Hints: Sample Exam 2

1.

a) False. For example the polynomial $x + i$ has a unique root $-i$. The statement would be true for a polynomial $p(x) \in \mathbb{R}[x]$.

b) True. The set is closed under $\star$. The identity element is 4. The operation $\star$ is associative since the usual multiplication is associative. Each element has an inverse $4^{-1} = 4, 2^{-1} = 2$.

c) False. The set is a subring, but not an ideal, for example

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \notin I.$$

d) False. For example the rings have different number of zero divisors (4 and 2 respectively).

e) False: the polynomial is divisible by $x - 1$, since 1 is its root.

f) False. If f is a non trivial homomorphism, then $f(2) = 1$, but then $f(4) = f(2 + 2) = 1 + 1 = 0$ and $f(4) = f(2 \cdot 2) = 1 \cdot 1 = 1$. Contradiction.

2. (15 points) This is done in your textbook, see page 131.

3. (15 points) Let $Z = Z(SL(2, \mathbb{R}))$. First we show that $Z = \{\pm I\}$ (here I denotes the identity matrix). Check that $\pm I \in Z$. We will show that $Z \subset \{\pm I\}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z$. First multiply it from both sides by a diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ to show that $b = c = 0$. Then multiply $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ from both sides by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to show that $a = d$. Finally note that the determinant of $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ has to be 1, i.e. $a = \pm 1$.

The order of I is 1, the order of $-I$ is 2.

4. (15 points) We have three cosets: $0 + I, 1 + I, 2 + I$.

Addition is commutative and $0+I$ is the neutral element. $(1+I)+(2+I)=0+I$, $(2+I)+(2+I)=1+I$. This determines the addition table.

Multiplication is commutative, and $1+I$ is the neutral element for multiplication. $(2+I)(2+I)=1+I$ and $(0+I)(1+I)=(0+I)(2+I)=0+I$. This determines the multiplication table.

The ring R/I is a field, since every non zero element has an inverse: $(2+I)^{-1}=2+I$, and $(1+I)^{-1}=1+I$.

5. (15 points) We have four symmetries:
e - the identity,
a - the symmetry that interchanges B with C and A with D,
b - the symmetry that interchanges A with B and D with C,
c - rotation by 180 degrees, it interchanges A with C and B with D.

Multiplication rule: $a^2=b^2=c^2=e, ab=ba=c, ac=ca=b, bc=cb=a$. Verify it! (Compare solutions to homework 11 for a method).