Hypergeometric decomposition of symmetric K3 pencils

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Collaborators

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An anniversary

Thanks to BIRS for hosting some of us in 2013 (!)

Where can you find this?

DKSSVW

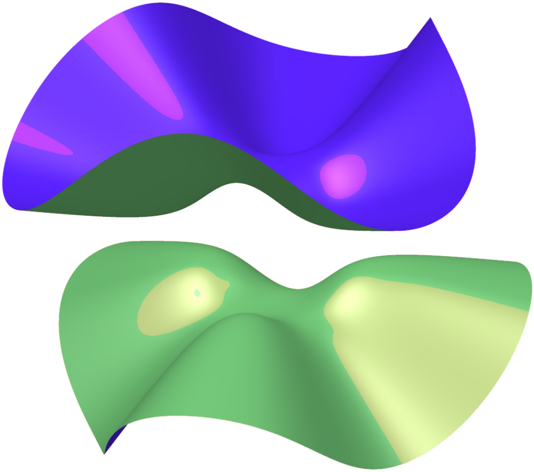
- Zeta functions of alternate mirror Calabi-Yau families (*Israel Journal of Mathematics* 2018)
- Hypergeometric decomposition of symmetric K3 quartic pencils (*Research in the Mathematical Sciences* 2020)

W

- Arithmetic mirror symmetry and hypergeometric structures. (*Crossing Walls in Enumerative Geometry*, AMS Contemporary Mathematics, 2021).

Grand goal

Use physics intuition to prove arithmetic theorems.



Five interesting quartics in $\mathbb{P}^3$

Family	Equation
F_4 (Fermat/Dwork)	$x_0^4 + x_1^4 + x_2^4 + x_3^4$
F_2L_2	$x_0^4 + x_1^4 + x_2^3x_3 + x_3^3x_2$
F_1L_3 (Klein-Mukai)	$x_0^4 + x_1^3x_2 + x_2^3x_3 + x_3^3x_1$
L_2L_2	$x_0^3x_1 + x_1^3x_0 + x_2^3x_3 + x_3^3x_2$
L_4	$x_0^3x_1 + x_1^3x_2 + x_2^3x_3 + x_3^3x_0$

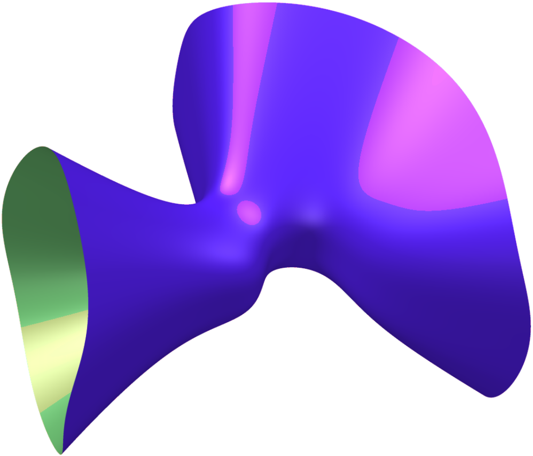
Intriguing group actions

We can extend each of our quartics to a pencil admitting a symplectic group action induced by multiplying coordinates by roots of unity.

Family	Equation	Group
F_4	$x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4\psi x_0 x_1 x_2 x_3$	$(\mathbb{Z}/4\mathbb{Z})^2$
F_2L_2	$x_0^4 + x_1^4 + x_2^3 x_3 + x_3^3 x_2 - 4\psi x_0 x_1 x_2 x_3$	$\mathbb{Z}/8\mathbb{Z}$
F_1L_3	$x_0^4 + x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1 - 4\psi x_0 x_1 x_2 x_3$	$\mathbb{Z}/7\mathbb{Z}$
L_2L_2	$x_0^3 x_1 + x_1^3 x_0 + x_2^3 x_3 + x_3^3 x_2 - 4\psi x_0 x_1 x_2 x_3$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
L_4	$x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_0 - 4\psi x_0 x_1 x_2 x_3$	$\mathbb{Z}/5\mathbb{Z}$

Concrete goal

Understand the relationship between the group actions and point counts over finite fields.



The zeta function

- Let $q = p^s$ and consider the finite field $\mathbb{F}_q$
- We can organize information about point counts on a variety X over extensions of $\mathbb{F}_q$ in a generating function.

Definition

The zeta function of X is

$$Z(X/\mathbb{F}_q, T) := \exp\left(\sum_{s=1}^{\infty} \#X(\mathbb{F}_{q^s})\frac{T^s}{s}\right) \in \mathbb{Q}[[T]].$$

Dwork and the Weil Conjectures

- $Z(X/\mathbb{F}_q, T)$ is rational
- We can factor $Z(X/\mathbb{F}_q, T)$ using polynomials with integer coefficients:

$$Z(X/\mathbb{F}_q, T) = \frac{\prod_{j=1}^{n} P_{2j-1}(T)}{\prod_{j=0}^{n} P_{2j}(T)}$$

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- $\dim X = n$
- $P_0(t) = 1 - T$ and $P_{2n}(T) = 1 - p^n T$
- For $1 \leq j \leq 2n-1$, $\deg P_j(T) = b_j$, where $b_j = \dim H^j_{dR}(X)$.

Projective hypersurfaces

For a smooth projective hypersurface X in $\mathbb{P}^n$, we have

$$Z(X, T) = \frac{P_X(T)^{(-1)^n}}{(1-T)(1-qT)\cdots(1-q^{n-1}T)},$$

with $P_X(T) \in \mathbb{Q}[T]$.

A pair of K3 surface examples

Let $p = 281$, $\psi = 5$. Using code by Edgar Costa, we compute P_X:

Fermat pencil

$$(1 - 281\,T)^{12}(1 + 281\,T)^{6}(1 - 281\,T)(1 + 418\,T + 281^2\,T^2)$$

Klein-Mukai pencil

$$(1 + 281\,T + 281^2\,T^2 + 281^3\,T^3 + 281^4\,T^4 + 281^5\,T^5 + 281^6\,T^6)^3(1 - 281\,T)(1 + 418\,T + 281^2\,T^2)$$

A common factor
DKSSVW18

- P_X has a common factor $R_\psi(T)$ of degree 3 for $\diamond \in \{\mathrm{F}_4, \mathrm{F}_2\mathrm{L}_2, \mathrm{F}_1\mathrm{L}_3, \mathrm{L}_2\mathrm{L}_2, \mathrm{L}_4\}$ and each appropriate ψ.

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- Similar collections of Calabi-Yau pencils with common factors in their zeta functions can be identified in $\mathbb{P}^n$ for any n.

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- Similar collections of Calabi-Yau pencils with common factors in their zeta functions can be identified in $\mathbb{P}^n$ for any n.
- The common factor corresponds to a common Picard-Fuchs equation satisfied by the holomorphic form.

Berglund-Hübsch-Krawitz duality

The techniques of [DKSSVW18] use Berglund-Hübsch-Krawitz (BHK) mirror symmetry and invertible polynomials.

Figure: Per Berglund

Figure: Tristan Hübsch

Figure: Marc Krawitz

An approach through quotients

Zeta functions of monomial deformations of Delsarte hypersurfaces

- Common zeta function factor (possibly larger) using a covering map
- Geometric information about the $\diamond$ family.

Figure: Remke Kloosterman

What about the rest of the zeta function?

In [DKSSVW20], we give a complete description of the zeta function for each $\diamond \in \{\mathsf{F}_4, \mathsf{F}_2\mathsf{L}_2, \mathsf{F}_1\mathsf{L}_3, \mathsf{L}_2\mathsf{L}_2, \mathsf{L}_4\}$ using:

- Hypergeometric Picard-Fuchs equations
- Generalizations of hypergeometric functions to finite fields
- Extensive computational validation.

Packaging zeta functions

Let S be a finite (specified) set of bad primes and for fixed $\diamond, \psi, p$, let $P_{\diamond,\psi,p}$ be the degree 21 factor of the zeta function. Define an incomplete L-series by

$$L_S(X_{\diamond,\psi}, s) := \prod_{p \notin S} P_{\diamond,\psi,p}(p^{-s})^{-1}.$$

This series converges for $s \in \mathbb{C}$ in a right half-plane.
We factor $L_S(X_{\diamond,\psi}, s)$ in terms of hypergeometric L-series.

Mirror quartics

Let Y_ψ be the mirror family to quartics in $\mathbb{P}^3$ (constructed using Greene-Plesser and the Fermat pencil). Then de la Ossa showed (see S. Kadir's thesis):

$$Z(Y_\psi/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-pT)^{19}(1-p^2T)R_\psi(T)}.$$

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$$Z(Y_\psi/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-pT)^{19}(1-p^2T)R_\psi(T)}.$$

The common factor $R_\psi(T)$ is invariant under mirror symmetry.

Changing fields

For $\mathbb{F}_q$ containing sufficiently many roots of unity, we show in DKSSVW20 that

$$Z(X_{\diamond,\psi}/\mathbb{F}_q, T) = \frac{1}{(1-T)(1-qT)^{19}(1-q^2T)R_\psi(T)}.$$

We may say our zeta functions are potentially equal.

Comparing to the mirror
DKSSVW20

Let Y_ψ be the family of mirror quartics. Then $Z(X_{\diamond,\psi})$ and $Z(Y_\psi)$ are potentially equal for any $\diamond$.

The transient factors

Recall that for $p = 281$, $\psi = 5$:

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Computing zeta functions

First attempts

- In principle, given an equation for X, one can compute $Z(X/\mathbb{F}_q, T)$ using $\#X(F_q), \#X(F_{q^2}), \ldots, \#X(F_{q^s})$, where s is approximately half the sum of the Betti numbers of X.

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- In principle, given an equation for X, one can compute $Z(X/\mathbb{F}_q, T)$ using $\#X(F_q), \#X(F_{q^2}), \ldots, \#X(F_{q^s})$, where s is approximately half the sum of the Betti numbers of X.
- In practice: combinatorial explosion!

Computing zeta functions

Specific examples

Figure: Edgar Costa

The state of the art

- K3 surfaces in $\mathbb{P}^3$ or realized as double covers of $\mathbb{P}^2$
- A handful of special threefolds or fourfolds

Fascinating experiments

Computational experiments show intriguing behavior in zeta function factors associated to Calabi-Yau holomorphic forms.

Figure: Philip Candelas

Figure: Xenia de la Ossa

Figure: Duco van Straten

The zeta function is given by the characteristic polynomial of the Frobenius action on a suitably defined p-adic cohomology space.

Computing zeta functions

The art

The zeta function is given by the characteristic polynomial of the Frobenius action on a suitably defined p-adic cohomology space.

- Costa's code: Monsky-Washnitzer cohomology
- DKSSVW18: Dwork cohomology
- Candelas–de la Ossa–van Straten '21: Dwork cohomology and (partially conjectural) exploitation of holomorphic Picard-Fuchs.

From p-adic cohomology to the zeta function

Key observation

A subspace of p-adic cohomology stable under the action of Frobenius yields a factor of the zeta function.

An explicit counting strategy

- Compute (hypergeometric) Picard-Fuchs equations for all elements of primitive cohomology.
- Decompose point counts for each family using finite field hypergeometric functions with related parameters.

Hypergeometric functions

Let $A, B \in \mathbb{N}$. Recall that a hypergeometric function is a function on $\mathbb{C}$ of the form:

$$
\begin{aligned}
{}_AF_B(\alpha;\beta|z) &= {}_AF_B(\alpha_1,\ldots,\alpha_A;\beta_1,\ldots,\beta_B|z) \\
&= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_A)_k}{(\beta_1)_k\cdots(\beta_B)_k k!} z^k,
\end{aligned}
$$

where $\alpha \in \mathbb{Q}^A$ are *numerator parameters*, $\beta \in \mathbb{Q}^B$ are *denominator parameters*, and the Pochhammer notation is defined by:

$$(x)_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}.$$

Hypergeometric differential operators

Consider the differential operator

$$\theta := z\frac{\mathrm{d}}{\mathrm{d}z}$$

and define the hypergeometric differential operator

$$D(\boldsymbol{\alpha};\boldsymbol{\beta}\,|\,z) := (\theta+\beta_1-1)\cdots(\theta+\beta_m-1) - z(\theta+\alpha_1)\cdots(\theta+\alpha_n).$$

Group actions on primitive cohomology

For each of our pencils, we can decompose the primitive cohomology using the characters of the symplectic group action on monomial representatives of cohomology classes.

Picard–Fuchs equations

Fermat pencil

Proposition

The primitive middle-dimensional cohomology group $H^2_{\mathrm{prim}}(X_{\mathsf{F}_4,\psi}, \mathbb{C})$ has 21 periods whose Picard–Fuchs equations are hypergeometric differential equations as follows:

Periods	Annihilated by
3 periods	$D(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1, 1 \mid \psi^{-4})$
6 periods	$D(\frac{1}{4}, \frac{3}{4}; 1, \frac{1}{2} \mid \psi^{-4})$
12 periods	$D(\frac{1}{2}; 1 \mid \psi^{-4})$

Naive expectations

Fermat pencil

Periods	Annihilated by	Corresponding factors
3 periods	$D(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1, 1 \mid \psi^{-4})$	1 degree 3 factor
6 periods	$D(\frac{1}{4}, \frac{3}{4}; 1, \frac{1}{2} \mid \psi^{-4})$	3 degree 2 factors
12 periods	$D(\frac{1}{2}; 1 \mid \psi^{-4})$	12 linear factors

Picard–Fuchs equations

Klein-Mukai pencil

Proposition

The group $H^2_{\text{prim}}(X_{\mathsf{F}_1\mathsf{L}_3,\psi})$ has 21 periods whose Picard–Fuchs equations are hypergeometric differential equations as follows:

Periods	Annihilated by
3 periods	$D(\frac{1}{4},\frac{1}{2},\frac{3}{4};1,1,1 \mid \psi^{-4})$
3 periods	$D(\frac{1}{14},\frac{9}{14},\frac{11}{14};\frac{1}{4},\frac{3}{4},1 \mid \psi^{4})$
3 periods	$D(\frac{-3}{14},\frac{1}{14},\frac{9}{14};0,\frac{1}{4},\frac{3}{4} \mid \psi^{4})$
3 periods	$D(\frac{-5}{14},\frac{-3}{14},\frac{1}{14};\frac{-1}{4},0,\frac{1}{4} \mid \psi^{4})$
3 periods	$D(\frac{3}{14},\frac{5}{14},\frac{13}{14};\frac{1}{4},\frac{3}{4},1 \mid \psi^{4})$
3 periods	$D(\frac{-1}{14},\frac{3}{14},\frac{5}{14};0,\frac{1}{4},\frac{3}{4} \mid \psi^{4})$
3 periods	$D(\frac{-11}{14},\frac{-1}{14},\frac{5}{14};\frac{-1}{4},0,\frac{1}{4} \mid \psi^{4})$

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Klein-Mukai pencil

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3 periods	$D(\frac{-1}{14}, \frac{3}{14}, \frac{5}{14}; 0, \frac{1}{4}, \frac{3}{4} \mid \psi^{4})$	1 degree 3 factor
3 periods	$D(\frac{-11}{14}, \frac{-1}{14}, \frac{5}{14}; \frac{-1}{4}, 0, \frac{1}{4} \mid \psi^{4})$	1 degree 3 factor

Naive expectations can fail

Recall that for $p = 281$, $\psi = 5$:

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$(1 + 281T + 281^2T^2 + 281^3T^3 + 281^4T^4 + 281^5T^5 + 281^6T^6)^3(1 - 281T)(1 + 418T + 281^2T^2)$

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We do not have 7 cubic factors!

The Gamma function and Gauss sums

One may define the rising factorials used in hypergeometric functions as ratios of Gamma functions.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

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The Gamma function and Gauss sums

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Over finite fields, a Gauss sum plays the role of a gamma function.

- Let $\omega\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a generator of the character group on $\mathbb{F}_q^\times$
- Let $\Theta\colon \mathbb{F}_q \to \mathbb{C}^\times$ be a nontrivial (additive) character.

For $m \in \mathbb{Z}$, define the Gauss sum

$$g(m) := \sum_{x \in \mathbb{F}_q^\times} \omega(x)^m \Theta(x). \qquad (1)$$

Finite field hypergeometric sums

The classic definition

Following Greene, Katz, McCarthy, Beukers–Cohen–Mellit. . .

Assumption

Set $q^\times = q - 1$ and suppose for all $i = 1, \ldots, d$

$$q^\times \alpha_i, q^\times \beta_i \in \mathbb{Z}.$$

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Definition

For $t \in \mathbb{F}_q^\times$, we define the finite field hypergeometric sum by

$$H_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t) := -\frac{1}{q^\times} \sum_{m=0}^{q-2} \omega((-1)^d t)^m G(m + \boldsymbol{\alpha} q^\times, -m - \boldsymbol{\beta} q^\times)$$

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$$G(m + \boldsymbol{\alpha} q^\times, -m - \boldsymbol{\beta} q^\times) := \prod_{i=1}^{d} \frac{g(m + \alpha_i q^\times) g(-m - \beta_i q^\times)}{g(\alpha_i q^\times) g(-\beta_i q^\times)}.$$

A plausibility check

- Our common factor corresponds to $D(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1, 1 \mid \psi^{-4})$.

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- Our common factor corresponds to $D(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1, 1 \mid \psi^{-4})$.
- We have $q^\times \alpha_i, q^\times \beta_i \in \mathbb{Z}$ when $4 | q - 1$.

Hypergeometric innovators

Figure: Frits Beukers

Figure: Henri Cohen

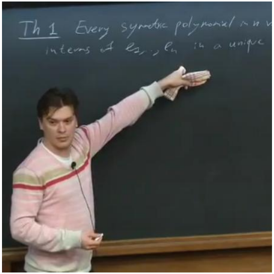

Figure: Anton Mellit

Defined over $\mathbb{Q}$

Definition (Beukers–Cohen–Mellit)

We say hypergeometric parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are defined over $\mathbb{Q}$ if the field generated by the coefficients of the polynomials

$$\prod_{j=1}^{d}(x - e^{2\pi i \alpha_j}) \quad \text{and} \quad \prod_{j=1}^{d}(x - e^{2\pi i \beta_j}).$$

is $\mathbb{Q}$.

The common factor parameters are defined over $\mathbb{Q}$

$$\boldsymbol{\alpha} : \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}$$

$$(x - e^{2\pi i/4})(x - e^{2\pi i/2})(x - e^{2\cdot 3\pi i/4}) = (x - i)(x + 1)(x + i)$$

$$\boldsymbol{\beta} : 1, 1, 1$$

$$(x - e^{2\pi i})(x - e^{2\pi i})(x - e^{2\pi i}) = (x - 1)^3.$$

Finite field hypergeometric sums

The Beukers–Cohen–Mellit definition

- We say q is good for $\boldsymbol{\alpha}, \boldsymbol{\beta}$ if q is coprime to the least common denominator of $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$.

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- If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are defined over $\mathbb{Q}$, we have positive integers $p_1, \ldots, p_r$, $q_1, \ldots, q_s$ such that

$$\prod_{j=1}^{d} \frac{(x - e^{2\pi i \alpha_j})}{(x - e^{2\pi i \beta_j})} = \frac{\prod_{j=1}^{r} x^{p_j} - 1}{\prod_{j=1}^{s} x^{q_j} - 1}.$$

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$$\prod_{j=1}^{d} \frac{(x - e^{2\pi i \alpha_j})}{(x - e^{2\pi i \beta_j})} = \frac{\prod_{j=1}^{r} x^{p_j} - 1}{\prod_{j=1}^{s} x^{q_j} - 1}.$$

- Using the $p_1, \dots, p_r$, $q_1, \dots, q_s$, Beukers–Cohen–Mellit define a finite field hypergeometric sum $H_q(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid t)$ that works for all good q.

A comparison

- The classic and BCM definitions of finite field hypergeometric sums agree when they are both defined.
- When α, β are defined over $\mathbb{Q}$, the BCM definition works for all but finitely many primes.

A practical difficulty

- The Klein-Mukai pencil has 3 periods annihilated by $D(\frac{1}{14}, \frac{9}{14}, \frac{11}{14}; \frac{1}{4}, \frac{3}{4}, 1 \mid \psi^4)$.

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- But $(x - e^{2\pi i/14})(x - e^{18\pi i/14})(x - e^{22\pi i/14}) \notin \mathbb{Q}[x]$, so we don't have a BCM hypergeometric sum corresponding to these parameters.

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- But $(x - e^{2\pi i/14})(x - e^{18\pi i/14})(x - e^{22\pi i/14}) \notin \mathbb{Q}[x]$, so we don't have a BCM hypergeometric sum corresponding to these parameters.
- The classic definition requires $q \equiv 1 \pmod{28}$.

Divide and conquer

Definition

We say that q is splittable for $\boldsymbol{\alpha}, \boldsymbol{\beta}$ if there exist partitions

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 \sqcup \boldsymbol{\alpha}' \text{ and } \boldsymbol{\beta} = \boldsymbol{\beta}_0 \sqcup \boldsymbol{\beta}'$$

where $\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0$ are defined over $\mathbb{Q}$ and

$$q^\times \alpha_i', q^\times \beta_j' \in \mathbb{Z}$$

for all $\alpha_i' \in \boldsymbol{\alpha}'$ and all $\beta_j' \in \boldsymbol{\beta}'$.

A hybrid definition

- In [DKSSVW20], we define a finite field hypergeometric sum $H_q(\boldsymbol{\alpha}; \boldsymbol{\beta} \mid t) \in K_{\boldsymbol{\alpha},\boldsymbol{\beta}} \subseteq \mathbb{C}$ for any q that is good and splittable for $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

A hybrid definition

- In [DKSSVW20], we define a finite field hypergeometric sum $H_q(\boldsymbol{\alpha};\boldsymbol{\beta}\,|\,t) \in K_{\boldsymbol{\alpha},\boldsymbol{\beta}} \subseteq \mathbb{C}$ for any q that is good and splittable for $\boldsymbol{\alpha},\boldsymbol{\beta}$.
- We package these sums in L-series for any prime power q such that $\boldsymbol{\alpha},\boldsymbol{\beta}$ is splittable:

$$L_q(H(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,t),T) := \exp\left(-\sum_{r=1}^{\infty} H_{q^r}(\boldsymbol{\alpha};\boldsymbol{\beta}\,|\,t)\frac{T^r}{r}\right) \in 1+TK_{\boldsymbol{\alpha},\boldsymbol{\beta}}[[T]]$$

What were those bad primes in the L-series?

Family	Group	Bad primes
F_4	$(\mathbb{Z}/4\mathbb{Z})^2$	2
F_2L_2	$\mathbb{Z}/8\mathbb{Z}$	2
F_1L_3	$\mathbb{Z}/7\mathbb{Z}$	2,7
L_2L_2	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2
L_4	$\mathbb{Z}/5\mathbb{Z}$	2,5

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Also bad

Primes dividing the numerator or denominator of either ψ^4 or $\psi^4 - 1$.

Main result

Fermat pencil

Let $t = \psi^{-4}$.
For the Fermat pencil F_4,

$$L_S(X_{\mathsf{F}_4,\psi}, s) = L_S(H(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}; 0, 0, 0 \,|\, t), s) \cdot L_S(H(\tfrac{1}{4}, \tfrac{3}{4}; 0, \tfrac{1}{2} \,|\, t), s-1, \phi_{-1})^3 \cdot L_S(H(\tfrac{1}{2}; 0 \,|\, t), \mathbb{Q}(\sqrt{-1}), s-1, \phi_{\sqrt{-1}})^6$$

where

$$\phi_{-1}(p) := \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \qquad \leftrightarrow \mathbb{Q}(\sqrt{-1}) \,|\, \mathbb{Q}$$

$$\phi_{\sqrt{-1}}(\mathfrak{p}) := \left(\frac{\sqrt{-1}}{\mathfrak{p}}\right) = (-1)^{(\mathrm{Nm}(\mathfrak{p})-1)/4} \qquad \leftrightarrow \mathbb{Q}(\zeta_8) \,|\, \mathbb{Q}(\sqrt{-1}).$$

A field of definition

Using our L-series for the Fermat pencil, one may deduce that the minimal field of definition for its Néron-Severi group is $\mathbb{Q}(\zeta_8, \sqrt{1-\psi^2}, \sqrt{1+\psi^2})$.

Main result

Stay tuned for a table

We can summarize our findings in the following table . . .

Pencil	Degree	α	β	Base Field
F_4	3	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	$0, 0, 0$	$\mathbb{Q}$
	$2 \cdot 3 = 6$	$\frac{1}{4}, \frac{3}{4}$	$0, \frac{1}{2}$	$\mathbb{Q}$
	$2 \cdot 6 = 12$	$\frac{1}{2}$	0	$\mathbb{Q}(\sqrt{-1})$, from $\mathbb{Q}$
F_1L_3	3	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	$0, 0, 0$	$\mathbb{Q}$
	18	$\frac{1}{14}, \frac{9}{14}, \frac{11}{14}$	$0, \frac{1}{4}, \frac{3}{4}$	$\mathbb{Q}(\zeta_7)$, from $\mathbb{Q}(\sqrt{-7})$
F_2L_2	3	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	$0, 0, 0$	$\mathbb{Q}$
	$3 \cdot 2 = 6$	-	-	$\mathbb{Q}(\zeta_8)$, from $\mathbb{Q}$
	2	$\frac{1}{4}, \frac{3}{4}$	$0, \frac{1}{2}$	$\mathbb{Q}$
	2	$\frac{1}{2}$	0	$\mathbb{Q}(\sqrt{-1})$, from $\mathbb{Q}$
	8	$\frac{1}{8}, \frac{5}{8}$	$0, \frac{1}{4}$	$\mathbb{Q}(\zeta_8)$, from $\mathbb{Q}(\sqrt{-1})$
L_2L_2	3	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	$0, 0, 0$	$\mathbb{Q}$
	$2 \cdot 4 = 8$	-	-	$\mathbb{Q}(\sqrt{-1})$, from $\mathbb{Q}$
	2	$\frac{1}{4}, \frac{3}{4}$	$0, \frac{1}{2}$	$\mathbb{Q}$
	8	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	$\mathbb{Q}(\sqrt{-1})$, from $\mathbb{Q}$
L_4	3	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	$0, 0, 0$	$\mathbb{Q}$
	$1 \cdot 2 = 2$	-	-	$\mathbb{Q}$
	16	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	$0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	$\mathbb{Q}(\zeta_5)$, from $\mathbb{Q}$