Solutions/ Hints to homework 6

February 15, 2008

section 3.3

10. a) f preserves addition but it fails to preserve multiplication, for example $f(1 \cdot 1) = f(1) = -1$ and $f(1)f(1) = 1$. Hence it is not a ring homomorphism.

b) f is a ring homomorphism. In fact, since $-1 = 1$ in $\mathbb{Z}_2$ we have that $f(x) = x$.

c) g does not preserve neither addition nor multiplication, hence it is not a homomorphism.

d) h does not preserve multiplication.

e) f is a ring homomorphism.

28. Clearly K is non-empty, since $0_R \in K$.

K is closed under subtraction: if $x, y \in K$ then $f(x-y) = f(x) - f(y) = 0_S - 0_S = 0_S$, i.e $x - y \in K$.

K is closed under multiplication: if $x, y \in K$ then $f(xy) = f(x)f(y) = 0_S \cdot 0_S = 0_S$, i.e $xy \in K$.

33. a) $\mathbb{Z}$ has a unit and E does not.

b) $\mathbb{R}^4$ is commutative and $M(\mathbb{R})$ is not.

c) $\mathbb{Z}_4 \times \mathbb{Z}_{14}$ has 56 elements and $\mathbb{Z}_{16}$ has 26 elements.

d) This one is fun. One way to argue is that $\mathbb{Q}$ is countable and $\mathbb{R}$ is not. But if you do not know what "countable" means, here is another argument:

Suppose that $f : \mathbb{Q} \to \mathbb{R}$ is an isomorphism. Then $f(1) = 1$ so $f(2) = 2$. Now, f is surjective, therfore there exists $x \in \mathbb{Q}$ such that $f(x) = \sqrt{2}$. But then $f(x^2) = 2$ (why?) and since f is injective we have that $x^2 = 2$. This is a contradiction.

e) $\mathbb{Z} \times \mathbb{Z}_2$ has zero divisors (find one!) and $\mathbb{Z}$ does not.

f) This one is also fun. Clearly the rings have the same number of elements, so we can not use the same argument as in part c). However we can try to repeat the proof given in example on page 75 ($\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$). Another way is show that the two rings have different numbers of zero divisors or units (i.e. write down all zero divisors in both rings).

section 4.1

6. Only sets described in a) and d) are subrings.

18. D is not a ring homomorphism, since it does not preserve the multiplication. For example, $D(x^2) = 2x$ but $D(x)D(x) = 1$.