Solutions/ Hints to homework 7

February 21, 2008

section 4.2

10. Let $f(x) = x^3 - 3abx + a^3 + b^3$. It's easy to check that $f(-a-b) = 0$, i.e. $x + a + b$ divides $f(x)$. Therefore the gcd is equal to $x + a + b$.

12, 13. Just adapt the proofs of 1.4 and 1.5.

section 4.3

9. A polynomial of degree 2 or 3 is irreducible iff it does not have a root.

a) $X^2 + X + 1$

b) $X^3 + X^2 + 1$ and $X^3 + X + 1$

13. a) $(3x + 1)(6x + 1) = 1$ in $\mathbb{Z}_9[x]$

b) Let $f(x) \neq 0, 3, 6$. Then $f(x) = 1 \cdot f(x) = (3x + 1)(6x + 1)f(x)$. Now we need to show that either $(3x+1)f(x)$ or $(6x+1)f(x)$ has positive degree.

Assume on the contrary that both $(3x + 1)f(x)$ and $(6x + 1)f(x)$ have degree 0. We will show that this is not possible. We need to consider two cases:

Case 1: the degree of $f(x)$ is greater or equal to 1. Write $f(x) = a_n x^n + \cdots + a_0$where $n \geq 1$ and $a_n \neq 0$ in $\mathbb{Z}_9$. Then

$$(3x + 1)f(x) = 3a_n x^{n+1} + (3a_{n-1} + a_n)x^n + \ldots + a_0,$$

$$(6x+1)f(x) = 6a_n x^{n+1} + (6a_{n-1} + a_n)x^n + \ldots + a_0.$$

Since $n \geq 1$ and since we assumed that $(3x+1)f(x)$ and $(6x+1)f(x)$ are constant we have that $3a_n, 6a_n, 3a_{n-1} + a_n, 6a_{n-1} + a_n$ are all equal to 0 in $\mathbb{Z}_9[x]$. In particular $9|((3a_{n-1}+a_n)+(6_{n-1}+a_n))$ i.e. $9|2a_n$. Since $(9,2) = 1$, we get that $9|a_n$, which contradicts the assumption that $a_n \neq 0$ in $\mathbb{Z}_9$.

Case 2: $f(x) = c$, where $c \in \mathbb{Z}_9$, $c \neq 0$. Then $c(3x+1)$ and $c(6x+1)$ are both constant iff $9|3c$. i.e. iff $c = 3$ or 6. That contradicts the assumption that $f(x) \neq 3, 6$.

14. $x(x+1)$ and $(x-2)(x+3)$