 Experimental and simulation result for full state feedback control

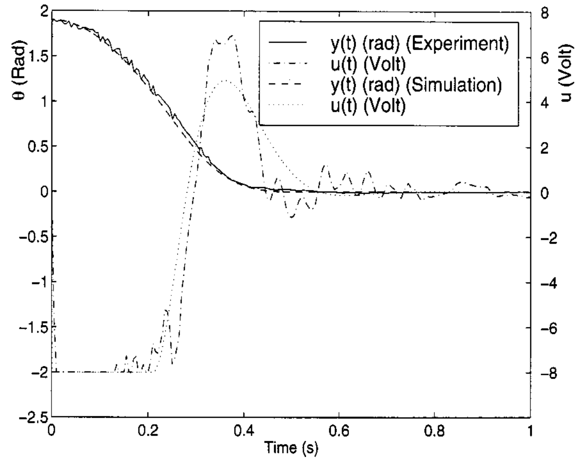

Fig. 11 Experimental and simulation result for full state feedback plus integral control

unmodeled dynamics. However, improved performance compared to a linear nonsaturating design is obtained. The simulation and experimental results are shown in Fig. 11. The feedback gains used are given by $\mathbf{K}=[3.38 \quad 68.8]$ and $K_i=0.499$. Preliminary results for how to incorporate the unmodeled dynamics directly into design are available [12,16].

9 Conclusions

In this paper, a new design method that directly incorporates the actuator constraints into the design procedure has been presented. Even though the controller saturates, the stability of the closed-loop system is guaranteed by finding a GDA such that all initial conditions are contained in the GDA. The design procedure can be applied to both full state feedback control, and full state feedback plus integral control for single input systems. Simulation and experimental results show that significant improvements in performance relative to a nonsaturating linear design can be achieved.

In this paper, it was assumed that the controller has access to all the plant states; in general, that will not be the case. When not all the states are measured, the proposed design method can still be applied by designing a state observer; this was successfully done for the experiment included in this paper. The effects on the proposed design method when using an observer to obtain the states are discussed in [16]. Preliminary results on robustness to unmodeled dynamics, parameter uncertainty, and disturbances are also available in [16] and [12].

A Matlab based design software package has been developed that implements the design approach presented in this paper.

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