Solutions/ Hints to homework 5

February 6, 2008

section 3.1

21. Clearly $\mathbb{Q}$ is closed under the operations $\oplus$ and $\odot$. First let's check that $\oplus, \odot$ are associative and commutative. Let $a, b, c \in \mathbb{Q}$.

$$a \oplus (b \oplus c) = a + (b + c + 1) + 1 = (a + b + 1) + c + 1 = (a \oplus b) \oplus c,$$

$$a \odot (b \odot c) = a(bc+b+c)+a+bc+b+c = (ab+a+b)c+ab+a+b+c = (a \odot b) \odot c,$$

$$a \oplus b = a + b + 1 = b + a + 1 = b \oplus a,$$

$$a \odot b = ab + a + b = ba + b + a = b \odot a.$$

We see that

$$a \oplus (-1) = a - 1 + 1 = a,$$

$$a \odot 0 = 0 + a = a,$$

hence -1 is the neutral element for the operation $\oplus$ and 0 is the neutral element for the operation $\odot$. To avoid confusion let's denote the neutral element for $\oplus$ by $\mathbf{0}_{\oplus}$ and the neutral element for $\odot$ by $\mathbf{1}_{\odot}$. Then $\mathbf{0}_{\oplus} = -1$ and $\mathbf{1}_{\odot} = 0$.

Now, notice that the equation $a \oplus x = \mathbf{0}_{\oplus}$ has a solution. Indeed:

$$a \oplus (-a - 2) = a + (-a - 2) + 1 = -1 = \mathbf{0}_{\oplus}.$$

Finally:

$$a \odot (b \oplus c) = a(b+c+1)+a+b+c+1 = ab+a+b+ac+a+c+1 = a \odot b \oplus a \odot b.$$

(Note that we do not need to check that $(b \oplus c) \odot a = b \odot a \oplus c \odot a$, since we already checked that $\odot$ is commutative).

Now the fun part: is $\mathbb{Q}$ with $\oplus, \odot$ an integral domain? I.e. is it possible that $x \neq \mathbf{0}_\oplus$ and $y \neq \mathbf{0}_\oplus$ but $x \odot y = \mathbf{0}_\oplus$? In other words: is it possible that $x, y \neq -1$ but $xy + x + y = -1$? Let's rewrite the last equation as $xy+x+y+1 = 0$ or even better $(x+1)(y+1) = 0$. This is true iff $x+1 = 0$ or $y + 1 = 0$. Conclusion: $\mathbb{Q}$ with $\oplus, \odot$ is an integral domain! Isn't that beautiful?

24.

$$yx = (x + x)x = xx + xx = y + y = w,$$

$$xy = x(x + x) = xx + xx = y + y = w,$$

$$yz = (z + z)z = zz + zz = y + y = w,$$

$$xz = x(y + x) = xy + xx = w + y = y$$

$$zx = (y + x)x = yx + xx = w + y = y.$$

26. Well, we do not have other choice but to check all the axioms. The set $M_2(\mathbb{Z}_2)$ is clearly closed under addition and multiplication. The addition is commutative, since the addition in $\mathbb{Z}_2$ is. To be more precise, if $a, a', b, b', c, c', d, d' \in \mathbb{Z}_2$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$$

$$= \begin{pmatrix} a'+a & b'+b \\ c'+c & d'+d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Similarly, the addition is associative (Write it down!). It is easy to check that the zero element is the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and the identity element is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (Check it!). The additive inverse of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ where $-a, -b, -c, -d$ denote the additive inverses of the elements a, b, c, d in $\mathbb{Z}_2$.

The multiplication is associative. This one is annoying, so let's do it:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}) \cdot \begin{pmatrix} a''' & b'' \\ c'' & d'' \end{pmatrix} =$$

$$\begin{pmatrix} (aa'+bc')a''+(ab'+bd')c'' & (aa'+bc')b''+(ab'+bd')d'' \\ (ca'+dc")a''+(cb'+dd')c'' & (ca'+dc')b''+(cb'+dd')d'' \end{pmatrix} =$$

$$\begin{pmatrix} a(a'a''+b'c'')+b(c'a''+d'c'') & a(a'b''+b'd'')+b(c'b''+d'd'') \\ c(a'a''+b'c'')+d(c'a''+d'c") & c(a'b''+b'd'')+d(c'b''+d'd'') \end{pmatrix} =$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}).$$

The distribution law is also satisfied (I leave this one for you to check). The ring is non-commutative, since for example

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Finally, the ring has 16 elements since this is in how many different ways we can fill out the four entries in a matrix with zero's and one's.

35. a) Let's just check that $\mathbf{i}^2 = -\mathbf{1}$, since the rest is similar.

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \cdot \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} i^2 & 0 \\ 0 & i^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{1}.$$

b) Check the axioms. Note that H is not commutative since $\mathbf{ij} = -\mathbf{ji}$.

c) d) Just follow the hints.

section 3.2

5a) Yes, $S \cap T$ is a subring. It is closed under subtraction and multiplication: if $x, y \in S \cap T$ then $x, y \in S$ and $x, y \in T$. Since S is a ring we have that $x - y, xy \in S$. Similarly $x - y, xy \in T$. Therefore $x - y, xy \in S \cap T$.

5b) No, it is not closed under subtraction. For example consider $2 \in 2\mathbb{Z}$ and $3 \in 3\mathbb{Z}$. Then clearly $2 - 3 = -1 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$.

10. False. It's not closed under subtraction. For example, 1_R is a unit, but $1_R - 1_R = 0_R$ is not a unit. However, the set of units is closed under multiplication This fact is **very important**. We will come back to it later on.

section 3.3

1. Let's just construct an isomorphism $f : \mathbb{Z}_6 \mapsto \mathbb{Z}_2 \times \mathbb{Z}_3$. By Thm 3.12 parts (1) and (4) we know that $f(0) = (0, 0)$ and $f(1) = (1, 1)$. To be more precise: I mean $f([1]_6) = ([1]_2, [1|_3)$ but for simplicity I will just write $f(1) = (1, 1)$. Therefore:

$$f(2) = f(1 + 1) = f(1) + f(1) = (1, 1) + (1, 1) = (2, 2) = (0, 2),$$

$$f(3) = f(2 + 1) = f(2) + f(1) = (0, 2) + (1, 1) = (1, 3) = (1, 0),$$

$$f(4) = \ldots = (0, 1),$$

$$f(5) = \ldots = (1, 2).$$

This map is clearly injective and surjective. Check, that it preserves addition and multiplication!