Solutions/ Hints to homework 8

March 6, 2008

section 4.4

8. Use Corollary 4.18.
a) reducible, the roots are $\pm\sqrt{7}$,

c) reducible, the roots are $\pm i\sqrt{7}$,

e) reducible, for example 4 is a root,

f) reducible, for example 1 is a root,

b), d) irreducible.

10. For example 17. Then $4^2 + 1 = 0$ in $\mathbb{Z}_{17}$.

13. (a) We will show that every root of $f(x)$ is also a root of $g(x)$ and vice versa. Write $g(x) = uf(x)$, where u is a unit. Alternatively $f(x) = u^{-1}g(x)$. Let a be a root of $f(x)$. Then $g(a) = uf(a) = u \cdot 0 = 0$. Let b be a root of $g(x)$. Then $f(b) = u^{-1}g(b) = u^{-1} \cdot 0 = 0$.

(b) False, for example the polynomials x and x^2 have the same roots but are not associate.

16. Suppose on the contrary that $f(x) \neq g(x)$ in $F[x]$. Then $h(x) = f(x) - g(x)$ is a non zero polynomial of degree $\leq n$. However, $h(x)$ has $n+1$ roots $c_1, \dots c_n$. Contradiction with Corollary 4.16.

17. For example $x^2 + x$. (Well, we can do even better: $3(x^2 + x)$ has five roots!) There is no contradiction, since $\mathbb{Z}_6$ is not a field!

section 4.6

6. Just follow the hint.

7. Suppose that $f(x) = ax^2 + bx + c$ is reducible in $\mathbb{R}$. Then we can decompose $f(x) = a(x - r_1)(x - r_2)$, with $r_1, r_2 \in \mathbb{R}$. However we know from ex. 6 that in $\mathbb{C}$ we have that $f(x) = a(x - z_1)(x - z_2)$, where z_1, z_2 have non trivial imaginary parts. By the uniqueness of the factorization (see Thm. 4.13) we have that $r_1 = z_1$ or $r_1 = z_2$. Contradiction.

8.The answer is no.

Let $f(x) = x^3 - 3x^2 + 2ix + i - 1$. First note that $f(x)$ has a root. (Why?) Then note that$f(x)$ has a complex root. (If all the roots were real, then since $f(x)$ is monic we would have that $f(x) \in \mathbb{R}[x]$).) Let $z = a + bi, b \neq 0$ be a complex root of $f(x)$. We have that $f(z) = 0$, and in particular

$$z^3 - 3z^2 = -2iz - i + 1.$$

Let's evaluate $f(\bar{z})$, where $\bar{z} = a - bi$.

$$f(\bar{z}) = \bar{z}^3 - 3\bar{z}^2 + 2i\bar{z} + i - 1 = \overline{z^3 - 3z^2} + 2i\bar{z} + i - 1 =$$

$$\overline{-2iz - i + 1} + 2i\bar{z} + i - 1 = 2i\bar{z} + i + 1 + 2i\bar{z} + i - 1 =$$

$$= 2i(2\bar{z} + 1) \neq 0.$$