Sample Exam 1 - Solutions

Math412, sections 1.1- 4.2

February 8, 2008

1 (a) False: $R \times S$ has zero divisors, for example $(1, 0) \cdot (0, 1) = (0, 0)$.

(b) False: the only units in $\mathbb{Z}_3[X]$ are constant non-zero polynomials.

(c) False: for example $x = 12$ satisfies the first congruence, but 144 is not congruent to 4 modulo 15.

(d) False: for example if $a = 2, b = 2, d = 4$ then $(2, 2) = 2$ and $d = 1 \cdot 2 + 1 \cdot 2$.

2 (a) $\mathbb{Z}$ or $F[x]$ where F is a field.

(b) Any subring is a ring.

(c) For example 5, because $5 \cdot 5 = 25 \equiv 1 \mod 12$. In general any element x such that $(x, n) = 1$ is a unit in $\mathbb{Z}_n$.

(d) $3^{2p} - 4 = (3^p - 2)(3^p + 2)$ and each of the factors is non trivial (i.e. greater than 2).

3. Consider an integer z. Write $z = a_n 10^n + \ldots + 10a_1 + a_0$. Note that $100 \equiv 0 \mod 4$, hence $10^k \equiv 0 \mod 4$ for $k \geq 2$. Hence $z \equiv 10a_1 + a_0 \mod 4$, i.e. z and $10a_1 + a_0$ have the same reminder when divided by 4.

4. $X^3 + 3X + 1 = (2X)(3X^2 + 2) + (-X + 1)$

$3X^2 + 2 = (-3X - 3)(-X + 1) + 0$

The last non-zero reminder is $-X+1$, so the gcd is the monic polynomial $X - 1$.

5. R is not a ring since the distributive law does not hold. For example, let f be a constant function $f(x) = 1, x \in \mathbb{R}$. Let g, h be any functions in R. Then $(f \odot (g \oplus h))(x) = 1$ but $(f \odot g)(x) \oplus (f \odot h)(x) = 1 + 1 = 2$.