Math 412 Exam 1

Name:

1. a)False. $\mathbb{Z}$ does not have zero divisors and $\mathbb{Z}\times\mathbb{Z}$ does (for example $(1,0)(0,1) = (0,0)$).

b) True: if $x^2 \equiv 4 \mod 40$ then certainly $x^4 \equiv 16 \mod 40$ and then (even more certainly) $x^4 \equiv 16 \mod 20$.

c) False. Let $a = c = 2, b = 1$. Then $(a, b) = 1, (b, c) = 1$ then $(a, c) = 2$.

d) Almost true. The greatest common divisor of polynomials $(x-1)(2x+3)(x+2)$ and $(x+1)(2x+3)(x-3)$ in $\mathbb{Z}_5[x]$ is the polynomial $\frac{1}{2}(2x+3)(x+2)$.

s 2. a) Does not exist. Let $f : \mathbb{Z}_2 \longrightarrow \mathbb{Z}$ and say $f(1) = n$ for some integer n. Then

$$0 = f(0) = f(1+1) = f(1) + f(1) = n + n = 2n,$$

hence $n = 0$.

b) Does not exists since $3^p - 1$ is an even number strictly greater than 2.

c) For example $2\mathbb{Z}$.

d) $\mathbb{Z}_{11}$ is a field, hence does not have zero divisors.

3. See homework solutions. (Homework 3, sec. 1.3 ex. 20)

4. a) see your textbook, pages 47-48.
b) see pages 69-70.

5. The new zero element is 2, since $a \oplus 2 = a$ for all $a \in \mathbb{Z}$.
The new identity element is 3, since $a \odot 3 = a$ for all $a \in \mathbb{Z}$.
Yes, $(\mathbb{Z}, \oplus, \odot)$ is an integral domain. (see solutions to Homework 5, sec. 31 ex. 21 for a mthod to show that).
No, it is not a field. For example there is no integer x such that $5 \odot x = 3$, i.e. 5 does not have an inverse in $(\mathbb{Z}, \oplus, \odot)$.