BHK Duality

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Math Reviews (American Mathematical Society)

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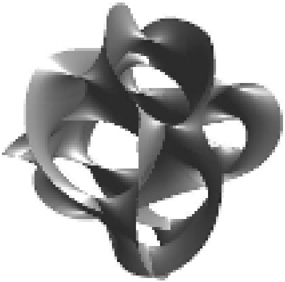

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- Within $GL(n+1, \mathbb{C})$, we can consider the diagonal matrices, with diagonal elements $(\lambda_0, \ldots, \lambda_n)$
- Also within $GL(n+1, \mathbb{C})$, we have the special linear group $SL(n+1, \mathbb{C})$ of invertible $(n+1) \times (n+1)$ matrices with determinant 1.

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- The intersection of $SL(n+1, \mathbb{C})$ and the diagonal matrices consists of diagonal matrices of determinant 1.

Diagonal Polynomial Symmetries

Definition

Let F_A be an invertible polynomial. The group $\mathrm{Aut}(F_A)$ consists of diagonal matrices $M \in GL(n+1, \mathbb{C})$ such that

$$F_A(M\vec{x}) = F_A(\vec{x})$$

for all $\vec{x} \in \mathbb{C}^{n+1}$.

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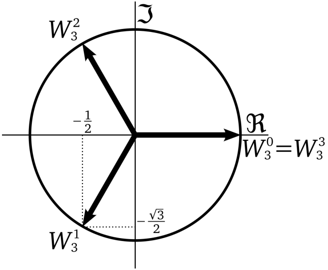
Exercise

1. Find $\mathrm{Aut}(x^2 + y^2)$.
2. Find $SL(x^2 + y^2)$.
3. Find $\mathrm{Aut}(x^4 + y^4 + z^4 + w^4)$.

Digression: Roots of Unity



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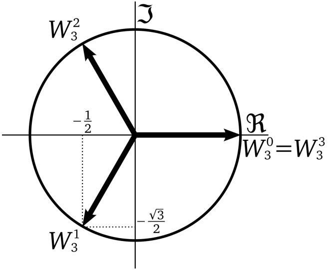

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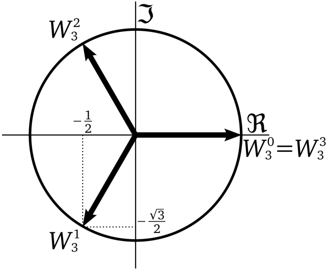

- We can write a complex number in polar form as $re^{i\theta}$, where r is the distance to the origin and θ is the angle from the positive x axis (in radians).
- Numbers on the unit circle are of the form $e^{i\theta}$.
- The n complex solutions to $x^n = 1$ are called roots of unity.

Roots of Unity and Diagonal Symmetries

1. Write the cube roots of unity in the form $e^{i\theta}$. (Hint: $1 = e^{2\pi i}$.)
2. Find $\mathrm{Aut}(x^3 + y^3 + z^3)$.
3. Find $SL(x^3 + y^3 + z^3)$.
4. Find $\mathrm{Aut}(x_0^{n+1} + \cdots + x_n^{n+1})$.
5. Find $\mathrm{Aut}(x^2 y + x y^2)$.

Diagonal Symmetry Facts

- $\mathrm{Aut}(F_A)$ is a finite abelian group.
- The coordinates of each element of $\mathrm{Aut}(F_A)$, written in the form $(\lambda_0, \ldots, \lambda_n)$, are roots of unity.

Diagonal Symmetry Facts

- $\mathrm{Aut}(F_A)$ is a finite abelian group.
- The coordinates of each element of $\mathrm{Aut}(F_A)$, written in the form $(\lambda_0, \ldots, \lambda_n)$, are roots of unity.
- The product of the coordinates of any element of $SL(F_A)$, written in the form $(\lambda_0, \ldots, \lambda_n)$, is 1.

A Matrix Shortcut

Fact

$\mathrm{Aut}(F_A)$ is generated by the columns $\rho_0, \ldots, \rho_n$ of A^{-1}:

$$\begin{bmatrix} r_0 \\ \vdots \\ r_n \end{bmatrix} \mapsto (e^{2\pi i r_0}, \ldots, e^{2\pi i r_n})$$

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Exercise

1. Find $\mathrm{Aut}(x^2y + y^3)$. What is the order of this group?
2. Find $\mathrm{Aut}(x^2 + xy^3)$. What is the order of this group?

Trivial Symmetries

Definition

Let F_A be an invertible polynomial. The trivial diagonal symmetries $J(F_A)$ are the elements of the subgroup of $SL(F_A)$ generated by $(e^{2\pi i q_0/d}, \ldots, e^{2\pi i q_n/d})$.

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Exercise

1. Find $J(x^3 + y^3 + z^3)$.

Our First Mirror

Input

- An invertible polynomial F_A satisfying the Calabi-Yau condition
- The trivial diagonal symmetry group $G = J(F_A)$.

Output

- Take the transpose matrix A^T.
- Consider the polynomial F_{A^T}.
- Let $G^T = SL(F_{A^T})$.
- Our mirror is given by the invertible polynomial F_{A^T} and the group G^T.

Mirror Practice

Exercise

Find the mirror of F_A with the trivial symmetry group.

1. $F_A = x^3 + y^3 + z^3$
2. $F_A = x^4 + y^4 + z^4 + w^4$
3. $F_A = x^2 y + xy^2$

More Symmetry Groups

Question

How can we describe $\mathrm{Aut}(F_{A^T})$?

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How can we describe $\mathrm{Aut}(F_{A^T})$?

Inverse and Transpose are Friends!

$\mathrm{Aut}(F_A)$ is generated by the columns of $(A^T)^{-1}$, which correspond to the rows of A^{-1}. We'll write these generators as $\rho_0^T, \ldots, \rho_n^T$.

A Dual Group

Fix a group G such that $J(F_A) \subset G \subset SL(F_A)$.

Definition

G^T is the subgroup of $\mathrm{Aut}(F_{A^T})$ given by

$$\{\prod_{j=0}^{n}(\rho_j^T)^{m_j} \mid g(\prod_{j=0}^{n} x_j^{m_j}) = \prod_{j=0}^{n} x_j^{m_j} \text{ for all } g \in G\}.$$

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Row and column description

The row vector $[m_0, \ldots, m_j]$ satisfies

$$[m_0, \ldots, m_j]A^{-1}\begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{Z}$$

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Dual Group Practice

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Exercise

1. Let $F_A = x^3 + y^3 + z^3$ and let $G = SL(F_A)$. Find G^T.

BHK Mirrors for Polynomials

Input

- An invertible polynomial F_A satisfying the Calabi-Yau condition
- A group G with $J(F_A) \subset G \subset SL(F_A)$.

Output

- Our mirror is given by the invertible polynomial F_{A^T} and the group G^T.

The Mirror of the Mirror

$$(A^T)^T = A$$

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$$(G^T)^T = G.$$

Toward Geometry

We'd like to associate some geometric meaning to our invertible polynomials.

Naive Solution

Set $F_A = 0$.

Exercise

1. Describe the solutions to $x^2 + y^2 = 0$.

Quasihomogeneous Polynomials

- $(0, \ldots, 0)$ is a solution to any invertible polynomial.
- Any invertible polynomial F_A is quasihomogeneous:

$$F_A(\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n) = \lambda^d (x_0, \ldots, x_n).$$

Visualizing Invertible Polynomial Solutions

Consider the solutions $(x_0, \ldots, x_n)$ to $F_A = 0$.

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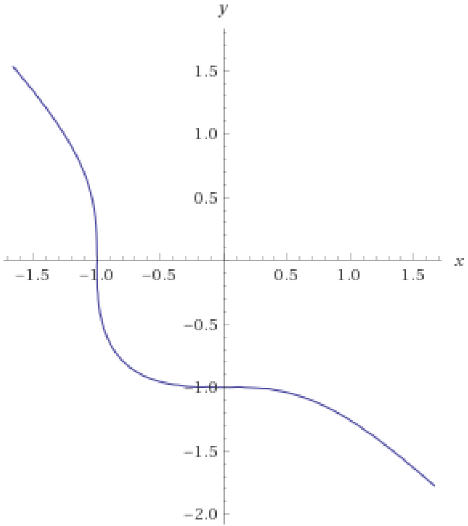
Exercise

Visualize the solutions to $x^2 + y^2 = 0$ in $\mathbb{C}$ and $\mathbb{R}$.

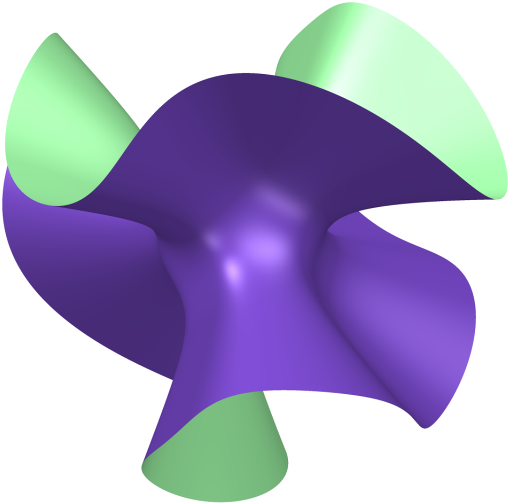
A Curve

We can visualize $x^3 + y^3 + z^3$ by setting $z = 1$:



A Surface

We can visualize $x^3y + y^3z + z^3x + w^4$ by setting $w = i$:



A Threefold

To visualize $x^5 + y^5 + z^5 + v^5 + w^5$, we have to take a lower-dimensional slice.

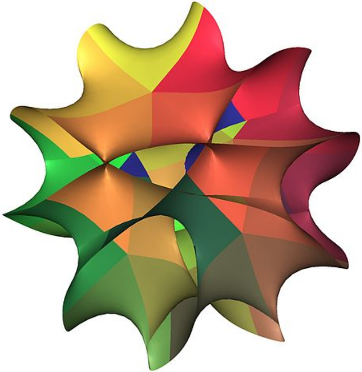


Figure: Slice of a Calabi-Yau threefold

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We can formalize the procedure we used for visualization using the notion of a weighted projective space.

- Start with $\mathbb{C}^{n+1}$. Fix a list of weights $q_0, \ldots, q_n$.

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- Start with $\mathbb{C}^{n+1}$. Fix a list of weights $q_0, \ldots, q_n$.
- Throw away $(0, \ldots, 0)$.
- Glue the remaining points according to the equivalence relation

$$(x_0, \ldots, x_n) \sim (\lambda^{q_0}, \ldots, \lambda^{q_n})$$

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  for $\lambda \in \mathbb{C} - \{0\}$.
- The resulting space is the weighted projective space $\mathbb{WP}(q_0, \ldots, q_n)$.

Compactifying

- We think of an invertible polynomial F_A as defining a subset X_A of $\mathbb{WP}(q_0, \ldots, q_n)$.
- As a topological space, X_A is compact.
- This is nice from both math and physics perspectives!

Quotient Groups

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Geometrically, if $J(F_A) \subset G \subset \mathrm{Aut}(F_A)$, we can think of $\widetilde{G}$ as symmetries of the geometric space X_A.